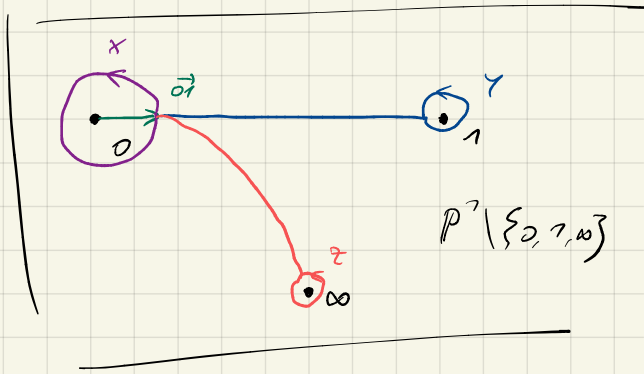
(4) The Drinfeld-Ihara map

lift action $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}(\hat{F}_2)$ to $\mathrm{Aut}(\hat{F}_2)$

need to choose (tangential) base pts



given $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

action on x: $\sigma(x) = x^{\alpha}$

for $\alpha \in \hat{\mathbb{Z}}^{\times}$

(Loop x around 0 generates inertia gp of 0 in $\overline{\mathbb{Q}(t)}/\overline{\mathbb{Q}}(t)$,
$\cong \hat{\mathbb{Z}}$ covering only ramified at 0 is locally $t \mapsto t^n$
then σ acts via cyclotomic character   for details w/ Puiseux series see Ihara paper)

Similar at 1 and ∞, but keep track of tangential base pt $\vec{01}$ at 0,

$$\Rightarrow \quad \sigma(y) = g^{-1} y^{\beta} g \quad , \quad \sigma(z) = h^{-1} z^{\gamma} h$$

$\beta, \gamma \in \hat{\mathbb{Z}}^{\times}$, $h, g \in \hat{F}_2$.

rewrite: $\alpha = \beta = \gamma$ since $x^{\alpha} y^{\beta} z^{\gamma} = 1$ in $\hat{F}_2^{ab} \simeq \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$

assume $g \equiv x^{\delta} y^{\varepsilon}$ in $\hat{F}_2^{ab}$, def $f = y^{-\varepsilon} g x^{\delta}$

$\Rightarrow$ lift of action has the form
$$\boxed{\begin{aligned} \sigma(x) &= x^{\alpha} \\ \sigma(y) &= f^{-1} y^{\alpha} f \end{aligned}}$$

w/ $\alpha = \chi(\sigma) \in \hat{\mathbb{Z}}^{\times}$

$f \in \hat{F}_2'$ ⟵ (commutator / derived subgp)

(lift unique w/ these conditions)

$\Rightarrow$ map $\boxed{Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \hat{\mathbb{Z}}^\times \times \hat{F}_2'}$

(not a gp homomorphism, $\sigma \longmapsto (\lambda_\sigma, f_\sigma)$,
$\tau \longmapsto (\lambda_\tau, f_\tau)$

then $\sigma\cdot\tau \longmapsto (\lambda_\sigma \lambda_\tau, f_\sigma F_\sigma(f_\tau))$ w/ $F_\sigma \in Aut(\hat{F}_2)$: $x \longmapsto x^{\lambda_\sigma}$, $y \longmapsto f_\sigma^{-1} y^{\lambda_\sigma} f_\sigma$

rather maps $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ into ext $1 \to \hat{F}_2' \longrightarrow \square \longrightarrow \hat{\mathbb{Z}}^\times \to 1$)

can see as evidence for Shafarevich conj.

Q : how to understand $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ using this map ?

(using dessins d'enfant, will see that action $Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to Out(\hat{F}_2)$

is faithful and Drinfeld-Ihara map injective using Belyi thm)

how can we determine the image ?

conj. given by Grothendieck-Teichmüller gp. $\widehat{GT}$

subgp of $\hat{\mathbb{Z}}^\times \times \hat{F}_2'$
defined in terms of
$\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) = \pi_1(M_{0,4})$
& $\pi_1(M_{0,5})$

(5) Visualizing the action : dessins d'enfant

Def: Dessin d'enfant: triple $X_0 \subset X_1 \subset X_2$

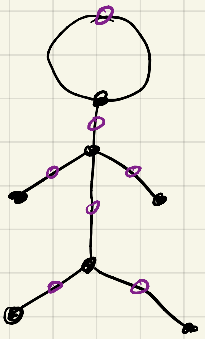
$$\begin{cases} X_0 \text{ finite set of pts}, \quad X_2 \text{ cpt. connected Riemann surface, genus } g \\ X_1 \text{ set of edges connecting vertices in } X_0 \text{ s.th.} \\ X_2 \setminus X_1 \text{ disjoint union of open cells (simply connected / contractible)} \end{cases}$$

+ bicolourable, i.e. $\exists$ partition $X_0 = Y \sqcup Z$
s.th. endpoints of each edge have different colour.

rem: alternative: clean dessins: only one type of cat. X, other type as mark on $X_1 \setminus X_0$

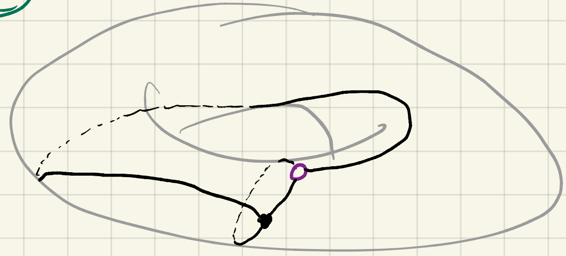
isomorphism of dessins: homeo $X_2 \to X_2'$ inducing homeos $X_1 \to X_1'$, $X_0 \to X_0'$

example: (1)



in $\mathbb{C}P^1$

(2)



Natural bijections:

$\{$ dessins d'enfant $\}$

(1) $\iff$ $\{$ finite covers of $\mathbb{P}^1$ unramified outside $\{0,1,\infty\}\}$

(2) $\iff$ $\{$ finite étale covers of $\mathbb{P}^1 \setminus \{0,1,\infty\}\}$

(3) $\iff$ $\{$ conjugacy classes of finite-index subgps $\Gamma \leq F_2\}$

$\iff$ $\{$ finite ext. of $\overline{\mathbb{Q}}(T)$ unramified outside $(T), (1-T), (\frac{1}{T})\}$

remarks on bijections:

(2)(3) standard covering space theory

& $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \simeq F_2$

(1) given cover $\beta: X \to \mathbb{P}^1$, dessin is $\beta^{-1}([0,1])$

other direction from: bicolored

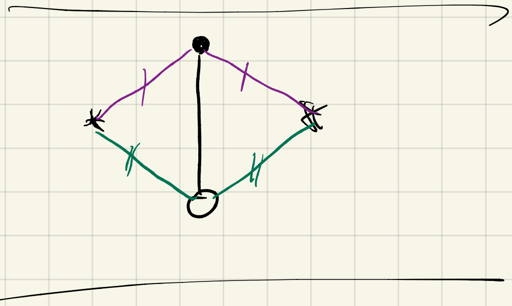
visualizing the covering

add vertex $*$ in each face, add edges to vertices $\circ, \bullet$

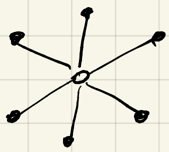
$\to$ triangulation of dessin

then identify marked edges

$\Rightarrow$ quotient $\mathbb{P}^1$ w/ 3 marked pts

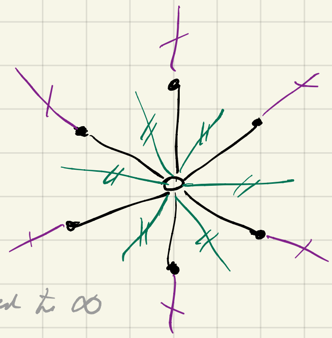


Expl: dessin



n edges

triangulation $\Rightarrow$



all joined to ∞

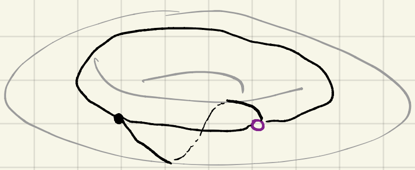
identify quotient w/ $\mathbb{P}^1$, 3 marked points,

covering map $z \longmapsto z^k$.

note: graph determines covering $X \to \mathbb{P}^1$

covering uniquely determines analytic structure on X

Expl:



ell. curve $E: y^2 = x(x-1)(x-\zeta_6)$

$\zeta_6 = \frac{1+\sqrt{-3}}{2}$

curve has order 3 autom. (rotation of lattice $\mathcal{O}_{\mathbb{Q}(\zeta_3)} \subseteq \mathbb{C}$)

Belyi map $E \to \mathbb{P}^1$ is quotient map.

Invariants of coverings / dessins d'enfant

Degree of covering $\beta : X \longrightarrow P^1$ = # edges in dessin

Genus of X from **Riemann-Hurwitz formula**

$f : Y \to X$ map of Riemann surfaces

$$\boxed{2g(Y) - 2 = (2g(X) - 2)\deg(f) + \sum_{p \in Y} (e_p - 1)}$$

$$\boxed{\text{for dessin: } \quad 2g - 2 = \deg\beta - \#\beta^{-1}(0) - \#\beta^{-1}(1) - \#\beta^{-1}(\infty)}$$

Expl: (1) 14 edges $\to$ $\deg\beta = 14$ $\qquad 2g_X - 2 = -2\deg\beta + \sum(e_p - 1)$

for each of $0, 1, \infty$
expected preimages (= deg)
— actual preimages

7 • – preimages of 0
7 • – preimages 1
2 cells – preimages of ∞

$\Rightarrow \quad -2 = -28 + 7 + 7 + 12$, so genus 0

(2) deg 3, 1 preimage for each of $0, 1, \infty$

$\Rightarrow \quad 0 = -6 + 3 \cdot 2$, so genus 1.

Belyi's theorem: connection to alg. curves / $\overline{\mathbb{Q}}$

X sm proj. cplx. curve

Then X defined over $\overline{\mathbb{Q}}$ iff there is a finite morphism $\beta : X \to \mathbb{C}P^1$ ramified at most over $0, 1, \infty$.

Idea: moduli field: fixed field of $\{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) \mid X^\sigma/\mathbb{C} \cong X/\mathbb{C}\} \leq \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$

"$\Leftarrow$" if ramified only over $0, 1, \infty$, only finitely many possibilities for fixed degree

$\Rightarrow$ subgp defining moduli field is finite index $\to$ moduli field is number field

Can show that then curve also defined over a number field

"$\Rightarrow$" if curve def. / $\overline{\mathbb{Q}}$, get $X \to P^1_{\overline{\mathbb{Q}}}$,

then explicit modification to get $\mathbb{Q}$-rational ramification, & reduction of number of ramification pts.

Galois action on dessins

in terms of action on Belyi morphism $\beta: X \to \mathbb{P}^1$.

$\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$\hat{=}$ autom. $\sigma^*: \mathrm{Spec}(\overline{\mathbb{Q}}) \to \mathrm{Spec}(\overline{\mathbb{Q}})$

$\varphi: X \to \mathrm{Spec}(\overline{\mathbb{Q}})$ structure morphism of scheme $\rightsquigarrow$ ${}^\sigma\varphi = (\sigma^{-1})^* \circ \varphi : X \xrightarrow{\varphi} \mathrm{Spec}(\overline{\mathbb{Q}}) \xrightarrow{(\sigma^{-1})^*} \mathrm{Spec}(\overline{\mathbb{Q}})$

scheme ${}^\sigma X$ described by σ applied to polynomials defining X.

Expl.: $E/\mathbb{C}$ ell. curve def^d by $y^2 = x^3 + ax + b$, def^d $/\overline{\mathbb{Q}}$ iff $a, b \in \overline{\mathbb{Q}}$

for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

${}^\sigma E$ def^d by $y^2 = x^3 + \sigma(a)x + \sigma(b)$

$\Rightarrow$ $E \cong {}^\sigma E$ as Riemann surfaces iff

$j(E) = 1728 \dfrac{4a^3}{4a^3 + 27b^2}$ is fixed by σ.

Expl.: $\mathbb{P}^1$ def^d $/\mathbb{Q}$ $\Rightarrow$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ get comm. diag

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{S_{\sigma^{-1}}} & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{\mathbb{Q}}) & \xrightarrow{(\sigma^{-1})^*} & \mathrm{Spec}(\overline{\mathbb{Q}}) \end{array}$$

i.e. $(\sigma^{-1})^*$ lifts to autom. of $\mathbb{P}^1$, $\hat{=}$ $\mathbb{P}^1 \rightsquigarrow {}^\sigma\mathbb{P}^1$.

$$\boxed{\begin{array}{ll} \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \text{ maps} & \beta: X \to \mathbb{P}^1 \\ \text{to} & {}^\sigma\beta = S_{\sigma^{-1}} \circ \beta : {}^\sigma X \to \mathbb{P}^1. \end{array}}$$

Galois action is faithful: $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\sigma \neq 1$

choose $\alpha \in \overline{\mathbb{Q}}$ w/ $\sigma(\alpha) \neq \alpha$

Choose E ell. curve $/\overline{\mathbb{Q}}$ w/ $j(E) = \alpha$

by expl above ${}^\sigma E \not\cong E$, assoc. Belyi morphism $\beta: E \to \mathbb{P}^1$ not fixed by σ.