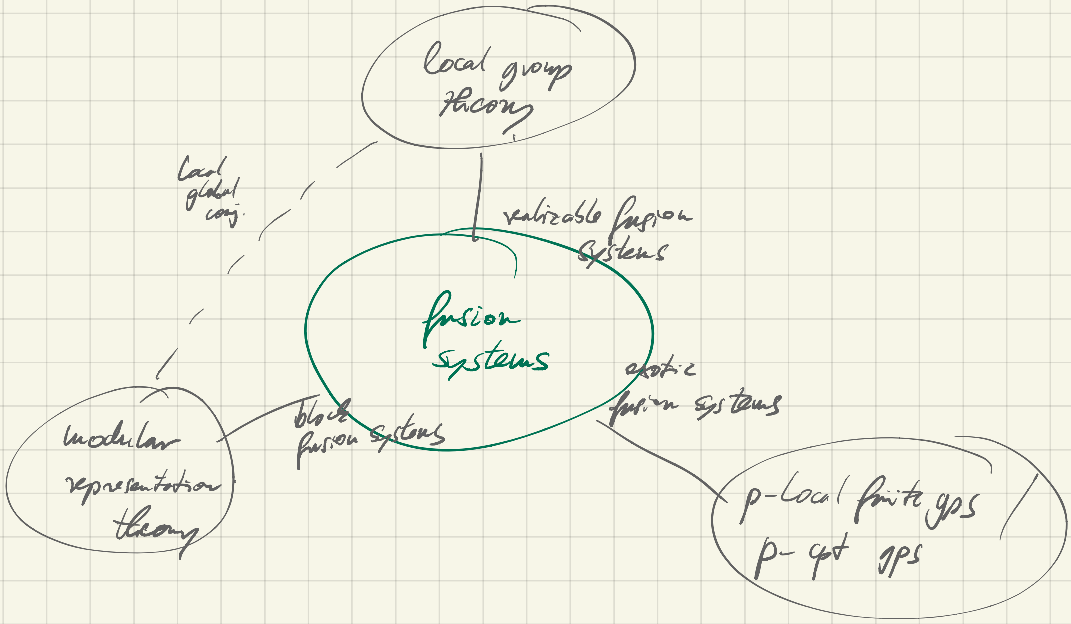
16 Oct 2025

Fusion systems in algebra & topology

§0 overview



Literature:

Aschbacher – Kessar – Oliver : Fusion systems in algebra & topology

Craven : theory of fusion systems : an algebraic approach

lecture notes by Craven, Linckelmann

Central question: What does it mean to "focus on a prime p" in gp theory, representation theory or topology?

goal of lecture: def / examples / properties of fusion systems

fusion systems in gp theory, modular repr. theory

realization of exotic fusion systems in topology

§1 what's a fusion system anyway?

what is fusion? $H \leq K \leq G$ subgp chain

- g, h not conjugate in H, fused in K if K-conjugate
- K controls fusion in H if whenever g, h fused in G, then already fused in K

there's also a subgp version of this.

fusion systems capture conjugacy of p-elements / p-subgps

G group, $H, K \leq G$ subgps

$$\boxed{\mathrm{Hom}_G(H,K) = \{ \varphi \in \mathrm{Hom}_{Grp}(H,K) \mid \varphi = c_g \text{ for some } g \in G \text{ s.th. } gHg^{-1} \leq K \}}$$

morphisms induced by conjugation

Def: S a p-group, fusion system on S is a category $\mathcal{F}$ s.th.

(1) $\mathrm{Obj}(\mathcal{F}) = \{ P \leq S \}$ set of subgps

(2a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$

(2b) each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ factors as composition of an $\mathcal{F}$-iso & an inclusion.

fusion system associated to finite gp G:

Expl: G finite group, $S \leq G$ a p-Sylow subgp.

def $\mathcal{F}_S(G)$: obj. subgps $P \leq S$

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$$

records conjugacy relationships between subgps of p-Sylow $S \leq G$.

Expl: $G = SL_3(\mathbb{F}_2)$, $S = Syl_2 G \cong D_8$ (e.g. strictly upper triang. matrices gen'd by $e_{12}(1)$, $e_{23}(1)$)

objects of $\mathcal{F}_S(G)$: subgps $P \leq S$

(2 elt. ab. C_2^2 $\begin{pmatrix} 1 & * & * \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ & $\begin{pmatrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{pmatrix}$, cyclic $C_4 = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$, 5 cyclic C_2)

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \{ \varphi \in \mathrm{Hom}(P,Q) \mid \varphi = c_g \text{ for some } g \in G \text{ w/ } gPg^{-1} \leq Q \}$$

what is $\mathrm{Aut}(P)$?

for $P = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: normalizer $\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \cong S_4$

centralizer $= P \cong C_2^2$

$$\rightsquigarrow \mathrm{Aut}_{\mathcal{F}_S(G)}(P) = N_G(P)/C_G(P) \cong SL_2(\mathbb{F}_2) \cong S_3$$

Note that $e_{12}(1)$ and $e_{23}(1)$ are conjugate in $N_G(P)$, but not conjugate in $N_G(S) = S$!

(fusion system encodes being conjugate in several steps)

What is $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q)$?

$P = \begin{matrix} 1 & * & * \\ & 1 & 0 \\ & & 1 \end{matrix}$, $Q = \begin{matrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{matrix}$

if $P \to Q$ in $\mathcal{F}_S(G)$, then (after adjusting by suitable autom. of Q)

$\exists\, g \in G$ s.th. $g e_{12}(1) g^{-1} = e_{23}(1)$, $g e_{13}(1) g^{-1} = e_{13}(1)$

$\rightsquigarrow$ $e_{12}(1)$, $e_{23}(1)$ conjugate in $C_G(e_{13}(1)) = S$,

but that's not true.

Exercise: similar to the above, describe $\mathcal{F}_S(G)$ for

$G = S_4$, $S = D_8$ 2-Sylow in S_4,

Why/how is it different from $\mathcal{F}_{D_8}(SL_3\mathbb{F}_2)$?

Rem:

① possible to classify fusion systems on D_8,

3 possibilities: $\mathcal{F}_{D_8}(D_8)$, $\mathcal{F}_{D_8}(S_4)$, $\mathcal{F}_{D_8}(SL_3\mathbb{F}_2)$

② fusion systems for finite gps, i.e., of the form $\mathcal{F}_S(G)$ above,

satisfy some more properties $\sim$ saturated fusion systems (see later)

Def (saturated) fusion system $\mathcal{F}$ is realizable

iff $\exists$ a finite gp G s.th. $\mathcal{F} \simeq \mathcal{F}_S(G)$

otherwise $\mathcal{F}$ is called exotic

Exp(!) – there are exotic fusion systems on S,

for S a 2-Sylow in $Spin_7(\mathbb{F}_q)$, q odd prime power.

(due to R. Solomon)

– Ruiz-Viruel examples: exotic fusion systems on extraspecial gp 7_+^{1+2}

§ 2 group theory

(in finite gp theory: "focus on a prime" = consider normalizers of p-subgps)

many statements from local gp theory can be phrased using fusion systems:

Burnside thm: G finite gp. w/ abelian p-Sylow $S \leq G$

$$\boxed{\mathcal{F}_S(G) \simeq \mathcal{F}_S(N_G(S))}$$

i.e. $N_G(S)$ controls fusion in S

Frobenius normal p-complement thm:

$\mathcal{F}_S(G) \simeq \mathcal{F}_S(S) \iff G$ has a normal p-complement

i.e. normal subgp $H \trianglelefteq G$ s.th. $G = H \rtimes S$.

Alperin fusion thm: (informal, precise statement later?)

$x, y \in S$, can decide conjugacy in G by only looking at p-local subgps (normalizers of subgps of S)

sort of the reason why fusion systems work.

See example $SL_3\mathbb{F}_2$ discussed above

many more results like this, Glauberman Z^* thm, ...

Rem: fusion systems also appear in Classification of Finite Simple Groups, Solomon discovered the exotic fusion systems on $Spin_7(\mathbb{F}_q)$ while trying to characterize Co_3 via its 2-fusion.

§3 representation theory

(in rep. theory: "focusing on a prime"
consider mod p representations,
local-global conj.)

- modular representation theory is the origin of the abstract notion of fusion system (L. Puig, Frobenius categories)

- there are fusion systems associated to p-blocks of finite grps
  (decompose $k[G]$, k of char $p>0$, as direct sum of 2-sided ideals,
  fusion system on defect gp, encodes conjugacy of Brauer pairs)

  Rem: it seems currently open but expected that block fusion systems are realizable.

- fusion systems can be used to formulate some results / conjectures in modular representation theory.
  (see Kessar pt IV in AKO book)

  Britta will say more about this

§4 topology

(in top "focusing on a prime"
// consider mod p cohomology or p-completions)

fusion systems capture

structure of p-completed classifying spaces, allow to realize exotic fusion systems.

classifying space: G discrete gp $\rightsquigarrow$ BG classifying space

(characterized by – connected, $\pi_1 BG \simeq G$,
universal cov. $\widetilde{BG}$ $\sim$ contractible)

p-completion: top. version of p-completion $\varprojlim A/p^n$ for ab gps.

(focus on prime p, π_* p-complete, $f: X \to Y$ iso on mod p-cohom.
iff homotopy equivalence after p-completion)

Martino-Priddy conjecture G, H fin. gps (proved by Oliver, Chermak, Glauberman-Lynd)

$$\mathcal{F}_{\mathrm{Syl}_p G}(G) \simeq \mathcal{F}_{\mathrm{Syl}_p H}(H) \iff BG_p^\wedge \sim BH_p^\wedge$$

$\rightsquigarrow$ Broto-Levi-Oliver p-local finite gps

(things that look like a finite gp at the prime p)

realization of exotic fusion systems

(Benson, Levi-Oliver)

(see also p-local Chevalley gps
Broto-Møller)

- Dwyer-Wilkerson space $BDI(4)$ (p-cpt gp, looks like classifying space of Lie gp at $p=2$, Weyl gp $G_{24} = C_2 \times SL_3 \mathbb{F}_2$)
- Adams operation ψ^q on $BDI(4)$

$\rightsquigarrow$ take homotopy fixed points is classifying space of 2-local finite gp

assoc. to Benson-Solomon exotic fusion system $Sol(q)$

(analogue of how $BGL_n(\mathbb{F}_q)$ is homotopy fixed pts of Adams op
on $BGL_n(\mathbb{C})$)