Poincaré on the way to his conjecture

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Poincaré und seine Vermutung

- Table of content:

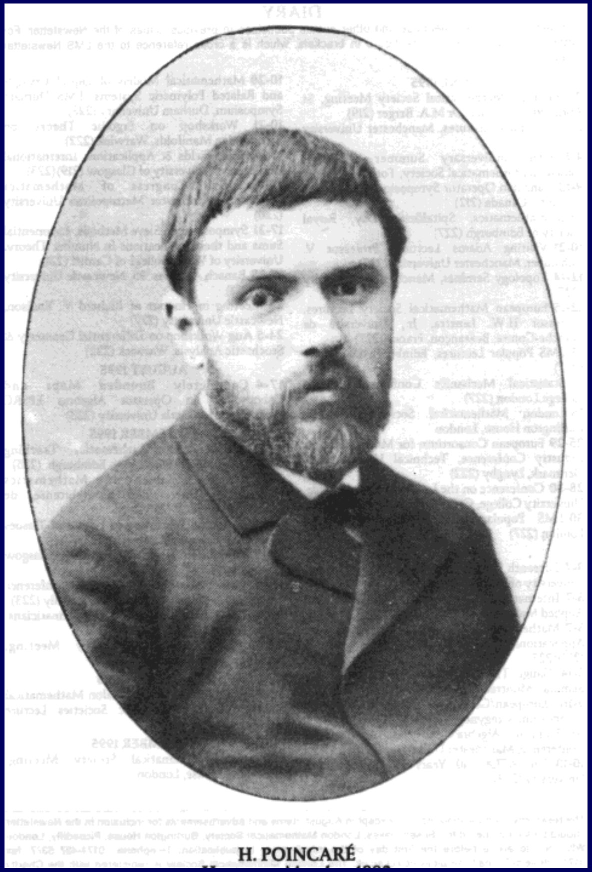
1. Life and Oeuvre

Life and Oeuvre

- 29.4.54 born in Nancy (Lorraine)
- 1873 – 75 Ecole polytechnique (Charles Hermite)
- 1875 – 1878 Ecole des mines
- 1878 First mathematical paper published
- 1879 Promotion
- 3.4.1879 Ingénieur des mines (Vésoul [Vosges])

Life and Oeuvre

- 1.12.1879 Chargé de cours (Caen)
- 29.10.81 Maître de conférences d'analyse (Paris)
- Chargé de cours de mathématique physique (Paris)
- 1886 Chaire de physique mathématique et de calcul des probabilités

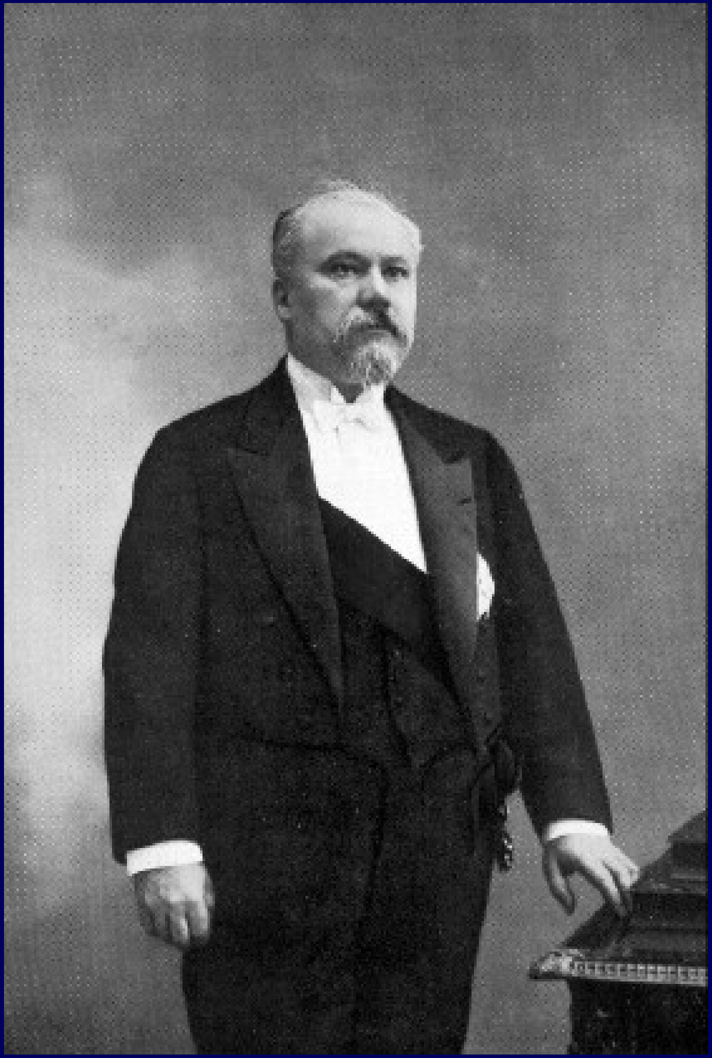


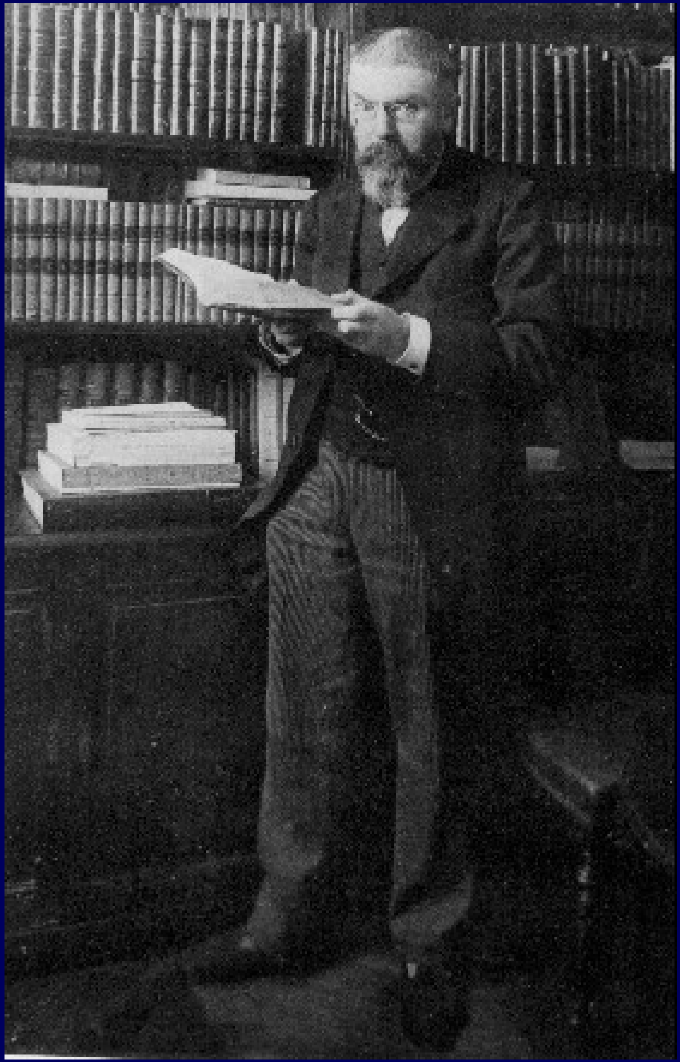
Life and Oeuvre

- 1887 Académie des sciences
- 1896 Chaire d'astronomie
- 1909 Académie francaise
- 1910 Inspecteur général des mines
- 17.7.1912 Poincaré dies at Paris

Life and Oeuvre

- 1881 Henri marries Eugénie Poulain d‘Andecy (of the Geoffroy Saint Hilaire family)
- Four childs: Jeanne (*1887), Yvonne (*1889), Henriette (*1891), Léon (*1893)
- Raymond Poincaré is a cousin of Henri





Life and Oeuvre

- Dr. Toulouse (1897):

  „M. Poincaré ist ein Mann mittlerer Größe (1,65m) und mittleren Gewichtes (70 kg mit Kleidern), mit einem leicht vorspringenden gewölbten Bauch. Sein Gesicht ist gebräunt, die Nase groß und rot. Haarfarbe dunkelblond, der Schnurrbart ist blond. Die Feinmotorik ist voll entwickelt. ... Handschuhgröße: 7 ¾, Schuhgröße: 42. ... Er raucht nicht und hat es auch nie versucht, weil er kein Interesse am Tabak empfand. Er ist nicht verfroren und ist für Kälte nicht empfindsamer als andere Menschen. Dennoch leidet er unter Erkältungen und Entzündungen

Leben und Werk

der Stirnhöhlen. Er schläft nicht bei geöffnetem Fenster. ... Seine Physiognomie wird beherrscht von andauernder Zerstreutheit. Man spricht mit ihm und hat den Eindruck, dass er das Gesagte weder aufnimmt noch versteht, selbst wenn er über eine gestellte Frage nachdenkt oder diese beantwortet. ... M. Poincaré meint, einen ruhigen, freundlichen und ausgeglichenen Charakter zu besitzen. Allerdings fehlt es ihm an jeglicher Geduld, selbst für seine Arbeit. Er lässt sich weder durch seine Gefühle noch durch seine Ideen hinreißen; er ist weder verbindlich noch vertrauensselig. Im praktischen Leben zeigt er sich diszipliniert. ... Er spielt nicht Schach und nimmt an, dass er kein guter Spieler sein würde. Er geht nicht jagen.“

2. Poincaré and topology

Poincaré and topology

- „All the fields in which I worked lead me to topology.“ (~1902)
- Curves defined by differential equations
- Functions of two variables
- Periods of multiple integrals
- Discret or finite subgroups of continuous groups

Poincaré and topology

- **Papers on topology**

1. Sur l‘analysis situs (Comptes rendus 1892)
2. Sur la généralisation d‘un théorème d‘Euler relatif aux polyèdres (Comptes rendus 1893)
3. Analysis situs (Journal de l‘Ecole Polytechnique 1895)
4. Sur les nombres de Betti (Comptes rendus 1899)
5. Complément à l‘analysis situs ((Rendiconti Circolo Palermo 1899)
6. Second complément à l‘analysis situs (Proceedings London Mathematical Society 1901)
7. Sur l‘analysis situs (Comptes rendus 1901)
8. Sur la connexion des surfaces algébriques (Comptes rendus1901)
9. Sur certaines surfaces algébriques; troisième complément à l‘analysis situs (Bulletin SMF 1902)

Poincaré and topology

10. Sur les cycles des surfaces algébriques; quatrième complément à l‘analysis situs (Journal des mathématiques 1902)
11. Cinquième complément à l‘analysis situs (Rendiconti Circolo Palermo 1904)

All (with the only exception of number 2) are to be found in volume VI of the „Oeuvres d‘ Henri Poincaré“.

1880 | 82

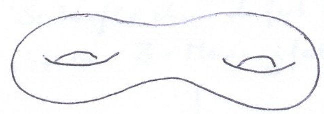
$$f(z) = f(\omega(z))$$

mit $\omega(z) = \dfrac{az+b}{cz+d}$

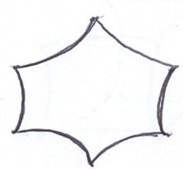
wobei $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$

„Fuchssche Funktionen"

Identifikation zugeordneter Kanten liefert geschlossene



Fläche vom Geschlecht 2g



Parkettierung von H^2 durch 4g − Ecke;

Ω ist Decktransformations-gruppe mit 2g Erzeugenden

$\omega \in \Omega$

$\Omega \subset PSL(2,\mathbb{R})$

Untergruppe

„Fuchssche Gruppe"

$PSL(2,\mathbb{R}) \cong Isom^+(H^2)$

Ω operiert auf H^2 eigentlich diskontinuierlich

1884

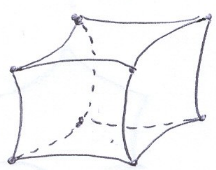
$$f(z) = f(\omega(z))$$

mit $\omega(z) = \dfrac{az+b}{cz+d}$

wobei $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C})$

„Kleinsche Funktionen"

Identifikation zugeordneter Seitenflächen liefert geschlossene 3-Mannigfaltigkeit



Raumteilung von H^3 durch Polyeder;

Ω ist Decktransformationsgruppe

$\omega \in \Omega$

Untergruppe

$\Omega \subset PSL(2,\mathbb{C})$

„Kleinsche Gruppe"

$PSL(2,\mathbb{C}) \cong Isom^{+}(H^3)$

Ω operiert auf H^3 eigentlich diskontinuierlich

3. First steps to the Poincaré conjecture

First steps to the Poincaré conjcture

Analysis situs (1895): Table of content

First steps to the Poincaré conjcture

10. Représentation géométrique
11. Représentation par un groupe discontinu
12. Groupe fondamental
13. Equivalences fondamentales
14. Conditions de l'homéomorphisme
15. Autres modes degénération
16. Théorème d'Euler
17. Cas où p est impair
18. Deuxième démonstration (page 282)

First steps to the Poincaré conjcture

- **Analysis situs (1892/1895)**

  A first question [Oeuvres VI, 189f]

  „One may ask oneself whether or not the Betti numbers suffice to characterize the closed manifolds from the point of view of Analysis situs. That is: Is it always possible to pass from one manifold to another with the same Betti numbers by a continuous deformation? This is true in three-dimensional space; one may think that it is true in arbitrary spaces. But the contrary is the case.“

First steps to the Poincaré conjcture

Poincaré now introduces the fundamental group.

Example 6: closed 3–manifolds with the same Betti numbers, but with non-isomorphic fundamental groups.

Cube with identifications on its faces (cf. the theory of automorphic functions (Fuchsian/Kleinian functions)).

First steps to the Poincaré conjcture

- **Poincaré‘s counter-example**:

  Take the unit cube 1 in ordinary 3-space and consider the following mappings („substitutions“):

$$S_1 : (x, y, z) \mapsto (x+1, y, z)$$

$$S_2 : (x, y, z) \mapsto (x, y+1, z)$$

$$S_3 : (x, y, z) \mapsto (ax+by, cx+dy, z+1)$$

The matrix (a,b,c,d) in the third mapping is an element of SL(2,Z).

First steps to the Poincaré conjcture

If we identify the faces of the cube using these mappings we get a whole series of closed 3-manifold:

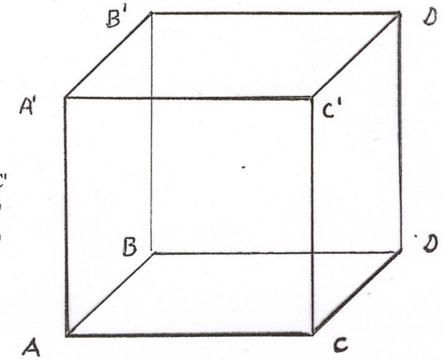
the **cube manifolds M(a,b,c,d)**

In general the topology of the manifold obtained depends on the matrix.

The simplest case is the unit-matrix (1,0,1,0). This yields the 3-**torus** (Poincaré's example no. 1):

M(1,0,1,0)

Beispiel 1:



(C_1) ABDC ≡ A'B'D'C'

(C_2) ABB'A' ≡ CDD'C'

(C_3) ACC'A' ≡ BDD'B'

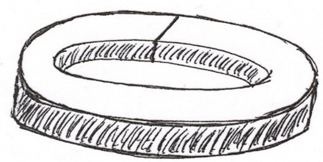
$$AC \overset{C_3}{\equiv} BD \overset{C_1}{\equiv} B'D' \overset{C_3^{-1}}{\equiv} A'C' \overset{C_1^{-1}}{\equiv} AC$$

$$AA' \overset{C_2}{\equiv} CC' \overset{C_3}{\equiv} DD' \overset{C_2^{-1}}{\equiv} BB' \overset{C_3^{-1}}{\equiv} AA'$$

$$AB \overset{C_2}{\equiv} CD \overset{C_1}{\equiv} C'D' \overset{C_2^{-1}}{\equiv} A'B' \overset{C_1^{-1}}{\equiv} AB$$

$$C_2 \cdot C_1 \cdot C_2^{-1} \cdot C_1^{-1} = 1$$
$$C_3 \cdot C_2 \cdot C_3^{-1} \cdot C_2^{-1} = 1$$
$$C_3 \cdot C_1 \cdot C_3^{-1} \cdot C_1^{-1} = 1$$

↓ C_2 (linke und rechte Seitenfläche identifizieren)



↓ C_1 (Boden- und Deckfläche identifizieren)

$T^2 \times I$

↓ C_3 (innerer Torus mit dem äußeren identifizieren)

M

$$\pi_1(M) = \langle C_1, C_2, C_3 \mid [C_1, C_2] = [C_2, C_3] = [C_1, C_3] = 1 \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

First steps to the Poincaré conjcture

In a similar way one gets the q**uaternion-space** (Poincaré‘s example no. 3 – the name was introduces by Threlfall and Seifert in 1930).

The fundamental group of the manifold M(a,b,c,d) is described byPoincaré as follows:

Generators: α, β, γ

Relations:

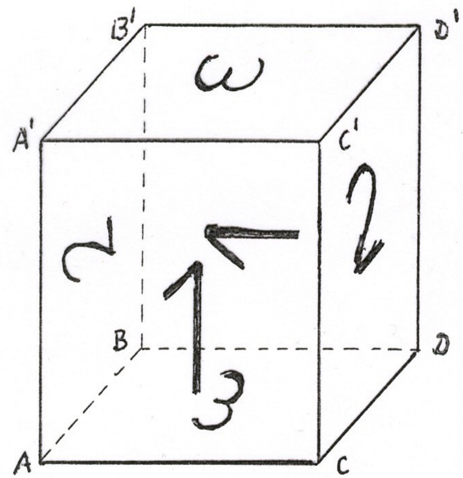
$$\beta\alpha\beta^{-1}\alpha^{-1} = \gamma\alpha\gamma^{-1}\alpha^{-1} = \gamma\alpha\gamma^{-1}\alpha^{-a}\beta^{-b} = \gamma\beta\gamma^{-1}\alpha^{-c}\beta^{-d} = 1$$

Beispiel 3 :

(c_1) $ABDC \equiv B'D'C'A'$

(c_2) $ABB'A' \equiv C'CDD'$

(c_3) $ACC'A' \equiv DD'B'B$



$$A \equiv B' \equiv C' \equiv D \text{ und } B \equiv D' \equiv A' \equiv C$$

$$AB \overset{c_1}{\equiv} B'D' \overset{c_3^{-1}}{\equiv} C'C \overset{c_2^{-1}}{\equiv} AB$$

$$AA' \overset{c_2}{\equiv} C'D' \overset{c_1^{-1}}{\equiv} DB \overset{c_3^{-1}}{\equiv} AA'$$

$$AC \overset{c_3}{\equiv} DD' \overset{c_2^{-1}}{\equiv} B'A' \overset{c_1^{-1}}{\equiv} AC$$

$$CD \overset{c_1}{\equiv} A'C' \overset{c_3}{\equiv} B'B \overset{c_2}{\equiv} CD$$

$$\pi_1(Q) = \langle c_1, c_2, c_3 \mid c_1^4 = 1,\ c_1^2 = c_2^2 = c_3^2 \rangle$$

First steps to the Poincaré conjcture

The Betti number B_1 of M(a,b,c,d) in dimension 1 is calculated by Poincaré by abelizing the fundamental group:

$$B_1(M(a,b,c,d)) = 2 \text{ if } (a-1)(d-1)-bc \neq 0$$

$$B_1(M(1,0,1,0)) = 4 \text{ (the 3-Torus)}$$

$$B_1(M(a,b,c,d)) = 3 \text{ else}$$

First steps to the Poincaré conjcture

Poincaré‘s duality theorem: $B_1 = B_2$,

Question:

Are the fundamental groups of two manifolds with the same Betti numbers B_1 always isomorphic?

Criterion:

If the fundamental groups of the manifolds M(a,b,c,d) and M(a‘,b‘,c‘,d‘) are isomorphic, then their matrices are conjugates in SL(2,Z).

[This isnot exactely correct, as was shown by Sarkaria in 1996, because we must consider conjugation in SL(2,R).]

First steps to the Poincaré conjcture

The matrices (1,h,0,1) and (1,h‘,0,1) are certainly not conjugated in SL(2,Z) if |h| ≠ |h‘|; but they both yield the same first and – by duality - also the same second Betti number (cf. above): $B_1 = B_2 = 3$.

Conclusion by **Poincaré:**

„It is not enough for two manifolds being homeomorphic to have the same Betti numbers.“ [Oeuvres VI, 258]

First steps to the Poincaré conjcture

A new question came to the mind of Poincaré:

„Are two manifolds of the same dimension with the same fundamental group always homeomorphic?“ [Poincaré VI, 258]

First steps to the Poincaré conjcture

State of the art in 1895:

The fundamental group is a stronger invariant in the case of orientable closed 3-manifolds than the Betti numbers.

But: How strong is it really?

The first and the second „Complément“ (1899/1900)

First Complément: the techniques used in the 1895 paper are made more rigorous; introduction of the incidence matrices.

Second „Complément“: Introduction of the torsion coefficients.

First steps to the Poincaré conjcture

The quaternion space can not be distinguished from the 3-sphere without using the fundamental group.

„To make this work not longer as it is, I restrict myself here to formulate the following postulate the proof of which needs some additional effort.“ [Poincaré VI, 370]:

A manifold with the same Betti numbers and the same torsion coefficients (in all dimensions) as the 3-sphere is homeomorphic to the sphere.

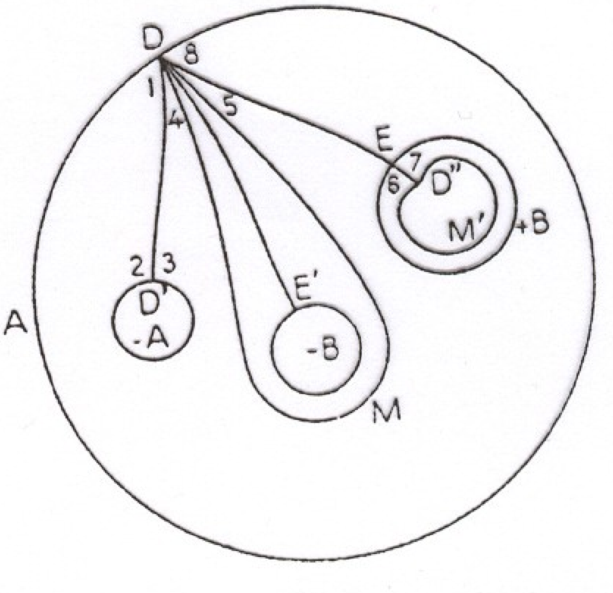
4. The homology sphere

The homology sphere

The fifth complément (1904)

In the fifth and last complément:

- Coming back to the classification problem for closed 3-manifolds in the special case of the 3-sphere.
- Systems of closed curves on surfaces
- Heegard splitting and Heegard diagram
- Elements of Morse theory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2 3
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M
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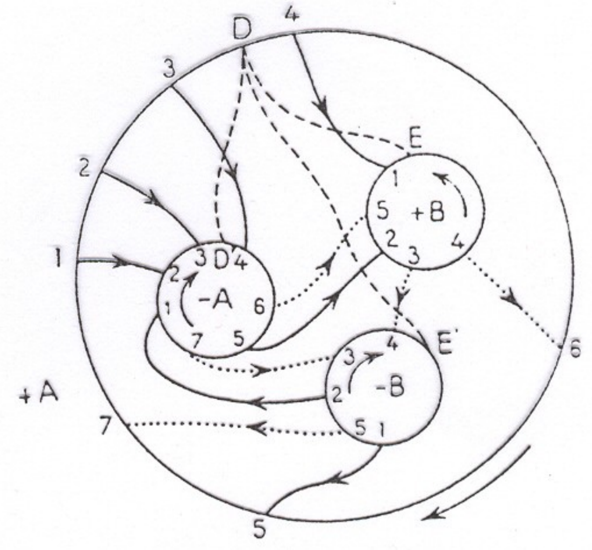
Fig. 2.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3 D 4
2
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1 6
7 5
E'
3 4
2 -B
5 1
6
+A
7
5

Fig. 4.

The homology sphere

Poincaré is now able to proof:

1. The fundamental group of the resulting 3-manifold is **not trivial**, because their is a subgroup in it isomorphic to the dodecahedron group.

2. But the Betti numbers and the torsion coefficients of this manifold are the same as those of the 3-sphere; so it is a homology sphere.

Conclusio**:** The idea formulated at the end of the 2. Complément is false: Betti numbers and torsion coefficients do not suffice to proof that a given manifold is homeomorphic to the 3-sphere.

The homology sphere

Perhaps the fundamental group is the solution?!

„There is a question to be studied: Is it possible, that the fundamental group of V reduces to the identical substution, whereas V is not simply connected?“

[simply connected means here homeomorphic to the 3-sphere]

Poincaré‘s conjecture (since ~1930)

Perelman‘s theorem (since ~2005)

The homology sphere

Since simply connected means here homeomorphic to the 3-sphere, Poincaré‘s conjecture reads as following:

Is a closed 3-manifold with trivial fundamental group always homeomorphic to the 3-sphere?

„Mais cette question nous entraînerait trop loin.“
[Oeuvres VI, 498]

The homology sphere

Alexander (1919):

The lens spaces L(5,1) and L(5,2) have isomorphic fundamental groups without being homeomorphic (Conjecture by H.Tietze [1908], proof byAlexander using the Eigenverschlingungszahlen).

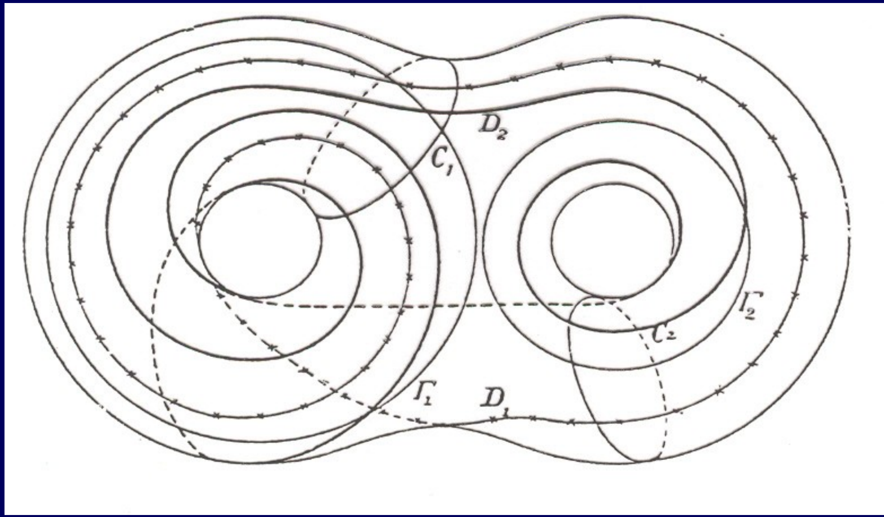
First partial solution of the Poincaré conjecture (1932) by Herbert Seifert:

Poincaré's conjecture is true for spaces fibered in the sense of Seifert

The homology sphere

Poincaré's homology sphere is a very interesting mathematical individual with a real biography

Representation of Poincaré's manifold given by Max Dehn (1907) in his article (with P. Heegard) for the „Encyclopedia":

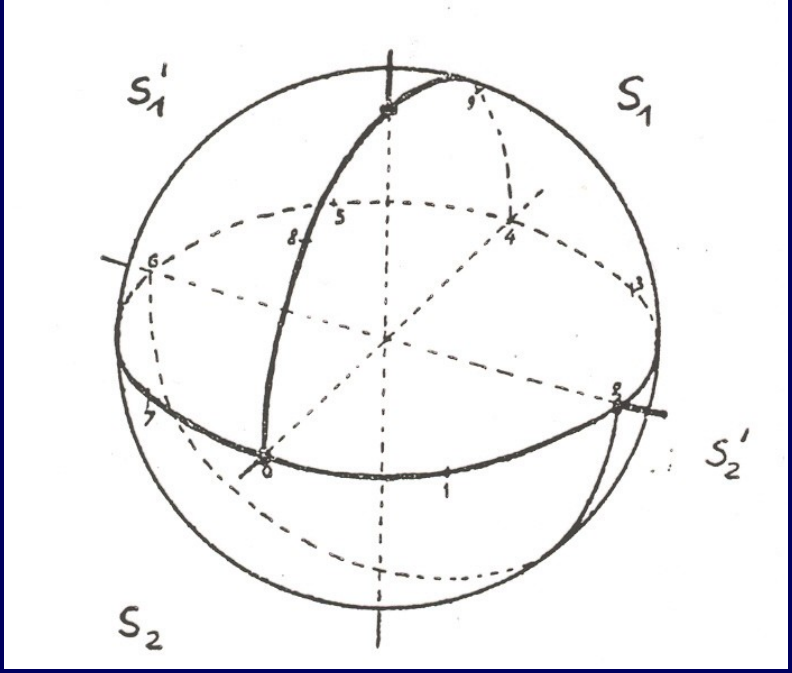


The homology sphere

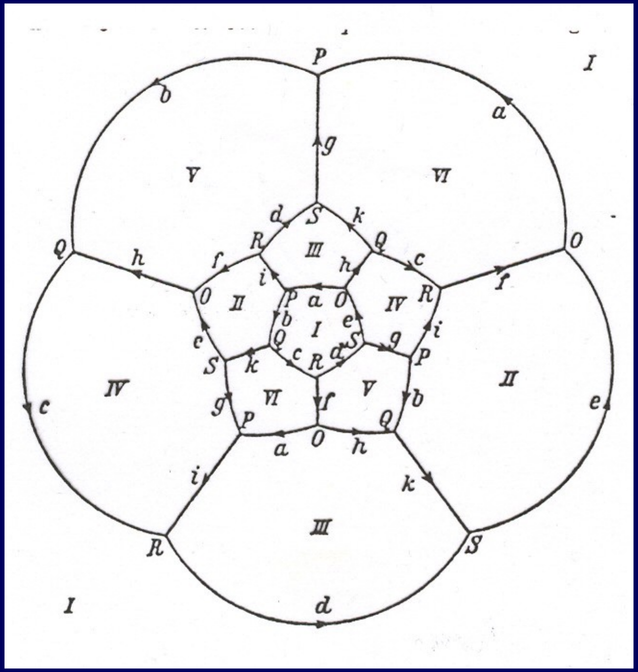
1931 a third representation of Poincaré‘s homology sphere was constructed by the Russian mathematician Kreines who used identifications on the 2-spehre.

1930 W. Threlfall and H. Seifert constructed the so called spherical dodecahedron space (hint by H. Kneser some years before). [There is also a hyperbolic dodecahedron space.]

All those manifolds are homeomorphic (Seifert and Threlfall 1933).



S1'
S1
S2'
S2



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5. Conclusions

Conclusion

Poincaré's way to his question which was named a conjecture afterwards is characterized by a **Leitidee** (the classification of the closed 3-manifolds) being motivated by an **analogy** (the classification of the closed surfaces) with interessesting applications (Kleinian functions). To reach his goal Poincaré constructed important tools (invariants) and interesting objects to test them. Poincaré's way to his conjecture underlines the importance of concrete mathematical objects.

Conclusion

So it is a good example to correct a little bit the strong tendency in the historiography of mathematics to look only to theories. It shows also the difficulties which may rise in connection with concrete objects.

6. References

Literatur und Dank

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