⑥ Teichmüller space, moduli of curves & the Grothendieck-Teichmüller gp

$\mathcal{M}_{g,n}$ – moduli space of genus g Riemann surfaces w/ n (distinct, ordered) marked points

($\hat{=}$ space of cplx structures on top. surface S

$\hat{=}$ space of hyperbolic metrics on top. surface S if $g \geq 2$)

"uniformization" by Teichmüller space $\mathcal{T}(S)$ for top. surface S

$\mathcal{T}(S_{g,n})$ = cplx structures on $S_{g,n}$ up to holomorphic diffeo isotopic to identity

= space of hyperbolic metrics

= space of monodromy representations $\pi_1(S) \longrightarrow PSL_2\mathbb{R}$.

mapping class gp $\quad \Gamma_{g,n} = \mathrm{Diff}^+(S_{g,n}) / \mathrm{Diff}^0(S)$

orientation-preserving diffeo up to those isotopic to identity

$$\boxed{\mathcal{M}_{g,n} = \mathcal{T}(S_{g,n}) / \Gamma_{g,n}}$$

as orbifolds ! & possibly exclude $g=0,1$ w/ small n.

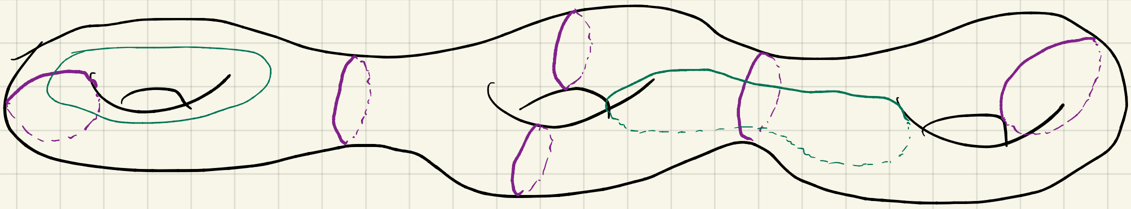
Topology of $\mathcal{T}(S)$: Fenchel–Nielsen coordinates

many ways to define the topology

closed surface S, genus $g \geq 2$

cut along $3g-3$ curves to get pair-of-pants decomposition

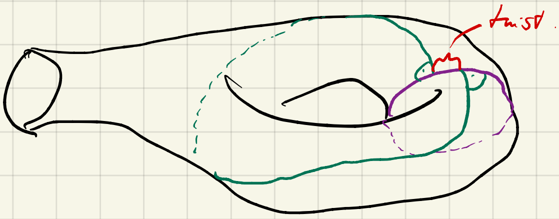
$g=3$
6 curves O



hyperbolic metric on S ⇝ each of the $3g-3$ curves has closed geodesic in homotopy class

⇝ $3g-3$ length parameters.

⇝ $3g-3$ twist parameters from $3g-3$ curves intersecting cut curves minimally



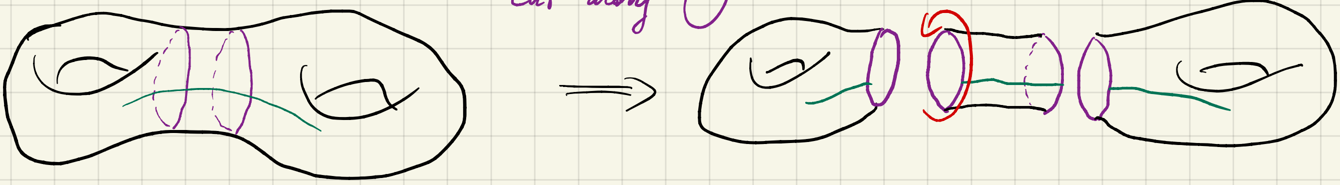
— geodesic fully contained in pair of pants & meeting O orthogonally
twist is distance between endpoints

$$\Longrightarrow \quad \mathcal{T}(S_g) = \mathbb{R}^{6g-6}$$

$$\mathcal{T}(S_{g,n}) \cong \mathbb{R}^{6g-6+2n}$$

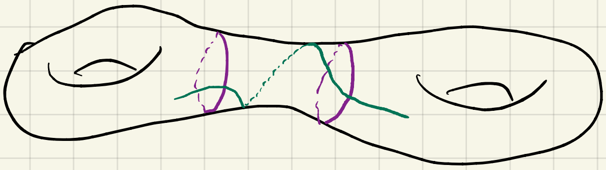
Generators of the mapping class gp : Dehn twists

cut along O



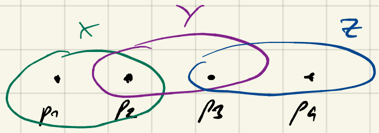
twist as indicated ⟹ & glue back

(full turn!)



The Grothendieck-Teichmüller gp

$$M_{0,4} \cong \mathbb{P}^1 \setminus \{0,1,\infty\}$$



$$\Gamma_{0,4} = \langle x, y, z \mid xyz = 1 \rangle$$

generators - Dehn twists along loops as indicated

Def Grothendieck-Teichmüller gp $\widehat{GT}$: gp of $(\lambda, f) \in \hat{\mathbb{Z}}^\times \times \hat{F}_2'$

s.th. $x \longmapsto x^\lambda$, $y \longmapsto f^{-1} y^\lambda f$ induces autom of $\hat{F}_2$

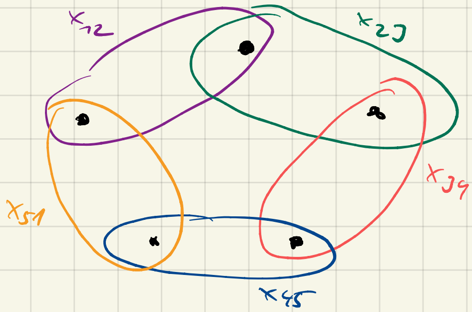
(I) $f(x,y)\,f(y,x) = 1$

(II) $f(x,y)\, x^m f(z,x)\, z^m f(y,z)\, y^m = 1$ w/ $xyz = 1$, $m = \frac{(\lambda-1)}{2}$

5-cycle relation (III) $f(x_{34}, x_{45})\, f(x_{51}, x_{12})\, f(x_{23}, x_{34})\, f(x_{45}, x_{51})\, f(x_{12}, x_{23}) = 1$

in $\hat{\Gamma}_{0,5}$, x_{ij} Dehn twists along loops as indicated

> for $\hat{F}_2 \longrightarrow G$ hom.
> $x, y \longmapsto a, b$
> write $f(a,b)$ for image of $f \in \hat{F}_2$



Ihara's thm: have injection $Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{GT}$

Remarkable: action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\hat{\Gamma}_{0,4}$, $\hat{\Gamma}_{0,5}$ defines $\widehat{GT}$

but then $\widehat{GT}$ extends $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-action on all $\hat{\Gamma}_{0,n}$!

game of Lego-Teichmüller: adding one more relation (possibly not necessary) extends action to all $\hat{\Gamma}_{0,n}$

Grothendieck conjecture: (from the Esquisse ? + Drinfel'd-Ihara stuff, and much more ...)

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\cong} Aut(\hat{T}) \xleftarrow{\cong} \widehat{GT}$$

Teichmüller tower, combines all $\Gamma_{0,n}$