Introduction to dessins d'enfant & Grothendieck-Teichmüller theory

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goal: explain what dessins d'enfant are & how they are used to study $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

① The absolute Galois gp $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

the absolute Galois gp $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) = \varprojlim_{K/\mathbb{Q} \text{ finite}} Gal(K/\mathbb{Q})$

profinite gp of autom. of $\overline{\mathbb{Q}}/\mathbb{Q}$.

What we know — what we don't know

- there are no unramified extensions $K/\mathbb{Q}$ (Hermite-Minkowski)
- every abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, (Kronecker-Weber)

$$1 \to Gal(\overline{\mathbb{Q}}/\mathbb{Q}^{ab}) \to Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to \hat{\mathbb{Z}}^\times \to 1$$

cyclotomic character from action on roots of unity

class field theory description of abelian ext. of global field via norm quotients of idele class gp.

$$Gal(L/K)^{ab} \simeq C_K / Nm_{L/K}(C_L)$$

generalization Langlands program: n-dim Galois reps $\Longleftrightarrow$ automorphic reps of $GL_n(\mathbb{A})$

Shafarevich conjecture: $Gal(\overline{\mathbb{Q}}/\mathbb{Q}^{ab})$ is a free profinite gp.

motivated by function field — number field analogy:

(true for $k(x)$, $k = \bar{k}$, Douady, Harbater, Pop, based on Riemann existence thm, & $\pi_1(\mathbb{C}P^1 \setminus n+1 \text{ pts})$ free gp of rk n.

Shafarevich conj. predicts that every finite gp. is a quotient of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

inverse Galois problem (known for function fields, but unknown over $\mathbb{Q}$)

(2) Covering spaces: Galois gps & fundamental gps

covering space:

in topology, $p : X \to B$, s.th. each pt $b \in B$ has nbhd. $U \subset B$ s.th.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr] & & \coprod_{p^{-1}(b)} U \arrow[dl] \\ & U & \end{tikzcd}$$

alternatively, unique lifting of paths

in both situations, covering space locally looks like the base space.

in algebraic geometry: étale maps $p : X \to B$, i.e. smooth of relative dim 0,

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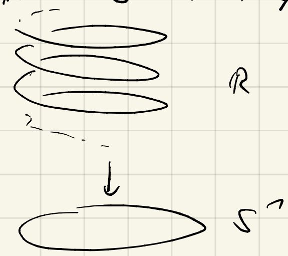
Examples:

(1) $\exp : \mathbb{R} \to S^1 : t \mapsto \exp(it)$



(2) Galois ext L/K $\hat{=}$ étale maps $\operatorname{Spec} L \to \operatorname{Spec} K$.

Theorem on classification of covering spaces

- have a category $\mathrm{Cov}(B)$ of covering spaces of B
- choose pt $b \in B$ $\Rightarrow$ fiber functor $F : \mathrm{Cov}(B) \to \mathrm{Sets}$
  $p : X \to B \longmapsto p^{-1}(b)$
- under suitable conditions, F induces equivalence of categories $\boxed{\mathrm{Cov}(B) \longrightarrow G\text{-Sets}}$
- under suitable conditions, $\boxed{G = \mathrm{Aut}(F)}$

in topology: $G = \pi_1(B, b)$, the fundamental gp.

in algebra: for $B = \operatorname{Spec}(K)$, $G = \mathrm{Gal}(\bar{K}/K)$, the Galois gp.

more generally *in alg. geom*: $G = \pi_1^{et}(B, b)$, the étale fundamental gp.

the algebraic examples are profinite gps
alg. finiteness conditions imply finite fibers

Similar situations: – Tannaka reconstruction of gp. G from fiber functor on cat. of finite-dim G-reps

– motivic Galois gp. from fiber functor on mixed Tate motives.

(3) Absolute Galois group acting on stuff

Grothendieck-Teichmüller theory:

$\boxed{\text{study } Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \text{ via its action on fundamental gps of geometric objects, in particular moduli spaces \& more specifically moduli spaces of curves.}}$

<u>how to get action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on fundamental</u> gps

k field, X/k connected variety (works more generally)

$x: \operatorname{Spec} \bar{k} \longrightarrow X$ geometric pt.

$\rightsquigarrow \quad \pi_1^{et}(X, \bar{x}) = \operatorname{Aut}\left(fib(\bar{x}) : FEt/_X \to Sets \right)$

étale fundamental gp. (SGA 1)

Expl: k field, $x: \operatorname{Spec} \bar{k} \to \operatorname{Spec} k$ alg / separable closure

$$\boxed{\pi_1^{et}(\operatorname{Spec} k, x) \simeq Gal(\bar{k}/k)}$$

Expl: $X/\mathbb{C}$ cplx variety, $x \in X(\mathbb{C})$ pt.

$$\boxed{\pi_1^{et}(X, x) \simeq \pi_1^{top}(X(\mathbb{C}), x)^\wedge}$$ Riemann existence thm.

profinite completion

<u>Fundamental exact sequence:</u> k field, X/k connected variety, $x: \operatorname{Spec} \bar{k} \to X$ geom. pt

$$\boxed{1 \to \pi_1^{et}(X_{\bar{k}}, x) \to \pi_1^{et}(X, x) \to Gal(\bar{k}/k) \to 1}$$

splits if X has k-rational pt,

<u>where sequence comes from:</u> think of $\pi_1^{et}(X, x)$ as

abs. Galois gp. of $k(X)$,

intermediate ext $\overline{k(X)} \;/\; \underbrace{\bar{k}(X) \;/\; k(X)}_{Gal(\bar{k}/k)}$, with $\overline{k(X)} / \bar{k}(X)$ corresponding to $\pi_1(X_{\bar{k}})$

Grothendieck section conjecture: splittings $\hat{=}$ $\mathbb{Q}$-rational pts.

anabelian geometry: if X sufficiently hyperbolic,

e.g. curves $C = \overline{C} \setminus n$ pts & $2-2g-n<0$,

fundamental gp determines X

$\Rightarrow$ for $X/\mathbb{Q}$ get canonical action

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}\left(\pi_1^{top}(X)^\wedge\right)$$

lift element to $\pi_1^{et}(X, x)$, then conjugation on $\pi_1^{top}(X)^\wedge$

$\parallel$

$\pi_1^{et}(X_{\overline{\mathbb{Q}}})$

outer automorphism: don't care about fixing a base point on $X_{\overline{\mathbb{Q}}}$,

change of base point $\hat{=}$ inner automorphisms

Special case $\mathbb{P}^1 \setminus \{0,1,\infty\}$

$\pi_1^{top}(\mathbb{CP}^1 \setminus \{0,1,\infty\}) \cong F_2$ free gp on 2 generators

$\cong \langle x, y, z \mid xyz = 1 \rangle$; x, y, z loops around $0, 1, \infty$

$$\Rightarrow \boxed{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}(\hat{F}_2)}$$