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Lurie's construction of Lubin-Tate spectra

goal: construction of Lubin-Tate spectra – realizing Lubin-Tate formal groups via spectral deformation theory, computation of homotopy gps.

§5 EC-2

§1 The Goerss-Hopkins-Miller theorem

Thm (Goerss-Hopkins-Miller)
Thm 5.0.1. EC-2

k - perfect field char $p > 0$

$\hat{G}_0$ - 1-dim formal gp, height $n < \infty$

Then there exists

- a complex periodic E_∞-ring $E \in CAlg(Sp)$ &
- an iso $\alpha : (k, \hat{G}_0) \simeq (\pi_0(E)/I_n^E, \hat{G}_E^{Q})$ of formal grps

s.th.:

1. E is even periodic &
   $(\pi_0(E), \hat{G}_E^{Q}) \to (\pi_0(E)/I_n^E, \hat{G}_E^{Q}) \to (k, \hat{G}_0)$
   is the universal deformation of $\hat{G}_0$.
   In particular $\pi_0(E) \simeq \underbrace{R_{LT}}_{\text{Lubin-Tate ring}} \simeq W(k)[[u_1, \dots, u_{n-1}]]$

2. E is $K(n)$-local & for every $K(n)$-local E_∞-ring A, α induces an equiv
   $Map_{CAlg}(E, A) \simeq Hom_{FGp}((k, \hat{G}_0), (\pi_0(A)/I_n^A, \hat{G}_A^{Q}))$

   discrete set of morphisms in category of formal gps.

§2. Recap / notation

Def's — spectrum A w/ unit $e: \mathbb{S} \to A$ complex orientable if

e factors as $\mathbb{S} \simeq \Sigma^{\infty-2}\mathbb{CP}^1 \longrightarrow \Sigma^{\infty-2}\mathbb{CP}^\infty \longrightarrow A$

defines c_1 & induces Thom iso for cplx v.b.

— E_∞ ring A weakly 2-periodic if $\Sigma^2 A$ rk 1 proj. A-mod.

— cplx periodic := cplx orientable + weakly 2-periodic.

— even periodic if $\pi_*(A) = \pi_0(A)[\tau^{\pm 1}]$ for $\tau \in \pi_{-2}(A)$

($\Rightarrow$ cplx periodic, by computation of $A^*(\mathbb{CP}^\infty)$)

Expl: KU, $\pi_*(KU) \simeq \mathbb{Z}[u^{\pm 1}]$

PMU $\pi_*(PMU) \simeq L[u^{\pm 1}]$

(periodic cplx bordism, Lazard ring see later)

§3 Remarks on the GHM thm

part ① of thm describes homotopy of E

part ② allows transition between $CAlg(Sp)$ & FG under $K(n)$-locality assumption.

spectral — $CAlg(Sp)$; algebraic — FG

used crucially in argument.

Consequences: — Lubin-Tate spectra are unique, if they exist.

in $CAlg(Sp)$

- maps between Lubin-Tate spectra completely determined by formal gp information
- if Lubin-Tate spectrum exists, any cplx periodic E_∞-ring satisfying either ① or ② is equivalent to it.

e.g. if E satisfies ②, get equivalence to L-T-spectrum from iso of formal grps. in char p

assume E' is L-T-spectrum, by ② get $E \to E'$ in $CAlg$

check equivalence by mapping into $K(n)$-local spectra A, for both E, E'

maps given by $Hom_{FG}((k, \hat{G}_0), (\pi_0(A)/I_A, G_A^Q))$

Corollary k perfect field char $p>0$, $\hat{G}_0$ 1-dim formal gp$/k$ ht $n<\infty$

Then $\boxed{Aut_{CAlg}(E(k,\hat{G}_0)) \simeq Aut_{FG}(k, \hat{G}_0)}$

Morava stabilizer gp acts on $E(k,\hat{G}_0)$, essentially unique

Devinatz-Hopkins thm $E_n^{hG_n} \simeq L_{K(n)} S$ — $K(n)$-local sphere.

cts homotopy fixed pts of Morava stab. gp action on Lubin-Tate spectrum E_n

§ 4 overview of pf

I given $(\Sigma, \hat{\mathbb{G}}_0)$, def E as ($K(n)$-localization of) orientation classifier of universal spectral deformation of $\hat{\mathbb{G}}_0$.

II prove mapping property (2) for E (Thm 5.1.5 in EC-2)

III compute homotopy grps of E (Thm 5.4.1 in EC-2, uses mapping property from II, proves property (1))

relation to Goerss-Hopkins:

alg. deformation, then Landweber exactness, E_∞-obstruction theory

E_∞ obstruction theory is replacement of cotangent cplx

cotangent cplx statements $\hat{=}$ vanishing of Goerss-Hopkins obstructions

Lurie does the deformation theory in spectra, deformation works because of statements about cotangent cplx

def of $E_\infty E$ ... structure in appears in computation of homotopy grps.

§5 formal groups, ht, Hasse inv, Landweber ideals

def: height of formal gp law.

R Cring, G 1-dim formal gp.

$[p] : G \to G$ mult by p map.

$n \geq 1$: G has height $\geq n$ if $[p]$ factors through $\hat{G} \xrightarrow{\varphi_G} \hat{G}^{(p^n)}$

$\hat{G}^{(p^n)}(A) = \hat{G}(A^{\frac{1}{p^n}})$

restriction of scalars along Frob $k \to A$, $x \mapsto x^{p^n}$.

(for FGL : power series expansion

$[p](x) = u x^{p^n} + \cdots$

or $[p] = 0$ for height ∞.

obs $\mathcal{E} = \bar{\mathcal{E}}$ for $p > 0$ classification in terms of height)

Remark on Hasse invariants

C alg curve

origin: rank of Frobenius action on Ω^1_C, determines p-rk of Jac

$\# A[p] = p^k$

for ell. curve	— 1	ordinary, fgl ht 1
	— 0	supersingular, fgl ht 2.

version for spectral formal gps in 4.4.9. EC-2

$\hat{G}$/R formal gp over comm. ring, height $\geq n > 0$

$$\begin{tikzcd} \hat{G} \arrow[r, "\varphi_{\hat{G}_n}"] \arrow[dr, "{[p]}"'] & \hat{G}^{(p^n)} \arrow[d, "T"] \\ & \hat{G} \end{tikzcd}$$

relative Frobenius.

$\rightsquigarrow T^* : \omega_{\hat{G}} \to \omega_{\hat{G}^{(p^n)}} \simeq \omega_{\hat{G}}^{\otimes p^n}$

$\rightsquigarrow v_n \in \omega_{\hat{G}}^{\otimes (p^n - 1)}$

Hasse invariant, $v_n = 0$ iff formal gp has ht $\geq n+1$

Def / Prop Landweber ideal

R CRing, $\hat{G}$ 1-dim formal gp $/R$,

There exists f.g. ideal $I_n^{\hat{G}} \subseteq R$ s.t. $I_n^{\hat{G}} \subseteq \ker(R \to R')$ ring hom.

iff $\hat{G}_{R'}$ has height $\geq n$.

(view v_m as R-mod hom $\omega_{\hat{G}}^{\otimes(1-p^m)} \to R$, image f.g. ideal

inductive construction starting w/ $I_0 = (0)$ if G has ht n)

in case $\omega_{\hat{G}}$ triv. $I_n = (p, v_1, \dots, v_{n-1})$)

I_n^{G} is called n-th Landweber ideal

for E_∞-rings, work in π_0

Def $\hat{G}/R$ ht $< n$ $\quad I_n^{G} = \pi_0(R)$

Quillen formal gp.

Thm A (p) local ep'ly periodic E_∞ ring, $n \geq 1$

A $K(n)$-local iff ① complete wrt. $I_n^A \subseteq \pi_0(A)$

② $I_{n+1}^A = \pi_0(A)$

§6 construction of Lubin-Tate spectra & mapping property

R_0 perfect $\mathbb{F}_p$-alg, $\hat{G}_0$ 1-dim fg ht n

construction steps: (I) given $(R_0, \hat{G}_0)$, $\hat{G}_0$ is id. component of connected p-divisible gp. G_0/R_0

(II) G_0 has universal deformation,

$\rightsquigarrow$ p-div. gp G over $R^{univ}_{G_0} \in CAlg(Sp)$, identity component G^0

motivation:
Serre-Tate thm. equiv p-div formal gps $\longleftrightarrow$ connected p-div. gp

$G \longmapsto G[p^\infty]$

conn. comp of id $G^0 \longleftarrow G$

essentially like recover gp from tangent space / infinitesimal data

$\rightsquigarrow$ so def. theories equivalent

Missing thm: if p loc. nilpotent, then p-div. gp formally smooth
$\Leftrightarrow$ unobstructed deformation theory

(III) $R^{or}_{G_0}$ = orientation classifier of G^0

constructed as follows

$\exists\, R^{univ}_{G_0}$-algebra A s.th. G^0 acquires pre-orientation over A

preorientation elt. in $Map_{Sp_*}(S^2, \Omega^\infty \hat{G}(\tau_{\geq 0}A)) \simeq Map_{Mod_{H\mathbb{Z}}}(\Sigma^2 H\mathbb{Z}, \hat{G}(\tau_{\geq 0}A))$
of $\hat{G}/A$

associated Bott map $\beta : \omega_{\hat{G}} \to \Sigma^{-2} A$

$R^{or}_{G_0} = A[\beta^{-1}]$

(IV) $E(\hat{G}_0) = L_{K(n)} R^{or}_{G_0}$ Lubin-Tate spectrum

(for $R_0 = k$ perfect field of char $p>0$ not necessary)

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by construction, G^0 has orientation over $R^{or}_{G_0}$, so does over $E = E(\hat{G}_0)$

$\Rightarrow$ iso $G^0_E \simeq G^Q_E$, after passing to $\pi_0(E)/I^E_n$

determines morphism $\alpha : (R_0, \hat{G}_0) \longrightarrow (\pi_0(E)/I^E_n, G^Q_E)$

Thm (mapping property for $E = E(\hat{G}_0)$)   Thm 5.1.5 EC-2

For every $K(n)$-local E_∞-ring A, composition w/ α induces an equiv

$$\boxed{\mathrm{Map}_{\mathrm{CAlg}}(E, A) \simeq \mathrm{Hom}_{\mathrm{FG}}\left((R_0, \hat{G}_0), (\pi_0(A)/I_n^A, \hat{G}_A^Q)\right)}$$

A $K(n)$-local $\Rightarrow A$ p-nilpotent p contained in ideal of def I_n

Pf sketch

- A adic E_∞ ring, $\pi_0(A)$ has I_n^A-adic top.
- G_A^Q Quillen p-div. gp. (conn. comp. of id. is Quillen formal gp) (Thm 4.4.14)
- morphism of E_∞-rings $E \to A$ maps I_n^E to I_n^A

  $R^{\mathrm{univ}}_{G_0} \to E \to A$ morphism of adic E_∞-rings

$$\Rightarrow \boxed{\mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ad}}}(R^{\mathrm{univ}}_{G_0}, A) = \mathrm{Def}_{G_0}(A) \simeq \operatorname*{colim}_{\substack{I \ni p, \\ \text{f.g. id. of def}}} \mathrm{BT}^p(A) \times_{\mathrm{BT}^p(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I)}$$

(by def of $R^{\mathrm{univ}}_{G_0}$)

Under this identification,

$$\mathrm{Map}_{\mathrm{CAlg}}(E, A) = \operatorname*{colim}_{I} \mathrm{BT}^p_{\mathrm{or}}(A) \times_{\mathrm{BT}^p(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I)$$

G_0 formally connected, so all defs G' appearing are formally connected

G_0 trapped p-div gp $/A$

formally con. oriented p-div gp $/A$ iso to G_A^Q

4.3.23

$$\cong$$

$$\operatorname*{colim}_{I} \{\hat{G}_A^Q\} \times_{\mathrm{FGrp}(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I)$$

formally connected p-div. gp. embed fully faithfully into formal gps (2.3.13) via id. component functor

diagram constant for f.g. ideals of def containing I_n with conclusive ideal

$$\Rightarrow \mathrm{Map}_{\mathrm{CAlg}}(E, A) \simeq \mathrm{Hom}_{\mathrm{FG}}\left((R_0, \hat{G}_0), (\pi_0(A)/I_n, G_A^Q)\right)$$

$\boxtimes$

§7: computation of homotopy gps

Thm (5.4.1 in EC-2)

R_0 perfect $\mathbb{F}_p$-alg (think perfect field)

$\hat{G}_0$ 1-dim formal gp. $/R_0$ exact height n

The spectrum $E = E(\hat{G}_0)$

satisfies the following:

from construction, classifies

① $\alpha : (R_0, \hat{G}_0) \longrightarrow (\pi_0(E)/I_n^E, \hat{G}_E^{\mathbb{Q}})$ is isom.

② $\pi_*(E)$ concentrated in even degrees — even periodic if ω_{G_0} trivial e.g. if $R_0 = k$

③ lifts $\{\bar{v}_m \in \pi_{2(p^m-1)} E\}$ of the Hasse invariants

form a regular seq $p = \bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{n-1}$ in $\pi_*(E)$.

Rmk: formal gp $\hat{G}_E^{\mathbb{Q}} / E \rightsquigarrow$ induced formal gp over $\pi_* E / I_m^E$

of height $\geq m$, so well def^d Hasse invariant $v_m \in \pi_* E / I_m^E$.

Rmk: for $R_0 = k$ perfect field of char $p > 0$, then E even periodic

$$\pi_*(E) = W(k)[\![\underbrace{u_1, \ldots, u_{n-1}}_{\text{in deg } 0,\ u_m = \frac{\bar{v}_m}{e^{p^m-1}}}]\!][e^{\pm 1}] \quad \text{in deg } 2$$

$\rightsquigarrow$ in this case $\pi_0(E)$ is Lubin-Tate ring,

$\hat{G}_E^{\mathbb{Q}}$ is Lubin-Tate formal gp law.

Sketch of proof structure: assume $R_0 = k$ field of char $p > 0$

⓪ preparation: from formal gp $\hat{G}_0 / R_0$ pass to FGL over $\tilde{R}_0$ — $\tilde{R}_0 / R_0$ faithfully flat.

base change of $\hat{G}_0$ has canonical coordinate $\frown$ $\mathcal{O}(\widetilde{\mathrm{Aut}(\hat{A}^1, +)})$

(I) spectral lift: there exists E_∞ ring $\tilde A$ s.th.

$$\begin{tikzcd} \tau_{\geq 0} PMU \arrow[r, "g"] \arrow[d] & \tilde A \arrow[d] \\ L/I_n \arrow[r, "g_0"] & \tilde R_0 \end{tikzcd}$$

connective periodic bordism

Lazard ring mod Landweber ideal

g is $\tau_{\geq 0}PMU$-thickening of g_0

then $A = \tilde A[u^{-1}]$ even periodic $K(n)$-local PMU-algebra.

& $\pi_0(A)/I_n \simeq \tilde R_0$

thickening: A connective E_∞ ring, $I \subseteq \pi_0(A)$ f.g. ideal, $A_0 = \pi_0(A)/I$.

$$\begin{tikzcd} \mathrm{CAlg}_A \arrow[r, phantom, "\rightarrow"] & A \arrow[r, "g"] \arrow[d] & B \arrow[d] \\ \mathrm{CAlg} \arrow[r, phantom, "\rightarrow"] & A_0 \arrow[r, "g_0"] & B_0 \end{tikzcd}$$

g is thickening of g_0 if

(a) B I-complete

(b) $\pi_0(B)/I \simeq B_0$

(c) $\mathrm{Map}_{\mathrm{CAlg}_A}(B,R) \simeq \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{A_0}}(B_0, \pi_0(R)/I)$

for any connective E_∞-alg R/A & I-complete.

Thm 5.2.5 EC-2

existence of thickenings under suitable perfectness assumptions.

get spherical Witt vectors

$$\begin{tikzcd} \mathbb{S}_p^\wedge \arrow[r] \arrow[d] & W^+(B) \arrow[d] \\ \mathbb{F}_p \arrow[r] & B_0 \end{tikzcd}$$

perfect $\mathbb{F}_p$ alg

Pf as for 3.4.1, p-div grp. deformation theory.

(II) identifying A:

$$\boxed{(PMU \wedge E)^\wedge_{I_n} \simeq A. \qquad (*)}$$

by applying mapping property of $E = E(R_0, \hat G_0)$

(a) get $\gamma: E \to A$ from morphism $(R_0, \hat G_0) \to (\tilde R_0, \tilde G_A)$ of formal grps, from construction of A.

(b) for claim $(*)$ suffices to show natural map induces equiv

$$\mathrm{Map}_{\mathrm{CAlg}}(A,B) \longrightarrow \mathrm{Map}_{\mathrm{CAlg}}(PMU \wedge E, B) \simeq \mathrm{Map}_{\mathrm{CAlg}}(PMU, B) \times \mathrm{Map}_{\mathrm{CAlg}}(E,B)$$

for any $K(n)$-local E_∞ ring B,

also follows from mapping property, reducing $\mathrm{Map}_{PMU}(A,B) \simeq \mathrm{Map}_{\mathrm{CAlg}}(E,B)$

to relation of $(\tilde R_0, \hat G_A^{Q})$ to $(R_0, \hat G_0)$

III Koszul cplx & computation of homotopy of E.

want to show $(p, \bar v_1, \ldots, \bar v_{n-1})$ regular seq. in $\pi_*(E)$

Koszul cplx R comm. ring, $(s_1, \ldots, s_n)$ seq of elts.

$$K_* = \bigotimes_{i=1}^{n} [R \xrightarrow{s_i} R] \quad . \quad \text{If } s_1, \ldots, s_n \text{ reg. seq.}$$

$$H^i(K_*) = \begin{cases} R/(s_1, \ldots, s_n) & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Spectral analogue: E as above, $\bar v_m : \Sigma^{2(p^m-1)} E \to E$

$$\boxed{E(m) = \bigotimes_{i=1}^{m-1} \mathrm{cofib}(\bar v_i)} \quad \rightsquigarrow \text{cofib seq} \quad \Sigma^{2(p^m-1)} E(m) \xrightarrow{\bar v_m} E(m) \to E(m+1)$$

(this happens in ∞-cat of E-mod.)

similarly for A as above: $A(m) = E(m) \otimes_E A$

- by construction $\pi_* A(m) \cong \pi_*(A)/I_m$ ($\sim v_i$ form regular seq in A)

  & $\pi_* A(n) \cong \tilde R_0 [u^{\pm 1}]$ (**)

- $PMU \wedge E(n)$ already I_n-complete, $\bar v_i$ act nilpotently

  & $PMU \wedge E$ flat $\Rightarrow$ homotopy of $E(n)$ concentrated in even deg

- descending induction: for all $0 \leq m \leq n$

  cofib seq. $\Sigma^{2(p^m-1)} E(m) \xrightarrow{\bar v_m} E(m) \to E(m+1)$ split into short exact seq in homotopy

  $\Rightarrow \pi_*(E(m))$ in even deg & $\bar v_m$ inj on $\pi_* E(m)$.

  (2) (3) reg. seq.

(1) check after faithfully flat base change to $\tilde R_0$

& $\tilde R_0 [u^{\pm 1}] \simeq \pi_*(PMU \wedge E(n))$ (**)