15 May 25
5 June 25

Six-functor formalism.

I shall not today attempt to define the kinds of material I understand to be embraced within that shorthand description, and perhaps I could never succeed in intelligibly doing so.

— Justice Potter Stewart

§0 overview / origin story

origin in SGA, proof of Weil conjectures
(Grothendieck, Deligne, Verdier, ...)

6FF as framework to formulate properties of cohomology, & in particular duality in different settings
(topological, ℓ-adic, ...)

further uses: - theory of perverse sheaves, Beilinson-Bernstein-Deligne-Gabber
- Fourier-Mukai transforms
- function-sheaf dictionary in geometric representation theory

more recently: - 6FF for motives / motivic stable homotopy
- ∞-categorical versions, w/ improved axiomatics encoding coherences via ∞-cats of correspondences (Mann)
⋮

§1 a categorical framework for cohomology coefficients

want to talk about cohomology $\boxed{H^*(X, \mathcal{F})}$

space we are interested in

appropriate coefficients, e.g. sheaf or cplx of sheaves, ...

- what properties should such an assignment have?
- how does cohomology vary w/ $\underline{X \ \& \ \mathcal{F}}$?

Setup: six-functor formalism is assignment (suitably functorial)

Spaces $\rightsquigarrow$ categories of coefficients for cohomology

$X \longmapsto \mathcal{C}(X)$

top. spaces
analytic varieties
schemes

triangulated or ∞ categories

e.g. q.coh. sheaves, ℓ-adic sheaves, mixed Hodge modules, $\mathcal{D}$-modules, motives, sheaves of (motivic) spectra, ...

Six-functors

$\otimes \dashv \mathrm{Hom}$ — on each $\mathcal{C}(X)$

$f^*: \mathcal{C}(Y) \rightleftarrows \mathcal{C}(X) : f_*$

$f_! : \mathcal{C}(X) \rightleftarrows \mathcal{C}(Y) : f^!$

for suitable maps $f: X \to Y$ of spaces.

+ a long list of axioms encoding natural / useful properties

we'll look at some of these & how they are applied.

Plan:

pt I
- how to get cohomology out of the formalism?
- formulation of duality & examples

pt II
- what we can do w/ 6FF:

trace formula & Weil conj. /
function-sheaf dictionary & geom. rep. theory

---

mostly ignore for now
- how to construct the six operations?

§2 Example: topological spaces & sheaf cohomology

(cf. e.g. Kashiwara – Schapira)

Spaces = loc. cpt. Hausdorff spaces & cts maps

$X \rightsquigarrow D^b(X)$ = bounded derived category of sheaves of ab. gps on X.

see Tommy's lectures

$(L)f^* \dashv Rf_*$ ordinary pullback & pushforward

$Lf_!$ pushforward w/ compact support.

$f_!\mathcal{F} \subseteq f_*\mathcal{F}$ subsheaf of sections w/ cpt support.
for $j: U \hookrightarrow X$ open immersion, $j_!\mathcal{F}$ is extension by zero.

$f^!$ – right adjoint (only on derived category level)

$\otimes \dashv \mathcal{H}om$ tensor product & internal Hom.

cohomology : $H^q(X, \mathcal{F}) = R^q\Gamma(X, \mathcal{F})$ derived global sections

coh. w/ cpt support : $H_c^q(X, \mathcal{F}) = R^q f_!(\mathcal{F})$

more examples

- $\mathcal{D}$-modules (topological & algebraic)
- mixed Hodge modules
- ℓ-adic sheaves
- coherent sheaves
- motives, motivic stable homotopy
- equivariant topology. (also stacks)
- parametrized spectra
- recently a lot of work in (p-adic) analytic settings

also realization functors / as morphisms between six functor formalisms

§ 3 (Co)homology from six functors

for $p: X \to pt$ recover cohomology etc. from six functors:

cohomology: $p_* p^* \mathbb{1} \in \mathcal{C}(pt)$ — in example above this is $R\Gamma(X, \underline{\mathbb{Z}})$

or coh. grps $\mathrm{Hom}_{\mathcal{C}(pt)}(\mathbb{1}, p_* p^* \mathbb{1}[n]) = H^n(X, \underline{\mathbb{Z}})$

coh. w/ cpt. support: $p_! p^* \mathbb{1}$

homology: $p_! p^! \mathbb{1}$

Borel-Moore homology: $p_* p^! \mathbb{1}$.

immediately generalize to $f: X \to Y$.

relative versions

functorialities recovered from adjunctions:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & pt & \end{tikzcd}$$

$f^* \dashv f_* \rightsquigarrow$ unit map

$\mathrm{id} \to f_* f^*$

$\mathrm{Hom}_{\mathcal{C}(X)}(f^*\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{C}(Y)}(\mathcal{F}, f_*\mathcal{G})$

corresponds to $\mathrm{id}: f^*\mathcal{F} \to f^*\mathcal{F}$ under adjunction.

$q_*\mathcal{F} \to q_* f_* f^* \mathcal{F} = p_* f^* \mathcal{F}$ induces $H^*(Y, \mathcal{F}) \to H^*(X, f^*\mathcal{F})$

analogous: $f_! \dashv f^! \rightsquigarrow$ counit map

$p_! p^! \mathbb{1} = q_! f_! f^! q^! \mathbb{1} \to q_! q^! \mathbb{1}$ induces $H_*(X, \mathbb{Z}) \to H_*(Y, \mathbb{Z})$

Cup product from f^* strong monoidal:

for $f: X \to Y \rightsquigarrow f^*: \mathcal{C}(Y) \to \mathcal{C}(X)$ is strong monoidal functor,

i.e. $f^*(M \otimes N) \simeq f^*(M) \otimes f^*(N)$

$\rightsquigarrow f_*$ lax monoidal, i.e. have natural transf. $f_*(M) \otimes f_*(N) \to f_*(M \otimes N)$

$\rightsquigarrow$ for $A \in \mathrm{CAlg}(\mathcal{C}(X))$, $f_* A \in \mathrm{CAlg}(\mathcal{C}(Y))$

so $p_* \mathbb{1}$ is CAlg

in example above, get cup product on

$H^*(X, \underline{\mathbb{Z}}) = \bigoplus_n \mathrm{Hom}_{\mathcal{C}(pt)}(\mathbb{1}, p_* p^* \mathbb{1}[n])$

§4 application: formulating duality

let's start w/ simple adjunction, for $f: X \to pt$

$$\boxed{\mathrm{Hom}(f_! f^* \mathbb{1}[n], \mathbb{1}) \simeq \mathrm{Hom}(\mathbb{1}, f_* f^! \mathbb{1}[-n])}$$

$H^n_c(X)^\vee$ $\qquad$ $H^{BM}_{-n}(X)$

can be taken as def of Borel-Moore homology

this becomes more interesting, for f smooth, proper

f proper: $\qquad$ cf §5

$\boxed{f_! = f_*}$, so LHS is $H^n(X)^\vee = H^n_c(X)^\vee$

f smooth: $\qquad$ cf §6

$\boxed{f^! \mathbb{1} = \omega_X[d]}$ $\left(\text{generally } f^! M \simeq f^! \mathbb{1} \otimes f^* M \right)$

orientation sheaf, local system from orientation covering underlying the line bundle $\omega_X = \Lambda^d \Omega_X$

so RHS is $\mathrm{Hom}(\mathbb{1}, f_* \omega_X[d-n]) = H^{d-n}(X, \omega_X)$

if in addition X orientable $(\omega_X \simeq \mathbb{Z})$ this is $H^{d-n}(X)$

$\Longrightarrow$ Poincaré duality $\boxed{H^n(X)^\vee \simeq H^{d-n}(X)}$

for cpt. orientable mfd, field coeffs, or dualize complexes & then take cohom.

Verdier duality

$$\boxed{\mathbb{D}_X : M \longmapsto \mathrm{Hom}_{\mathcal{C}(X)}(M, f^!\mathbb{1})}$$

- duality on constructible (= compact) objects

$\mathbb{D}_X \circ \mathbb{D}_X \simeq \mathrm{id}$

- exchanges $f_*, f_!$ & $f^*, f^!$

$$\boxed{\mathbb{D} f_! \xrightarrow{\simeq} f_* \mathbb{D}}, \quad \boxed{f^* \mathbb{D} \xrightarrow{\simeq} f^! \mathbb{D}}$$

version of proj. formula

expl: $f_! f^! \mathbb{1}$ is the Verdier dual of $f_* f^* \mathbb{1}$, in $\mathcal{C}(pt)$

duality between cohomology and homology

useful for relative versions & generalizations of duality

benefit of formulating duality

via six functors:

- similar patterns in different settings
- once we have sheaf formulation of Poincaré duality compatible w/ six functors, can prove by local computation.

§5 proper pushforward & base change

pushforward functors related for proper maps

there is a 2-natural transformation $(-)_! \to (-)_*$

which is an isomorphism for proper maps

consequence: proper pushforwards in cohomology (w/ cpt support)

induced by $q_! \mathcal{F} \to q_! f_* f^* \mathcal{F} = q_! p_! f^* \mathcal{F} = p_! f^* \mathcal{F}$

base change: for a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

we have a canonical isom.

$$\boxed{f^* p_! \xrightarrow{\sim} q_! g^*}$$

(+ suitable assumptions, sep / loc. fin. type)

consequence: proper base change theorem

for $f: X \to Y$ proper, $\mathcal{F}$ sheaf on X — stalk of $R^n f_* \mathcal{F}$ at $y \in Y$ is $H^n(f^{-1}(y), \mathcal{F})$ — coh. of fiber

projection formula:

$$\boxed{M \otimes f_!(N) \xrightarrow{\sim} f_!(f^* M \otimes N)}$$

iso left/right adj. / ambidexterity

proj. formula for other classes.

$f_!(f^* y \cup x) = y \cup f_!(x)$.

§6 Smooth pullback, purity & fundamental classes

(pullback functors related for smooth maps)

factors as regular closed immersion + smooth map

twist by Thom space of relative cotangent cplx.

for lci morphisms $f: X \to Y$ there is a natural transformation

$$f^*(-) \otimes \mathrm{Th}_X(L_f) \longrightarrow f^!(-)$$

which is an isomorphism if f is smooth.

this is called relative purity.

Consequence:

— Gysin maps (transfers, wrong-way functoriality)

for $f: X \to Y$ smooth proper, rel dim d, have pushforward in cohomology

$$H^{*+d}(X, f^*\mathcal{F}) \longrightarrow H^*(Y, \mathcal{F})$$

induced from $p_* f^* \mathcal{F}[d] = q_* f_! f^! \mathcal{F} \xrightarrow{\text{counit}} q_* \mathcal{F}$

Similar for regular codim d immersion.

— trace map $f_! \Sigma^{L_f} f^* \longrightarrow \mathrm{id}$ by adjunction

— fundamental classes

$$\boxed{f^*(-) \otimes \mathrm{Th}(T_f) \simeq f^!(-)}$$

$\rightsquigarrow$ iso $\mathrm{Th}(T_f) \simeq f^! \mathbb{1}$

$\rightsquigarrow$ Thom iso, Euler classes as $s^! s_*(1)$ for zero section $s: X \to E$ of vb.

base change: for a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow[f]{} & S \end{array}$$

we have a canonical iso.

$$\boxed{q_* g^! \xrightarrow{\sim} f^! p_*}$$

(+ suitable assumptions, sep / loc. fin. type)

§ 7 Example: (quasi) coherent sheaves

$$\boxed{X \in Sch_{/R} \rightsquigarrow D(QCoh(X))}$$

Grothendieck, Hartshorne Residues & Duality, but no $f_!$

can be fixed by solid modules in Clausen-Scholze condensed math.

duality: coherent setting

$f: X \to Y$ proper morphism noetherian fin.-dim schemes

$$\boxed{R\mathcal{H}om_Y(Rf_* \mathcal{F}, \mathcal{G}) \xrightarrow{\sim} Rf_* \, R\mathcal{H}om_X(\mathcal{F}, f^! \mathcal{G})}$$

$\mathcal{F}$ bounded above cplx of $\mathcal{O}_X$-mod., quasi-coh. cohom.
$\mathcal{G}$ bounded below " " coherent cohom.

Serre duality for $f: X \to \operatorname{Spec} k$, $X/_k$ smooth, proj, dim d

$$LHS = H^q(X, \mathcal{F})^\vee \qquad \mathcal{G} = \mathcal{O}_{\operatorname{Spec} k}$$

$$RHS = \operatorname{Ext}^{d-q}(\mathcal{F}, \Omega^d_{X/k}) \qquad f^!\mathcal{G} = \Omega^d_{X/k}[d]$$

dualizing sheaf.

adjunction map $f_! f^! \to \mathrm{id}$ induces

trace map $H^d(X, \Omega^d_{X/k}) \to k$, iso for X smooth proper

preimage of $1 \in k$ fundamental class / volume form.

(more generally $f: X \to Y$ smooth, rel. dim d

$$f^!(-) \simeq \Omega^d_{Y/X}[d] \otimes f^*(-) \;)$$