5 Jun 25

Six-functor formalisms, pt II

§1 recall

(six functors & axioms, some of which I hadn't mentioned so far)

Spaces $\rightsquigarrow$ categories of coefficients for cohomology

$X \longmapsto \mathcal{C}(X)$

connected by six functors $\otimes$, $\mathcal{H}om$, $\underbrace{f^* \dashv f_*, \quad f_! \dashv f^!}_{\text{for suitable } f: X \to Y \text{ of spaces.}}$

encode (co)homology & cup product.

some axioms:

- f^* strongly monoidal (encodes cup-product compatibility of f^* & Künneth formula)
- natural transformation $(-)_! \to (-)_*$, equiv. for proper maps
- for lci morphisms natural transformation

  $$f^*(-) \otimes Th_X(L_f) \longrightarrow f^!(-),$$

  equivalence for smooth maps.
- base change

  $$\boxed{f^* p_! \xrightarrow{\sim} q_! g^*}$$

  $$\boxed{q_* g^! \xrightarrow{\sim} f^! p_*}$$

  for cartesian square

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

- projection formula

  $$\boxed{M \otimes f_!(N) \xrightarrow{\sim} f_!(f^* M \otimes N)}$$

  + variants.

Verdier duality : $D_X : M \longmapsto \mathrm{Hom}_{\mathcal{C}(X)}(M, p^! \mathbb{1})$

duality on constructible / cpt. objects $p: X \to pt.$

$$\boxed{\mathbb{D} f_! \xrightarrow{\simeq} f_* \mathbb{D} \quad , \quad f^* \mathbb{D} \xrightarrow{\simeq} f^! \mathbb{D}}$$

§ 2: $1\frac{?}{2}$ more examples

toy example
(not even a half example)

Spaces = finite sets

E some field, maybe $\mathbb{Q}_\ell$.

coefficients : $X \longmapsto \mathrm{Map}(X, E)$

for $f: X \to Y$ get $f^* : \mathrm{Map}(Y, E) \longrightarrow \mathrm{Map}(X, E)$

$Y \to E \longmapsto X \xrightarrow{f} Y \to E$

actually a ring homomorphism!

$f_! : \mathrm{Map}(X, E) \longrightarrow \mathrm{Map}(Y, E)$

$X \xrightarrow{g} E \longmapsto f_! g(y) = \sum_{x \in f^{-1}(y)} g(x)$

"integration over fibers"

$$\boxed{\text{Rem: for } f: X \to \{*\} \quad f_! f^* \mathbb{1} = \# X.}$$

more serious example ℓ-adic sheaves

Spaces = varieties over a finite field $\mathbb{F}_q$.

$$X \in \mathrm{Var}/_{\mathbb{F}_q} \longmapsto \mathbb{D}^{(b)}_{(c)}(X, \mathbb{Q}_\ell)$$

(bounded constructible) derived category of ℓ-adic sheaves on X, built from étale top. on X, $\varprojlim_n$ of system of sheaves w/ $\mathbb{Z}/_{\ell^n\mathbb{Z}}$-coeffs.

then $- \otimes \mathbb{Q}_\ell$

assoc. six functor formalism for ℓ-adic cohomology

well-behaved cohomology theory for varieties over finite fields now can try to translate fun stuff from finite sets to ℓ-adic sheaves, e.g. convolution, Fourier transform, point counting.

§3 Correspondences

we've seen how six functors relate (co)homology of X, Y whenever we have a map $f: X \to Y$

some aspects work more generally for

correspondences

name origin
incidence correspondence?
$\{(p,\ell) \mid p \in \ell\} \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^\vee$

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

Fourier-Mukai transform for kernel $\mathcal{K} \in \mathcal{C}(Z)$

functor $\mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$

$$\boxed{\mathcal{F}/X \longmapsto g_!(f^*\mathcal{F} \otimes \mathcal{K})}$$

(modelled on classical Fourier transform

$pr_{2!}(pr_1^* \otimes ev^* \exp(-2\pi i -))$, for
$$\begin{tikzcd} & \mathbb{R}\times\mathbb{R}^\vee \arrow[dl, "pr_1"'] \arrow[d, "ev"] \arrow[dr, "pr_2"] & \\ \mathbb{R} & \mathbb{R} & \mathbb{R}^\vee \end{tikzcd}$$
)

$\|$

$\hat{f}(\xi) = \int f(x) \exp(-2\pi i \xi x)\, dx$

also, ℓ-adic Fourier transform appears in pf of Weil conjectures (Deligne, Laumon)

cohomological correspondence

between $\mathcal{F}/X$, $\mathcal{G}/Y$

$$\mathcal{K} \in \mathrm{Hom}_{\mathcal{C}(Z)}(f^*\mathcal{F}, g^!\mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{C}(Y)}(g_! f^*\mathcal{F}, \mathcal{G})$$

acts on cohomology (f g proper)

$$R\Gamma(X,\mathcal{F}) \xrightarrow{\text{unit } f^* \dashv f_*} R\Gamma(X, f_*f^*\mathcal{F}) \xrightarrow{\alpha} R\Gamma(X, f_* g^!\mathcal{G})$$

$$\begin{tikzcd}[row sep=small, column sep=small] & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & pt & \end{tikzcd}$$

$$\cong$$

$$R\Gamma(Y,\mathcal{G}) \longleftarrow R\Gamma(Y, g_* g^!\mathcal{G})$$

counit $g_! \dashv g^!$
& g proper $g_* = g_!$

dually, action on cpt. supp. cohom. of f proper.

$$R\Gamma_c(Y,\mathcal{G}) \to R\Gamma(X,\mathcal{F})$$

well-def'd composition of correspondences : base-change

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there is a general six functor formalism def of
trace of an endomorphism (in correspondences)

$$\alpha : f^*\mathcal{F} \longrightarrow g^!\mathcal{F} \quad \text{on} \quad X \xleftarrow{f} Z \xrightarrow{g} X$$

lives in coeff category of fixed-pt. scheme

$$\begin{tikzcd} \mathrm{Fix}(c) \arrow[r, "\delta"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "c"] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

can be applied to graph of Frobenius as correspondence
for varieties over finite fields $\rightsquigarrow$ Weil conjectures

§4 function-sheaf dictionary

(relates the two examples)

Setup: X variety $/\mathbb{F}_q$ $\rightsquigarrow$ $D^b_c(X, \mathbb{Q}_\ell)$

derived cat. of ℓ-adic sheaves on X_0

$\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$

has Frobenius action.

{ informal }

consider trace

$$\boxed{\begin{aligned} tr : D^b_c(X ; \mathbb{Q}_\ell) &\to \operatorname{Map}(X(\mathbb{F}_q), \mathbb{Q}_\ell) \\ \mathcal{F}/X &\longmapsto \underbrace{\left(x \in X(\mathbb{F}_q) \mapsto \sum (-1)^i \, tr\left(\mathrm{Frob}^* \mid H^i \mathcal{F}_x\right)\right)}_{= \, tr_x(\mathcal{F})} \end{aligned}}$$

properties: – traces are additive, compatible w/ composition

– compatible w/ pullbacks

– Grothendieck–Lefschetz–Verdier trace formula

$$\begin{tikzcd} D^b_c(X, \mathbb{Q}_\ell) \arrow[r, "f_!"] \arrow[d, "tr"'] & D^b_c(Y, \mathbb{Q}_\ell) \arrow[d, "tr"] \\ \operatorname{Map}(X(\mathbb{F}_q), \mathbb{Q}_\ell) \arrow[r, "f_!"'] & \operatorname{Map}(Y(\mathbb{F}_q), \mathbb{Q}_\ell) \end{tikzcd}$$

for $f : X \to Y$ morphism of varieties $/\mathbb{F}_q$.

Specialise to $c : X \to \operatorname{Spec} \mathbb{F}_q$

count points

$$\boxed{\begin{aligned} \# X(\mathbb{F}_q) &= c_! c^* 1 = c_! c^* tr(\mathbb{Q}_\ell) \\ &\overset{\text{trace formula}}{=} tr(c_! c^* \mathbb{Q}_\ell) = \sum (-1)^i \, tr\left(\mathrm{Frob}^* \mid H^i_c(\overline{X}, \mathbb{Q}_\ell)\right) \end{aligned}}$$

trace on cohomology

§5 example: categorification of Hecke algebra

$G = GL_n(\mathbb{F}_q)$, $\quad B$ = upper tri-ag matrices

Hecke alg: $\quad \mathcal{H}(G,B)$ = B-biinvariant functions on G w/ convolution as multiplication

convolution in six functor language:

$$\begin{tikzcd} & G \times G \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] \arrow[r, "\mu"] & G \\ G & & G \end{tikzcd}$$

in finite set example: $f, g : G \to \mathbb{Q}$

$$f * g = \mu_!\left((pr_1^* f) \cdot (pr_2^* g)\right)$$

usual def

$(f*g)(c) = \int f(x) g(c-x)\,dx$

for suitable functions on $\mathbb{R}$.

in ℓ-adic sheaf example (for alg. gp $/\mathbb{F}_q$)

$$\boxed{\mathcal{F} * \mathcal{G} = \mu_!(\underbrace{\mathcal{F} \boxtimes \mathcal{G}}_{pr_1^*\mathcal{F} \otimes pr_2^*\mathcal{G}})}$$

$\rightsquigarrow$ can "categorify" Hecke alg., i.e.

trace induces isom.

$$\boxed{K_0\left(D^{b,c}_{B\times B}(G, \mathbb{Q}_\ell)\right) \cong \mathcal{H}(G,B)_{\mathbb{Q}_\ell}}$$

maps constant sheaf on double coset to Hecke operator

- can be applied to geom. rep. theory, Khovanov homology, motivic aspects in geom. rep. theory ...