# Semisimple Langlands for $GL_2(\mathbb{Q}_p)$ and mod $p$ Hecke modules

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December 12, 2024

**Abstract**

Let $p \geq 5$ and let $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ be the center of the mod $p$ pro-$p$-Iwahori Hecke algebra of $\mathbf{GL}_2(\mathbb{Q}_p)$. Let $\zeta$ be a central character of $\mathbf{GL}_2(\mathbb{Q}_p)$. Let $X_\zeta$ be the projective curve parametrizing 2-dimensional mod $p$ semi-simple representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with determinant $\omega\zeta$. We construct a quotient morphism of schemes $\mathscr{L}_\zeta : \mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})_\zeta \to X_\zeta$. We then show that the correspondence between the specialization $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p,z}$ of the spherical $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ from [PS] in closed points $z \in \mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})_\zeta$ and the Galois representation $\rho_{\mathscr{L}_\zeta(z)}$ *is* the semi-simple mod $p$ local Langlands correspondence for the group $\mathbf{GL}_2(\mathbb{Q}_p)$.

## Contents



## 1 Introduction

*Background.* The mod $p$ (and the $p$-adic) Langlands correspondence for $\mathbf{GL}_2(\mathbb{Q}_p)$ was conjectured by Breuil, and has been fully established by Colmez-Dospinescu-Paškūnas [CDP14], building on work of Breuil, Colmez, Emerton, Kisin, Paškūnas and many others. Its *semisimple* version was established by Breuil in [Br03]. It is an explicit map $\rho \mapsto \pi(\rho)$, from the set of semisimple continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on 2-dimensional $\overline{\mathbb{F}}_p$-vector spaces, to the set of semisimple smooth representations of $\mathbf{GL}_2(\mathbb{Q}_p)$ on $\overline{\mathbb{F}}_p$-vector spaces.

Set $G := \mathbf{GL}_2(\mathbb{Q}_p)$, let $Z(G) = \mathbb{Q}_p^\times$ be the center of $G$ and $\zeta : Z(G) \to \overline{\mathbb{F}}_p^\times$ be a central character. Assume $p \geq 5$. In [DEG22], Dotto-Emerton-Gee introduce a curve $X_\zeta$ over $\mathbb{F}_p$ (denoted by $X$ in loc.cit.), which is a chain of projective lines with ordinary double points and of length $(p \pm 1)/2$, where the sign is equal to $-\zeta(-1)$. The definition of $X_\zeta$ is motivated by the Galois side of Breuil's semisimple correspondence: the closed $\overline{\mathbb{F}}_p$-points of $X_\zeta$ parametrize isomorphism classes of semisimple 2-dimensional continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$ with determinant $\omega\zeta$:

$$X_\zeta(\overline{\mathbb{F}}_p) \cong \big\{\text{semisimple continuous } \rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{GL}_2}(\overline{\mathbb{F}}_p) \ \text{ with } \det\rho = \omega\zeta\big\}/\sim .$$

Here $\omega$ is the mod $p$ cyclotomic character and $\widehat{\mathbf{GL}_2}$ is the Langlands dual group of $\mathbf{GL}_2$ over $\overline{\mathbb{F}}_p$. See [DEG22, 1.4] for further discussion on the curve $X_\zeta$. In the the sequel, we let $X$ be the disjoint union over all $X_\zeta$, base changed to $\overline{\mathbb{F}}_p$.

Let $I^{(1)} \subset G$ be the standard pro-$p$ Iwahori subgroup consisting of integral matrices which are upper unipotent mod $p$, and let $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$ is the pro-$p$-Iwahori Hecke algebra of $G$ with coefficients in $\overline{\mathbb{F}}_p$. By work of Ollivier [O09], the functor of $I^{(1)}$-invariants $\pi \mapsto \pi^{I^{(1)}}$ is an equivalence from the category of mod $p$ smooth representations of $G$ which are generated by their $I^{(1)}$-invariants, to the category of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules. Thus the composed map $\rho \mapsto \pi(\rho)^{I^{(1)}}$ is a correspondence from the set of semisimple mod $p$ 2-dimensional continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to the set of semisimple $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules. We refer to this latter correspondence as the semisimple mod $p$ local Langlands correspondence for Hecke modules.

*Statement of the result.* Let $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ be the center of the algebra $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$. It is fibered over the central characters $\zeta$ of $\mathbf{GL}_2(\mathbb{Q}_p)$. In [PS, 7.4.1], we constructed the mod $p$ spherical module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$. It is a distinguished $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-action on a maximal commutative subring of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$, which is a mod $p$ analogue (plus extension to the pro-$p$ Iwahori level) of the classical (anti)spherical module appearing in complex Kazhdan-Lusztig theory [KL87, 3.9]. The quasi-coherent module associated to $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ on $\mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$, when specialized at closed points, gives rise to a parametrization of *all* irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules. This is the spherical map from [PS, 7.4.2]. We recall the definition of $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ and the spherical map in 4.4 below.

Here we prove the following theorem. For a closed point $z \in \mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$, viewed as a character $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \to \overline{\mathbb{F}}_p$, we denote by $\mathcal{M}^{(1),\mathrm{ss}}_{\overline{\mathbb{F}}_p,z}$ the semisimplication of the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module

$$\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p,z} := \mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}),z} \overline{\mathbb{F}}_p.$$

**Theorem** (cf. Theorem 4.11). *Let $G = \mathbf{GL}_2(\mathbb{Q}_p)$ with $p \geq 5$. There exists, for any $\zeta$, a quotient morphism of $\overline{\mathbb{F}}_p$-schemes*

$$\mathscr{L}_\zeta : \mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})_\zeta \longrightarrow X_\zeta,$$

*with the following property: given a closed point $z \in \mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})_\zeta$, the correspondence*

$$\mathcal{M}^{(1),\mathrm{ss}}_{\overline{\mathbb{F}}_p,z} \longleftrightarrow \rho_{\mathscr{L}_\zeta(z)}$$

*between the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module $\mathcal{M}^{(1),\mathrm{ss}}_{\overline{\mathbb{F}}_p,z}$ and the Galois representation $\rho_{\mathscr{L}_\zeta(z)}$ associated to the point $\mathscr{L}_\zeta(z) \in X$ is the semisimple mod $p$ local Langlands correspondence for Hecke modules.*

We remark straightaway that the morphism $\mathscr{L}_\zeta$ is first defined set-theoretically on the level of $\overline{\mathbb{F}}_p$-points and many of its properties follow already from this definition.

The above result implies an interesting interpolation property (see Cor. 4.15). For simplicity, we also write $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ for the quasi-coherent module on the affine variety $\mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ associated to the $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$-module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$. Let $\mathscr{L}$ be the union over all $\mathscr{L}_\zeta$. Let $\mathscr{L}_*\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ be pushforward along $\mathscr{L}$. On the other hand, we may view the semisimple Langlands correspondence for Hecke modules as a function $x \mapsto \pi(\rho_x)^{I^{(1)}}$ on the set $X(\overline{\mathbb{F}}_p)$. In this situation, the above theorem implies that the quasi-coherent $\mathcal{O}_X$-module $\mathscr{L}_*\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$, interpolates the semisimple Langlands correspondence: for any $x \in X(\overline{\mathbb{F}}_p)$, one has an isomorphism of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules

$$\Big(\mathscr{L}_*\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{\mathcal{O}_X} k(x)\Big)^{\mathrm{ss}} \cong \pi(\rho_x)^{I^{(1)}}.$$

As a byproduct of our constructions, we also obtain an interpolation of Paškūnas' parametrization of the blocks of the category $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(\overline{\mathbb{F}}_p[G])$ of locally admissible smooth $G$-representations over $\overline{\mathbb{F}}_p$ with central character $\zeta$ [Pas13]. See 7.3 for the precise statement.

*More details on the construction.* The construction of the morphism $\mathscr{L}$ is a consequence of our results from [PS] on the geometry of the *generic* pro-$p$-Iwahori-Hecke algebra (with coefficients in the ring $\mathbb{Z}[\mathbf{q}]$ where $\mathbf{q}$ is a formal variable) for $\mathbf{GL}_2(\mathbb{Q}_p)$, specialized at $\mathbf{q} = p = 0 \in \overline{\mathbb{F}}_p$. To give more details, let $\widehat{\mathbf{G}}$ be the Langlands dual group of $\mathbf{GL}_2$ over $\overline{\mathbb{F}}_p$, with maximal torus $\widehat{\mathbf{T}}$. We consider the special fibre at $\mathbf{q} = 0$ of the Vinberg fibration $V_{\widehat{\mathbf{T}}} \xrightarrow{\mathbf{q}} \mathbb{A}^1$ associated to $\widehat{\mathbf{T}} \subset \widehat{\mathbf{G}}$ followed by base change to $\overline{\mathbb{F}}_p$. This yields the $\overline{\mathbb{F}}_p$-semigroup scheme

$$V_{\widehat{\mathbf{T}},0} := \mathrm{SingDiag}_{2\times 2} \times_{\overline{\mathbb{F}}_p} \mathbb{G}_m,$$

where $\mathrm{SingDiag}_{2\times 2}$ represents the semigroup of singular diagonal $2 \times 2$-matrices over $\overline{\mathbb{F}}_p$, cf. [PS, 7.1]. Let $\mathbb{T}^\vee$ be the finite abelian group dual to $\mathbb{T} = \mathbf{T}(\mathbb{F}_p)$, and consider the extended semigroup

$$V^{(1)}_{\widehat{\mathbf{T}},0} := \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}},0}.$$

It has a natural diagonal $W_0$-action. In [PS, 7.2.1] we established the mod $p$ pro-$p$-Iwahori Satake isomorphism

$$\mathrm{Spec}\, \mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p} : V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \xrightarrow{\sim} \mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$$

identifying the center $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ with the ring of regular functions on the quotient $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. It encodes the duality between $\mathbf{GL}_2$ and the dual group $\widehat{\mathbf{G}}$. The morphism $\mathscr{L}$ is then a composition of the inverse of $\mathrm{Spec}\, \mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p}$ with a certain morphism $L$ (see below) from $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ to $X$:

$$\mathscr{L} := L \circ (\mathrm{Spec}\, \mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p})^{-1}.$$

The morphism $L = \coprod_\zeta L_\zeta$ is fibered over the central characters $\zeta$ of $\mathbf{GL}_2(\mathbb{Q}_p)$. Each morphism

$$L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \to X_\zeta$$

is first defined set-theoretically on the level of $\overline{\mathbb{F}}_p$-points. This uses Paškūnas' parametrization of the blocks of the category $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(\overline{\mathbb{F}}_p[G])$ from [Pas13] and is responsible for our restriction to $p \geq 5$. The interpolation property of the map $\mathscr{L}$ with respect to the $p$-adic Langlands correspondence follows already from this definition. Then one checks on each connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, that $L_\zeta$ extends to a morphism of varieties. In fact, any fibre $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ is a naturally ordered union of connected components, which generically[1] are equal to two affine lines $\mathbb{A}^1 \cup_0 \mathbb{A}^1$ intersecting at the origin. It turns out that $L_\zeta$ is locally given by the toric construction of the projective line: it identifies the open subset $\mathbb{G}_m$ in the "first" irreducible component $\mathbb{A}^1$ of the connected component $\mathbb{A}^1 \cup_0 \mathbb{A}^1$ with the open subset $\mathbb{G}_m$ in "second" irreducible component $\mathbb{A}^1$ of the "next" connected component $\mathbb{A}^1 \cup_0 \mathbb{A}^1$ via the map $z \mapsto z^{-1}$, thus forming a $\mathbb{P}^1$. This produces an irreducible component of the chain $X_\zeta$.

*Organization of the article.* In section 2 we recall some results from [PS], notably that the quotient $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ is naturally fibered over the central characters $\zeta$ of $\mathbf{GL}_2(\mathbb{Q}_p)$. In section 3 we recall some properties of $X_\zeta$. Whereas in [DEG22] the irreducible components of $X_\zeta$ are labeled by certain cuspidal types, we chose a labelling of irreducible components by certain pairs of Serre weights, which is inspired from [Em19][2] and which is more suitable for our purposes. In section 4 we state the existence and interpolation properties of the morphism $L_\zeta$. We define $L_\zeta$ on the

[1] There occur also connected components equal to $\mathbb{A}^1$ corresponding to *non-regular* components of $\mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$.

[2] The idea of relating the curve $X$ and the spherical module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ came to the authors when listening to the talk [Em19], and led to the first preprint [PS2] in 2020. We thank M. Emerton for this enlightening talk. This article is a revised version of [PS2].

level of $\overline{\mathbb{F}}_p$-points and reduce the verification of being a morphism of varieties to two basic cases. In section 5 resp. 6 we prove all stated properties of $L_\zeta$ in the two basic even resp. odd cases. This is the main technical work of the paper. Finally, in section 7 we explain the interpolation of Paškūnas' block parametrization.

*Notation.* We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let $k$ be its residue field, an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. We let $G = \mathbf{GL}_2(\mathbb{Q}_p)$. We let $\mathbf{T}$ denote the diagonal torus in $\mathbf{GL}_2$ and $W_0$ its Weyl group. Let $\mathbb{T} = \mathbf{T}(\mathbb{F}_p)$. If $H$ is a finite group, then $H^\vee := \mathrm{Hom}(H, k^\times)$. Finally, $\widehat{\mathbf{G}}$ denotes the dual group of $\mathbf{GL}_2$ over $k$, with maximal torus $\widehat{\mathbf{T}}$.

# 2 Mod $p$ Satake parameters with fixed central character

We recall some results from [PS, 7.5] in the special case $F = \mathbb{Q}_p$.

**2.1.** Let $\omega : \mathbb{F}_p^\times \to k^\times$ be given by the embedding $\mathbb{F}_p \subset k$. The group $(\mathbb{F}_p^\times)^\vee = \langle\omega\rangle$ is cyclic of order $p-1$. Any element $\omega^r$ gives rise to a non-regular character of $\mathbb{T}$ via $\omega^r(t_1, t_2) := \omega^r(t_1)\omega^r(t_2)$ for all $(t_1, t_2) \in \mathbb{T} = \mathbb{F}_p^\times \times \mathbb{F}_p^\times$. Composition with multiplication in $\mathbb{T}^\vee$ produces an action of $(\mathbb{F}_p^\times)^\vee$ on $\mathbb{T}^\vee$, which factors through the quotient $\mathbb{T}^\vee/W_0$:

$$\mathbb{T}^\vee/W_0 \times (\mathbb{F}_q^\times)^\vee \longrightarrow \mathbb{T}^\vee/W_0,\ (\gamma, \omega^r) \mapsto \gamma\omega^r.$$

A $W_0$-orbit $\gamma \in \mathbb{T}^\vee/W_0$ is called regular if it consists of two elements. Otherwise, it is called non-regular. If $\gamma \in \mathbb{T}^\vee/W_0$ is regular (non-regular), then $\gamma\omega^r$ is regular (non-regular).

**2.2.** We may restrict characters to the subgroup $\mathbb{F}_p^\times \simeq \{\mathrm{diag}(a, a) : a \in \mathbb{F}_p^\times\} \subset \mathbb{T}$ and this gives a homomorphism $\mathbb{T}^\vee \to (\mathbb{F}_p^\times)^\vee$ which factors into a restriction map

$$\mathbb{T}^\vee/W_0 \to (\mathbb{F}_p^\times)^\vee,\ \gamma \mapsto \gamma|_{\mathbb{F}_p^\times}.$$

The relation to the $(\mathbb{F}_p^\times)^\vee$-action on the source $\mathbb{T}^\vee/W_0$ is $(\gamma\omega^r)|_{\mathbb{F}_p^\times} = \gamma|_{\mathbb{F}_p^\times}\,\omega^{2r}$. We recall the fibers of the restriction map $\gamma \mapsto \gamma|_{\mathbb{F}_q^\times}$. Let $(\cdot)|_{\mathbb{F}_p^\times}^{-1}(\omega^{2r})$ be the fibre at a square element $\omega^{2r}$. The action of $\omega^{-r}$ on $\mathbb{T}^\vee/W_0$ induces a bijection with the fibre $(\cdot)|_{\mathbb{F}_q^\times}^{-1}(1)$. The fibre

$$(\cdot)|_{\mathbb{F}_q^\times}^{-1}(1) = \{1 \otimes 1\} \coprod \{\omega \otimes \omega^{-1}, \omega^2 \otimes \omega^{-2}, ..., \omega^{\frac{q-3}{2}} \otimes \omega^{-\frac{q-3}{2}}\} \coprod \{\omega^{\frac{q-1}{2}} \otimes \omega^{-\frac{q-1}{2}}\}$$

has cardinality $\frac{p+1}{2}$ and, in the above list, we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. The $W_0$-orbits represented by the characters $\omega^r \otimes \omega^{-r}$ for $r = 1, ..., \frac{p-3}{2}$, are all regular $W_0$-orbits. The two orbits at the two ends of the list are non-regular orbits. Since the action of $\omega^{-r}$ preserves regular (non-regular) orbits, any fibre at a square element (there are $\frac{p-1}{2}$ such fibres) has the same structure. On the other hand, let $(\cdot)|_{\mathbb{F}_p^\times}^{-1}(\omega^{2r-1})$ be the fibre at a non-square element $\omega^{2r-1}$. The action of $\omega^{-r}$ induces a bijection with the fibre $(\cdot)|_{\mathbb{F}_p^\times}^{-1}(\omega^{-1})$. The fibre

$$(\cdot)|_{\mathbb{F}_p^\times}^{-1}(\omega^{-1}) = \{1 \otimes \omega^{-1}, \omega \otimes \omega^{-2}, ..., \omega^{\frac{p-1}{2}-1} \otimes \omega^{-\frac{p-1}{2}}\}$$

has cardinality $\frac{p-1}{2}$ and we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. All elements of the fibre are regular $W_0$-orbits. Since the action of $\omega^{-r}$ preserves regular (non-regular) orbits, any fibre at a non-square element (there are $\frac{p-1}{2}$ such fibres) has the same structure.

**2.3.** We have the commutative $k$-semigroup scheme

$$V_{\widehat{\mathbf{T}},0}^{(1)} = \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}},0} = \mathbb{T}^\vee \times \mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m.$$

cf. [PS, 7.5.3]. It has a natural $W_0$-action: the natural action of $W_0$ on the factors $\mathbb{T}^\vee$ and $\mathrm{SingDiag}_{2\times 2}$ and the trivial one on $\mathbb{G}_m$. In addition to this, there is a commuting action of the $k$-group scheme

$$\mathcal{Z}^\vee := (\mathbb{F}_p^\times)^\vee \times \mathbb{G}_m$$

on $V^{(1)}_{\widehat{\mathbf{T}},0}$: the (constant finite diagonalizable) group $(\mathbb{F}_p^\times)^\vee$ acts only on the factor $\mathbb{T}^\vee$ and in the way described in 2.1; an element $z_0 \in \mathbb{G}_m$ acts trivially on $\mathbb{T}^\vee$, by multiplication with the diagonal matrix $\mathrm{diag}(z_0, z_0)$ on $\mathrm{SingDiag}_{2\times 2}$ and by multiplication with the square $z_0^2$ on $\mathbb{G}_m$. Therefore the quotient $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ inherits a $\mathcal{Z}^\vee$-action. We have a decomposition

$$V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 = \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\mathrm{reg}}} V_{\widehat{\mathbf{T}},0} \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\text{non-reg}}} V_{\widehat{\mathbf{T}},0}/W_0.$$

In this optic, the $(\mathbb{F}_p^\times)^\vee$-action is by permutations on the index set $\mathbb{T}^\vee/W_0$. It preserves the subsets of regular and non-regular components. The $\mathbb{G}_m$-action on $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ preserves connected components.

**2.4.** According to [PS, 7.5.4], we have two projection morphisms

$$\begin{array}{ccc} & V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 & \\ {\scriptstyle \mathrm{pr}_{\mathbb{T}^\vee/W_0}}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_{\mathbb{G}_m}} \\ \mathbb{T}^\vee/W_0 & & \mathbb{G}_m. \end{array}$$

Composing $\mathrm{pr}_{\mathbb{T}^\vee/W_0}$ with the restriction map $(\cdot)|_{\mathbb{F}_p^\times} : \mathbb{T}^\vee/W_0 \to (\mathbb{F}_p^\times)^\vee$, setting

$$\theta := \big((\cdot)|_{\mathbb{F}_p^\times} \circ \mathrm{pr}_{\mathbb{T}^\vee/W_0}\big) \times \mathrm{pr}_{\mathbb{G}_m}$$

yields

$$\begin{array}{c} V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \\ \big\downarrow{\scriptstyle \theta} \\ \mathcal{Z}^\vee. \end{array}$$

The relation to the $\mathcal{Z}^\vee$-action on the source $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ is given by the formula

$$\theta(x.(\omega^r, z_0)) = \theta(x)(\omega^{2r}, z_0^2) = \theta(x)(\omega^r, z_0)^2$$

for $x \in V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ and $(\omega^r, z_0) \in \mathcal{Z}^\vee$. The following definition is [PS, Def. 7.5.1].

**2.5. Definition.** *Let $\zeta \in \mathcal{Z}^\vee$. The* space of mod $p$ Satake parameters with central character $\zeta$ *is the $k$-scheme*

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta := \theta^{-1}(\zeta).$$

**2.6.** Let $\zeta = (\zeta|_{\mathbb{F}_p^\times}, z_2) \in \mathcal{Z}^\vee(k) = (\mathbb{F}_p^\times)^\vee \times k^\times$. Denote by $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{z_2}$ the fibre of $\mathrm{pr}_{\mathbb{G}_m}$ at $z_2 \in k^\times$. Recall from [PS, 7.5.5] that

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta = \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\mathrm{reg}},\gamma|_{\mathbb{F}_p^\times}=\zeta|_{\mathbb{F}_p^\times}} V_{\widehat{\mathbf{T}},0,z_2} \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\text{non-reg}},\gamma|_{\mathbb{F}_p^\times}=\zeta|_{\mathbb{F}_p^\times}} V_{\widehat{\mathbf{T}},0,z_2}/W_0.$$

There are standard coordinates $x, y$ such that $V_{\widehat{\mathbf{T}},0,z_2} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1$, two affine lines crossing at the origin. There is a Steinberg coordinate $z_1$ such that

$$V_{\widehat{\mathbf{T}},0,z_2}/W_0 \simeq \mathbb{A}^1.$$

**2.7. Lemma.** *Let $\zeta, \eta \in \mathcal{Z}^\vee$. The action of $\eta$ on $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ induces an isomorphism of $k$-schemes $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \simeq (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta\eta^2}$.*

*Proof.* This is [PS, Lem. 7.5.2]. □

# 3 Mod $p$ Langlands parameters with fixed determinant

**3.1.** We normalize local class field theory $\mathbb{Q}_p^\times \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}}$ by sending $p$ to a geometric Frobenius. In this way, we identify the $k$-valued smooth characters of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and of $\mathbb{Q}_p^\times$. Finally, $\omega : \mathbb{Q}_p^\times \to k^\times$ denotes the extension of the character $\omega : \mathbb{F}_p^\times \to k^\times$ to $\mathbb{Q}_p^\times$ satisfying $\omega(p) = 1$, and $\mathrm{unr}(x) : \mathbb{Q}_p^\times \to k^\times$ denotes the character trivial on $\mathbb{F}_p^\times$ and sending $p$ to $x$.

**3.2.** Let $\zeta : \mathbb{Q}_p^\times \to k^\times$ be a character. Recall from [DEG22] the projective curve $X_\zeta$ over $\mathbb{F}_p$ whose $\overline{\mathbb{F}}_p$-points parametrize (isomorphism classes of) two-dimensional semisimple continuous Galois representations over $k$ with determinant $\omega\zeta$:

$$X_\zeta(k) \cong \big\{\text{semisimple continuous } \rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k) \ \text{ with } \det\rho = \omega\zeta\big\}/\sim .$$

The curve $X_\zeta$ is a chain of projective lines over $k$ of length $\frac{p\pm1}{2}$, whose irreducible components intersect at ordinary double points. The sign $\pm1$ is equal to $-\zeta(-1)$. We refer to $\zeta$ in the case $-\zeta(-1) = -1$ resp. $-\zeta(-1) = +1$ as an *even character* resp. *odd character*. From now on, we let $X_\zeta$ denote its base change to $k$. There is a finite set of closed points $X_\zeta^{\mathrm{irred}} \subset X_\zeta$ which correspond to the classes of irreducible representations. Its open complement $X_\zeta^{\mathrm{red}} = X_\zeta \setminus X_\zeta^{\mathrm{irred}}$ parametrizes the reducible representations (i.e. direct sums of characters). Let $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to k^\times$ be a character. Since $\det(\rho\otimes\eta) = (\det\rho)\eta^2$, twisting representations with $\eta$ induces an isomorphism

$$(\cdot)\otimes\eta : X_\zeta \xrightarrow{\sim} X_{\zeta\eta^2}.$$

Hence one is reduced to consider only two 'basic' cases: the even case where $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = 1$ and the odd case where $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$. Indeed, if $\zeta|_{\mathbb{F}_p^\times} = \omega^r$ for some even $r$, then choosing $\eta$ with $\eta(p)^2 = \zeta(p)^{-1}$ and $\eta|_{\mathbb{F}_p^\times} = \omega^{-\frac{r}{2}}$, one finds that $(\zeta\eta^2)(p) = 1$ and $(\zeta\eta^2)|_{\mathbb{F}_p^\times} = 1$; if $\zeta|_{\mathbb{F}_p^\times} = \omega^r$ for some odd $r$, then choosing $\eta$ with $\eta(p)^2 = \zeta(p)^{-1}$ and $\eta|_{\mathbb{F}_p^\times} = \omega^{-\frac{r+1}{2}}$, one finds that $(\zeta\eta^2)(p) = 1$ and $(\zeta\eta^2)|_{\mathbb{F}_p^\times} = \omega^{-1}$.

**3.3.** We make explicit some structure elements of $X_\zeta$ in the even case $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = 1$. Every irreducible component of $X_\zeta$ is isomorphic to $\mathbb{P}^1$ and there are $\frac{p-1}{2}$ components. They are labelled by pairs of Serre weights of the following form:

$$\begin{array}{ccc}
\mathrm{Sym}^0 & | & \mathrm{Sym}^{p-3}\otimes\det \\
\mathrm{Sym}^2\otimes\det^{-1} & | & \mathrm{Sym}^{p-5}\otimes\det^2 \\
\mathrm{Sym}^4\otimes\det^{-2} & | & \mathrm{Sym}^{p-7}\otimes\det^3 \\
\vdots & \vdots & \vdots \\
\mathrm{Sym}^{p-3}\otimes\det^{\frac{p+1}{2}} & | & \mathrm{Sym}^0\otimes\det^{\frac{p-1}{2}} .
\end{array}$$

The component with label " $\mathrm{Sym}^0 \mid \mathrm{Sym}^{p-3}\otimes\det$ " intersects the next component at the point of $X_\zeta^{\mathrm{irred}}$ parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^2\otimes\det^{-1}, \mathrm{Sym}^{p-3}\otimes\det\}$. The component with label " $\mathrm{Sym}^2\otimes\det^{-1} \mid \mathrm{Sym}^{p-5}\otimes\det^2$ " intersects the next component at the point of $X_\zeta^{\mathrm{irred}}$ parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^4\otimes\det^{-2}, \mathrm{Sym}^{p-5}\otimes\det^2\}$. Continuing in this way, one finds $\frac{p-3}{2}$ points of $X_\zeta^{\mathrm{irred}}$, which correspond to the $\frac{p-3}{2}$ double points of the chain $X_\zeta$. There are two more points in $X_\zeta^{\mathrm{irred}}$: they are smooth points, each one lies on one of the two 'exterior' components and corresponds there to the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^0, \mathrm{Sym}^{p-1}\}$ and $\{\mathrm{Sym}^0\otimes\det^{\frac{p-1}{2}}, \mathrm{Sym}^{p-1}\otimes\det^{\frac{p-1}{2}}\}$ respectively. So $X_\zeta^{\mathrm{irred}}$ has cardinality $\frac{p+1}{2}$. Suppose we are on one of the two exterior components $\mathbb{P}^1$. There is a canonical affine coordinate $z_1$ on the open complement of the double point, identifying this open complement with $\mathbb{A}^1$. We call the four points where $z_1 = \pm1$ *the four exceptional points* of $X_\zeta$.

**3.4.** We make explicit some structure elements of $X_\zeta$ in the odd case $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$. Every irreducible component of $X_\zeta$ is isomorphic to $\mathbb{P}^1$ and there are $\frac{p+1}{2}$ components. Except for

the two outer components, they are labelled by pairs of Serre weights of the following form:

$$\begin{array}{c|c} \mathrm{Sym}^{p-2} & \text{" } \mathrm{Sym}^{-1} \text{ "} \\ \mathrm{Sym}^{p-4}\otimes\det & \mathrm{Sym}^{1}\otimes\det^{-1} \\ \mathrm{Sym}^{p-6}\otimes\det^{2} & \mathrm{Sym}^{3}\otimes\det^{-2} \\ \vdots & \vdots \\ \mathrm{Sym}^{1}\otimes\det^{\frac{p-3}{2}} & \mathrm{Sym}^{p-4}\otimes\det^{\frac{p+1}{2}} \\ \text{" } \mathrm{Sym}^{-1}\otimes\det^{\frac{p-1}{2}} \text{ "} & \mathrm{Sym}^{p-2}\otimes\det^{\frac{p-1}{2}}. \end{array}$$

The weight $\mathrm{Sym}^{-1}$ does not exist, which is why we put it into quotation marks, so the first and the last line of the above table consist of only of *one* weight. On the geometric side, this reflects the fact that these lines describe the two outer components. Each of them intersects the rest of the chain in only *one* point.

The component with label " $\mathrm{Sym}^{p-2}$ | " $\mathrm{Sym}^{-1}$ "" intersects the next component at the point of $X_\zeta^{\mathrm{irred}}$ parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^1\otimes\det^{-1}, \mathrm{Sym}^{p-2}\}$. The component with label " $\mathrm{Sym}^{p-4}\otimes\det$ | $\mathrm{Sym}^1\otimes\det^{-1}$ " intersects the next component at the point of $X_\zeta^{\mathrm{irred}}$ parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^3\otimes\det^{-2}, \mathrm{Sym}^{p-4}\otimes\det\}$. Continuing in this way, one finds $\frac{p-1}{2}$ points of $X_\zeta^{\mathrm{irred}}$, which correspond to the $\frac{p-1}{2}$ double points of the chain $X_\zeta$. There are no more points in $X_\zeta^{\mathrm{irred}}$ and $X_\zeta^{\mathrm{irred}}$ has cardinality $\frac{p-1}{2}$. Suppose we are on one of the two exterior components $\mathbb{P}^1$. There is a canonical affine coordinate $t$ on the open complement of the double point, identifying this open complement with $\mathbb{A}^1$. We call the four points where $t = \pm 2$ *the four exceptional points* of $X_\zeta$. [3]

# 4 A morphism from Hecke to Galois

**4.1.** We let $I \subset G$ be the standard Iwahori subgroup of $G$ consisting of integral matrices which are upper triangular mod $p$. Let $I^{(1)} \subset I$ be its $p$-Sylow subgroup, i.e. matrices which are upper unipotent mod $p$. We identify $W_0$ with the subgroup of $G$ generated by the matrix $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We let $W^{(1)} = (\mathbb{T} \times X_\bullet(\mathbf{T})) \rtimes W_0$ be the extended Iwahori-Weyl group, wheren $X_\bullet(\mathbf{T}) = \mathbb{Z}\times\mathbb{Z}$ equals the cocharacter lattice of $\mathbf{T}$. We identify cocharacters with a subgroup of diagonal matrices in the usual way, i.e. $(1,0) \mapsto \mathrm{diag}(p^{-1}, 1)$ and $(0,1) \mapsto \mathrm{diag}(1, p^{-1})$.

We also put

$$u = \begin{pmatrix} 0 & p^{-1} \\ 1 & 0 \end{pmatrix}, \quad u^{-1} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \quad us = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad su = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}.$$

Moreover, $u^2 = \mathrm{diag}(p^{-1}, p^{-1})$.[4] Since

$$\begin{pmatrix} 0 & p^{-1} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} = \begin{pmatrix} d & p^{-1}c \\ pb & a \end{pmatrix}$$

the element $u \in G$ normalizes the group $I^{(1)}$. Note that $u = (1,0)s \in W^{(1)}$.

**4.2.** Let $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$ be the pro-$p$-Iwahori-Hecke algebra of $G$ relativ to $I^{(1)}$ with coefficients in $k = \overline{\mathbb{F}}_p$, cf. [V04]. It equals the convolution $k$-algebra generated by the $I^{(1)}$-double cosets in $G$. A system of representatives for these double cosets is given by $W^{(1)}$ and we denote the Hecke operator attached to $w \in W^{(1)}$ by $T_w$. The $T_w$ form a $k$-basis of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$. We write $U := T_u$.

[3] The Galois representations living on the two exterior components in the odd case are *unramified* (up to twist), i.e. of type $\rho = \begin{pmatrix} \mathrm{unr}(x) & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$ and $t$ equals the 'trace of Frobenius' $x + x^{-1}$. Hence $t = \pm 2$ if and only if $x = \pm 1$.

[4] Note that our element $u$ equals the element $u^{-1}$ in [Be11],[Br07] and [V04].

Recall further that $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$ decomposes into a product of algebras $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_p}$ indexed by the elements $\gamma \in \mathbb{T}^\vee/W_0$, cf. [V04, 3.1]. For simplicity, we denote the image of $U$ and $T_s$ in a direct factor $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_p}$ by the same letters. In the case of a regular orbit $\gamma$, one chooses an ordering $(\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}})$ on the set $\gamma$. The associated standard coordinates $X, Y \in H^{\gamma}_{\overline{\mathbb{F}}_p}$ compute the center $Z(H^{\gamma}_{\overline{\mathbb{F}}_p})$ as $\overline{\mathbb{F}}_p[U^2][]X,Y]/(XY)$, cf. [V04, 2.2]. Vignéras' standard module $M(x,y,z,\chi|_{\mathbb{T}})$ is then the unique $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_p}$-module, where the central elements $X, Y, U^2$ act as the scalars $x, y, z \in \overline{\mathbb{F}}_p$ respectively [V04, 2.3]. In the case of a non-regular orbit $\gamma$, the center $Z(H^{\gamma}_{\overline{\mathbb{F}}_p})$ is generated by $U^2$ and a certain second element $Z$. Vignéras' standard module $M(a,z,\chi|_{\mathbb{T}})$ is then the unique $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_p}$-module, where the central elements $Z, U^2$ act as the scalars $a, z \in \overline{\mathbb{F}}_p$ respectively [V04, 1.4].

We denote by $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ the category of smooth $G$-representations over $k$. We have the functor of $I^{(1)}$-invariants $\pi \mapsto \pi^{I^{(1)}}$ from $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ to the category $\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$. It gives a bijection between the irreducible $G$-representations and the irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules. Thereby, supersingular representations correspond to supersingular Hecke modules [V04].

We recall the $I^{(1)}$-invariants for some classes of representations. If $\pi = \mathrm{Ind}_B^G(\chi)$ is a principal series representation with $\chi = \chi_1 \otimes \chi_2$, then $\pi^{I^{(1)}}$ is a standard module in the component $\gamma := \{\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}}\}$. In the case of a regular orbit $\gamma$, one chooses an ordering $(\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}})$ on the set $\gamma$ and associated standard coordinates. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(0, \chi(su), \chi(u^2), \chi|_{\mathbb{T}}) = M(0, \chi_2(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1}), \chi|_{\mathbb{T}}).$$

In the case of a non-regular orbit $\gamma$ we obtain

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(\chi(su), \chi(u^2), \chi|_{\mathbb{T}}) = M(\chi_2(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1}), \chi|_{\mathbb{T}}).$$

These standard modules are irreducible if and only if $\chi \neq \chi^s$ [V04, 4.2/4.3].[5]

If $\pi = \pi(r, 0, \eta)$ is a standard supersingular representation with parameter $r = 0, ..., p-1$ and a character $\eta : \mathbb{Q}_p^\times \to k^\times$, then $\pi^{I^{(1)}}$ is a supersingular module in the component $\gamma = \{\chi, \chi^s\}$ represented by the character $\chi := (\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times})$, cf. [Br07, 5.1/5.3]. If $\pi$ is the trivial representation $\mathbb{1}$ or the Steinberg representation St, then $\gamma = 1$ and $\pi^{I^{(1)}}$ is the character $(0,1)$ or $(-1,-1)$ respectively.

**4.3.** Let $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$. Since $u \in G$ normalizes the group $I^{(1)}$, one has $I^{(1)}uI^{(1)} = uI^{(1)}$. It follows that the convolution action of the Hecke operator $U$ (resp. $U^2$) on $\pi^{I^{(1)}}$ is therefore induced by the action of $u$ (resp. $u^2$ on $\pi$). Similarly, the group $I^{(1)}$ is normalized by the Iwahori subgroup $I$ and $I/I^{(1)} \simeq \mathbb{T}$. It follows that the convolution action of the operators $T_t, t \in \mathbb{T}$ on $\pi^{I^{(1)}}$ is the factorization of the $\mathbf{T}(\mathbb{Z}_p)$-action on $\pi$.

**4.4.** Recall from [PS, Def. 7.4.1] the definition of the mod $p$ spherical module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$. Let $\mathcal{A}^{(1)}_{\overline{\mathbb{F}}_p}$ denote the maximal commutative subring of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$ (associated with the dominant orientation, say). Then $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} = \mathcal{A}^{(1)}_{\overline{\mathbb{F}}_p}$ as $\mathcal{A}^{(1)}_{\overline{\mathbb{F}}_p}$-modules and the missing action of the Hecke operator $T_s$ is then given by a mod $p$ and pro-$p$ analogue (based on results of M.-F. Vignéras) of the classical Demazure operator. For more details, we refer to loc.cit. Recall also from [PS, 7.4.2] the spherical map

$$\mathrm{Sph} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k) \longrightarrow \{\text{left } \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}\text{-modules}\}/\sim,$$

given as

$$\mathrm{Sph}(v) = \text{class of } \mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p, z(v)} := \mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}), z(v)} \overline{\mathbb{F}}_p,$$

[5] Our formulas differ from [V04, 4.2/4.3] by $\chi(\cdot) \leftrightarrow \chi(\cdot)^{-1}$, since we are working with left modules; also compare with the explicit calculation with right convolution given in [V04, Appendix A.5].

where $z(v) \in \operatorname{Spec} Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ denotes the image of $v$ under the mod $p$ pro-$p$ Iwahori-Satake isomorphism [PS, 7.2.1]

$$\operatorname{Spec} \mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p} : V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \xrightarrow{\sim} \operatorname{Spec} Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}). \tag{1}$$

Let $\mathrm{Sph}^{\mathrm{ss}}$ be the map Sph followed by semisimplification.

Recall also the twisting action of $\mathcal{Z}^\vee(k)$ on semisimple $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules [PS, 7.5.6]: let $(\eta, z) \in \mathcal{Z}^\vee(k)$. In the case of regular $\gamma$, the actions of $X, Y, U^2$ get multiplied by $z, z, z^2$ respectively and the component $\gamma$ gets multiplied by $\eta$; in the case of non-regular $\gamma$, the action of $U$ gets multiplied by $z$, the action of $T_s$ remains unchanged and the component $\gamma$ gets multiplied by $\eta$.

**4.5. Lemma.** *The map* $\mathrm{Sph}^{\mathrm{ss}}$ *is* $\mathcal{Z}^\vee(k)$*-equivariant.*

*Proof.* This is [PS, Prop. 7.5.3]. □

**4.6.** We identify $\mathbb{Q}_p^\times$ with the center $Z(G)$ in the usual way. A (smooth) character $\zeta : Z(G) = \mathbb{Q}_p^\times \to k^\times$ is determined by its value $\zeta(p^{-1}) \in k^\times$ and its restriction $\zeta|_{\mathbb{Z}_p^\times}$. Since the latter is trivial on the subgroup $1+p\mathbb{Z}_p$, we may view it as a character of $\mathbb{F}_p^\times$; we will write $\zeta|_{\mathbb{F}_p^\times}$ for this restriction in the following. Thus the group of characters of $Z(G)$ gets identified with the group of $k$-points of the group scheme $\mathcal{Z}^\vee = (\mathbb{F}_p^\times)^\vee \times \mathbb{G}_m$:

$$Z(G)^\vee \xrightarrow{\sim} \mathcal{Z}^\vee(k),\ \zeta \mapsto (\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})). \tag{2}$$

The mod $p$ pro-$p$-Iwahori Satake isomorphism (1) allows us to view $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$-modules $M$ (and therefore also $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules) as quasi-coherent sheaves $S(M)$ on $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. The rule $M \mapsto S(M)$ is the mod $p$ parametrization functor $P : \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \to \mathrm{QCoh}(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)$ from [PS, Def. 7.3.4] in the special case $F = \mathbb{Q}_p$.

**4.7. Lemma.** *Suppose that* $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$ *has a central character* $\zeta : Z(G) \to k^\times$*. Then the Satake parameter* $S(\pi^{I^{(1)}})$ *of* $\pi^{I^{(1)}} \in \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ *has central character* $\zeta$*, i.e. it is supported on the closed subscheme*

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1}))} \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0.$$

*Proof.* This is [PS, Prop. 7.5.4] in the case $F = \mathbb{Q}_p$. □

Next, recall the twisting action of the group $\mathcal{Z}^\vee(k)$ on the standard $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules $M$ (and their simple constituents) from 4.4, which we write as $M.(\eta, z)$ for $(\eta, z) \in \mathcal{Z}^\vee(k)$.

**4.8. Proposition.** *Let* $\pi \in \mathrm{Mod}^{\mathrm{ladm}}(k[G])$ *be irreducible or a reducible principal series representation. Let* $\eta : \mathbb{Q}_p^\times \to k^\times$ *be a character and view it as a pair in* $\mathcal{Z}^\vee(k)$ *via formula (2). Then*

$$(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p^{-1}))$$

*as* $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$*-modules.*

*Proof.* For future reference, we remark that the statement holds true, mutatis mutandis, with $\mathbb{Q}_p$ replaced by a finite extension. We therefore give a proof and references that work in this generality. An irreducible locally admissible representation, being a finitely generated $k[G]$-module, is admissible [Em10, 2.2.19]. A principal series representation (irreducible or not) is always admissible [Em10, 4.1.7]. The list of irreducible admissible smooth $G$-representations is given in [H11b, Thm. 1.1]. There are four families: principal series representations, supersingular representations, characters and twists of the Steinberg representation.

We first suppose that $\pi$ is a principal series representation (irreducible or not), i.e. of the form $\mathrm{Ind}_B^G(\chi)$ with a character $\chi = \chi_1 \otimes \chi_2$. Then $\pi \otimes \eta \simeq \mathrm{Ind}_B^G(\chi_1\eta \otimes \chi_2\eta)$. We use the results from 4.2 (which hold over a finite extension, cf. [V04]). The modules $\pi^{I^{(1)}}$ and $(\pi \otimes \eta)^{I^{(1)}}$ are standard

modules in the components $\gamma := \{\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}}\}$ and $\gamma(\eta|_{\mathbb{F}_p^\times})$ respectively. Suppose that $\gamma$ is regular. We choose the ordering $(\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}})$ and standard coordinates $x, y$. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(0, \chi_2(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1}), \chi|_{\mathbb{T}})$$

and

$$\mathrm{Ind}_B^G(\chi_1\eta \otimes \chi_2\eta)^{I^{(1)}} = M(0, \chi_2(p^{-1})\eta(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1})\eta(p^{-2}), (\chi|_{\mathbb{T}}).(\eta|_{\mathbb{F}_p^\times})).$$

This shows $(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p^{-1}))$ in the regular case. Suppose that $\gamma$ is non-regular. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(\chi_2(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1}), \chi|_{\mathbb{T}})$$

and

$$\mathrm{Ind}_B^G(\chi_1\eta \otimes \chi_2\eta)^{I^{(1)}} = M(\chi_2(p^{-1})\eta(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1})\eta(p^{-2}), (\chi|_{\mathbb{T}}).(\eta|_{\mathbb{F}_p^\times})).$$

This shows $(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p^{-1}))$ in the non-regular case.

We now treat the case where $\pi$ is a character or a twist of the Steinberg representation. Consider the exact sequence

$$1 \to \mathbb{1} \to \mathrm{Ind}_B^G(1) \to \mathrm{St} \to 1.$$

According to [V04, 4.4] the sequence of invariants

$$(S) : 1 \to \mathbb{1}^{I^{(1)}} \to \mathrm{Ind}_B^G(1)^{I^{(1)}} \to \mathrm{St}^{I^{(1)}} \to 1$$

is still exact and $\mathbb{1}^{I^{(1)}}$ resp. $\mathrm{St}^{I^{(1)}}$ is the trivial character $(0,1)$ resp. sign character $(-1,-1)$ in the Iwahori component $\gamma = 1$. Tensoring the first exact sequence with $\eta$ produces the exact sequence

$$1 \to \eta \to \mathrm{Ind}_B^G(1) \otimes \eta \to \mathrm{St} \otimes \eta \to 1.$$

Since the restriction $\eta|_{\mathbb{Z}_p^\times}$ is trivial on $1 + p\mathbb{Z}_p$, one has $(\eta \circ \det)|_{I^{(1)}} = 1$ and so, as a sequence of $k$-vector spaces with $k$-linear maps, the sequence of invariants

$$1 \to \eta^{I^{(1)}} \to (\mathrm{Ind}_B^G(1) \otimes \eta)^{I^{(1)}} \to (\mathrm{St} \otimes \eta)^{I^{(1)}} \to 1$$

coincides with the sequence $(S)$. It is therefore an exact sequence of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules, with outer terms being characters of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$. From the discussion above, we deduce

$$(\mathrm{Ind}_B^G(1) \otimes \eta)^{I^{(1)}} = \mathrm{Ind}_B^G(1)^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p)^{-1}) = M(\eta(p^{-1}), \eta(p^{-2}), 1.(\eta|_{\mathbb{F}_p^\times})).$$

It follows then from [V04, 1.1] that $\eta^{I^{(1)}}$ must be the trivial character $(0, \eta(p^{-1}))$ in the component $1.(\eta|_{\mathbb{F}_p^\times})$ and $(\mathrm{St} \otimes \eta)^{I^{(1)}}$ must be the sign character $(-1, -\eta(p^{-1}))$ in the component $1.(\eta|_{\mathbb{F}_p^\times})$. This implies

$$\eta^{I^{(1)}} = \mathbb{1}^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p)^{-1}) \quad \text{and} \quad (\mathrm{St} \otimes \eta)^{I^{(1)}} = \mathrm{St}^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(\varpi)^{-1}).$$

This proves the claim in the cases $\pi = \mathbb{1}$ or $\pi = \mathrm{St}$. If, more generally, $\pi = \eta'$ is a general character of $G$, then

$$(\pi \otimes \eta)^{I^{(1)}} = (\eta'\eta)^{I^{(1)}} = \mathbb{1}^{I^{(1)}}.((\eta'\eta)|_{\mathbb{F}_q^\times}, (\eta'\eta)(p)^{-1}) = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p)^{-1}).$$

On the other hand, if $\pi = \mathrm{St} \otimes \eta'$ is a twist of Steinberg, then

$$(\pi \otimes \eta)^{I^{(1)}} = (\mathrm{St} \otimes (\eta'\eta))^{I^{(1)}} = \mathrm{St}^{I^{(1)}}.((\eta'\eta)|_{\mathbb{F}_p^\times}, (\eta'\eta)(p)^{-1}) = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p)^{-1}).$$

It remains to treat the case where $\pi$ is a supersingular representation. In this case $\pi \otimes \eta$ is also supersingular and the two modules $\pi^{I^{(1)}}$ and $(\pi \otimes \eta)^{I^{(1)}}$ are supersingular $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules [V04, 4.9]. Let $\gamma$ be the component of the module $\pi^{I^{(1)}}$. By 4.3, the component of $(\pi \otimes \eta)^{I^{(1)}}$ equals $\gamma(\eta|_{\mathbb{F}_p^\times})$. Moreover, if $U^2$ acts on $\pi^{I^{(1)}}$ via the scalar $z_2 \in k^\times$, then $U^2$ acts on $(\pi \otimes \eta)^{I^{(1)}}$ via $z_2(\eta \circ \det)(u^2) = z_2\eta(p)^{-2}$, cf. 4.3. Since the supersingular modules are uniquely characterized by their component and their $U^2$-action, we obtain $(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p)^{-1})$, as claimed. $\square$

**4.9.** Let $p \geq 5$. We let $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$ be the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ consisting of locally admissible representations having central character $\zeta$. By work of Paškūnas [Pas13], the blocks $b$ of the category $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$, defined as certain equivalence classes of simple objects, can be parametrized by the set of isomorphism classes $[\rho]$ of semisimple continuous Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$ having determinant $\det \rho = \omega\zeta$, i.e. by the $k$-points of $X_\zeta$. There are three types of blocks. Blocks of type 1 are supersingular blocks. Each such block contains only one irreducible $G$-representation, which is supersingular. Blocks of type 2 contain only two irreducible representations. These two representations are two generic principal series representations of the form $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2\omega^{-1})$ and $\mathrm{Ind}_B^G(\chi_2 \otimes \chi_1\omega^{-1})$ (where $\chi_1\chi_2 \neq 1, \omega^{\pm 1}$). There are four blocks of type 3 which correspond to the four exceptional points. In the even case, each such block contains only three irreducible representations. These representations are of the form $\eta, \mathrm{St} \otimes \eta$ and $\mathrm{Ind}_B^G(\omega \otimes \omega^{-1}) \otimes \eta$. In the odd case, each block of type 3 contains only one irreducible representation. It is of the form $\mathrm{Ind}_B^G(\chi \otimes \chi\omega^{-1})$.

**4.10.** Let $p \geq 5$. Paškūnas' parametrization $[\rho] \mapsto b_{[\rho]}$ is compatible with Breuil's semisimple mod $p$ local Langlands correspondence

$$\rho \mapsto \pi(\rho)$$

for the group $G$ [Br07, Be11], in the sense that if $\rho$ has determinant $\omega\zeta$, then the simple constituents of the $G$-representation $\pi(\rho)$ lie in the block $b_{[\rho]}$ of $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$. The correspondence and the parametrizations (for varying $\zeta$) commute with twists: for a character $\eta : \mathbb{Q}_p^\times \to k^\times$, $\pi(\rho \otimes \eta) = \pi(\rho) \otimes \eta$ and $b_{[\rho]} \otimes \eta = b_{[\rho\otimes\eta]}$.

**4.11.** Fix a character $\zeta : Z(G) = \mathbb{Q}_p^\times \to k^\times$, corresponding to a point $(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) \in \mathcal{Z}^\vee(k)$ under the identification $\mathcal{Z}(G)^\vee \cong \mathcal{Z}^\vee(k)$ from 4.6. Recall the spherical module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ from 4.4. Its Satake parameter $S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})$ is a quasi-coherent module on the affine variety $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. It restricts to a quasi-coherent module $S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})_\zeta$ on the closed subvariety $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ of $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. On the other hand, we may view the semisimple local Langlands correspondence with character $\zeta$ for Hecke modules as a function

$$x \mapsto \pi(\rho_x)^{I^{(1)}}$$

on the set $X_\zeta(k)$.

**Theorem.** *Suppose $p \geq 5$. There exists a finite morphism of $k$-schemes*

$$L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$$

*such that the quasi-coherent $\mathcal{O}_{X_\zeta}$-module $L_{\zeta *}S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})_\zeta$ interpolates the semisimple mod $p$ Langlands correspondence: for all $x \in X_\zeta(k)$, there is an isomorphism of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules*

$$\Big(\big(L_{\zeta *}S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})_\zeta\big) \otimes_{\mathcal{O}_{X_\zeta}} k(x)\Big)^{\mathrm{ss}} \cong \pi(\rho_x)^{I^{(1)}}.$$

Note that $\Big(\big(L_{\zeta *}S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})_\zeta\big) \otimes_{\mathcal{O}_{X_\zeta}} k(x)\Big)^{\mathrm{ss}} = \Big(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})} (\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p})^{-1}(\mathcal{O}_{L_\zeta^{-1}(x)})\Big)^{\mathrm{ss}}$, by unravelling all definitions involved.

**4.12.** The connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ are either regular and then of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$, or non-regular and then of type $\mathbb{A}^1$. The morphism $L_\zeta$ appearing in the theorem depends on the choice of an order of the two affine lines in each regular component. It is surjective and quasi-finite. Moreover, writing $L_\zeta^\gamma$ for its restriction to the connected component $(V^\gamma_{\widehat{\mathbf{T}},0}/W_0)_\zeta \subset (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, one has:

(e) *Even case.* All connected components are of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$, except for the two 'exterior' components which are of type $\mathbb{A}^1$. $L_\zeta^\gamma$ is an open immersion for any $\gamma$.

(o) *Odd case.* All connected components are of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. $L_\zeta$ is an open immersion on all connected components, except for the two 'exterior' ones. On an 'exterior' component $\gamma$, the restriction of $L_\zeta^\gamma$ to one irreducible component $\mathbb{A}^1$ is an open immersion, and its restriction to the open complement $\mathbb{G}_m$ is a degree 2 finite flat covering of its image, with branched locus equal to the intersection of this image with the exceptional locus of $X_\zeta$.

**4.13.** We set $L := \coprod_\zeta L_\zeta$. This is the morphism

$$L : V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \longrightarrow X$$

referred to in the introduction.

**4.14.** Note that the semisimple mod $p$ Langlands correspondence associates with any semisimple $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$ a semisimple smooth $G$-representation $\pi(\rho)$ of length $1, 2$ or $3$, hence whose semisimple $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module of $I^{(1)}$-invariants $\pi(\rho)^{I^{(1)}}$ has length $1, 2$ or $3$. On the other hand, the antispherical map

$$\mathrm{Sph} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k) \longrightarrow \{\text{left } \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}\text{-modules}\}$$

has an image consisting of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules are of length 1 or 2, cf. [PS, 7.5.6]. Theorem 4.11 combined with the properties 4.12 of the morphism $L_\zeta$ provide the following case-by-case elucidation of the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules $\pi(\rho)^{I^{(1)}}$.

**4.15. Corollary.** *Let $x \in X_\zeta(k)$, corresponding to $\rho_x : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$. Then the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module $\pi(\rho)^{I^{(1)}}$ admits the following explicit description.*

*(i) If $x \in X^{irred}_\zeta(k)$, then the fibre $L_\zeta^{-1}(x) = \{v\}$ has cardinality 1 and*

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v).$$

*It is irreducible and supersingular.*

*(ii) If $x \in X^{red}_\zeta(k) \setminus \{$the four exceptional points$\}$, then $L_\zeta^{-1}(x) = \{v_1, v_2\}$ has cardinality 2 and*

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2).$$

*It has length 2.*

*(iiie) If $x \in X^{red}_\zeta(k)$ is exceptional in the even case, then $L_\zeta^{-1}(x) = \{v_1, v_2\}$ has cardinality 2 and*

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v_1)^{\mathrm{ss}} \oplus \mathrm{Sph}(v_2).$$

*It has length 3.*

*(iiio) If $x \in X^{red}_\zeta(k)$ is exceptional in the odd case, then $L_\zeta^{-1}(x) = \{v\}$ has cardinality 1 and*

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v) \oplus \mathrm{Sph}(v).$$

*It has length 2.*

**4.16.** Now we proceed to the proof of 4.11, 4.12 and 4.15.

We start by defining the morphism $L_\zeta$ at the level of $k$-points. Let $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k)$ and let the connected component which contains $v$ indexed by $\gamma \in \mathbb{T}^\vee/W_0$.

1. Suppose that $\gamma$ is regular. Then $\mathrm{Sph}(v) = \mathrm{Sph}^\gamma(v)$ is a simple two-dimensional $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-module, cf. [PS, Thm. 7.4.6]. Let $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$ be the simple module, unique up to isomorphism, such that $\pi^{I^{(1)}} \simeq \mathrm{Sph}^\gamma(v)$, cf. 4.2. Then $\pi \in \mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$ with

$$\zeta = (\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) = (\gamma|_{\mathbb{F}_p^\times}, z_2)$$

by 4.7. Let $b$ be the block of $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$ which contains $\pi$. We define $L_\zeta(v)$ to be the point of $X_\zeta(k)$ which corresponds to $b$.

2. Suppose that $\gamma$ is non-regular.

(a) If $v \in D(2)_\gamma(k)$, then $\mathrm{Sph}(v) = \mathrm{Sph}^\gamma(2)(v)$ is a simple two-dimensional $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-module, cf. [PS, Thm. 7.4.10]. As in the regular case, there is a simple module $\pi$, unique up to isomorphism, such that $\pi^{I^{(1)}} \simeq \mathrm{Sph}^\gamma(2)(v)$. It has central character $\zeta = (\gamma|_{\mathbb{F}_p^\times}, z_2)$ and there is a block $b$ of $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$ which contains $\pi$. We define $L_\zeta(v)$ to be the point of $X_\zeta(k)$ which corresponds to $b$.

(b) If $v \in D(1)_\gamma(k)$, then $\mathrm{Sph}(v)^{\mathrm{ss}}$ is the direct sum of the two characters forming the antispherical pair $\mathrm{Sph}^\gamma(1)(v) = \{(0, z_1), (-1, -z_1)\}$ where $z_2 = z_1^2$, cf. [PS, Thm. 7.4.10]. As in the regular case, there are two simple modules $\pi_1$ and $\pi_2$, unique up to isomorphism, such that $\pi_1^{I^{(1)}} \simeq (0, z_1)$ and $\pi_2^{I^{(1)}} \simeq (-1, -z_1)$ and $\pi_1, \pi_2$ have central character $\zeta = (\gamma|_{\mathbb{F}_p^\times}, z_2)$. Moreover, we claim that there is a unique block $b$ of $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$ which contains both $\pi_1$ and $\pi_2$. Indeed, if $\gamma = \{1 \otimes 1\}$ and $z_1 = 1$, then $\pi_1 = \mathbb{1}$ and $\pi_2 = \mathrm{St}$, cf. 4.2. Then by 4.8 it follows more generally that if $\gamma = \{\omega^r \otimes \omega^r\}$, then $\pi_1 = \eta$ and $\pi_2 = \mathrm{St} \otimes \eta$ with $\eta = (\eta|_{\mathbb{F}_p^\times}, \eta(p^{-1})) := (\omega^r, z_1)$. Consequently $\pi_1, \pi_2$ are contained in a unique block $b$ of type 3, cf. 4.9. We define $L_\zeta(v)$ to be the point of $X_\zeta(k)$ which corresponds to $b$.

Thus we have a well-defined map of sets $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) \longrightarrow X_\zeta(k)$.

We show property (i) of 4.15. Let $x \in X_\zeta^{\mathrm{irred}}(k)$ and suppose $L_\zeta(v) = x$. Then $b_x$ is a supersingular block, contains a unique irreducible representation $\pi$, which is supersingular, and $\pi = \pi(\rho_x)$, cf. 4.9-4.10. By definition of $L_\zeta$, one has $\mathrm{Sph}(v) \simeq \pi^{I^{(1)}}$. Since the spherical map Sph is $1:1$ over supersingular modules, cf. [PS, Thm. 7.4.6/ Thm. 7.4.10], such a preimage $v$ of $x$ exists and is uniquely determined by $x$. Summarizing, we have $L_\zeta^{-1}(x) = \{v\}$ and $\mathrm{Sph}(v) \simeq \pi(\rho_x)^{I^{(1)}}$. This is property (i).

As a next step, we take a second character $\eta : \mathbb{Q}_p^\times \to k^\times$ and show that the diagram

$$\begin{array}{ccc} (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) & \xrightarrow{L_\zeta} & X_\zeta(k) \\ {\scriptstyle \cdot\eta}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle (\cdot)\otimes\eta} \\ (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta\eta^2}(k) & \xrightarrow{L_{\zeta\eta^2}} & X_{\zeta\eta^2}(k) \end{array}$$

commutes. Here, the vertical arrows are the bijections coming from 2.7 and 3.2. To verify the commutativity, let $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^\vee/W_0$. Suppose that $\gamma$ is regular or that $\gamma$ is non-regular with $v \in D(2)_\gamma(k)$. Let $\pi$ be the simple $G$-module with $\pi^{I^{(1)}} \simeq \mathrm{Sph}(v)$ and let $b_{[\rho]}$ be the block corresponding to the point $L_\zeta(v)$. By the equivariance property 4.5, one has $\mathrm{Sph}(v.\eta) \simeq \mathrm{Sph}(v).\eta$. Taking $I^{(1)}$-invariants is compatible with twist, cf. 4.8, and so $L_{\zeta\eta^2}(v.\eta)$ corresponds to the block which contains the representation $\pi \otimes \eta$, i.e. to $b_{[\rho]} \otimes \eta = b_{[\rho\otimes\eta]}$, cf. 4.10, and so $L_{\zeta\eta^2}(v.\eta) = [\rho \otimes \eta] = L_\zeta(v).\eta$.

If $v \in D(1)_\gamma(k)$, let $\pi_1$ and $\pi_2$ be the simple modules such that $(\pi_1 \oplus \pi_2)^{I^{(1)}} \simeq \mathrm{Sph}^\gamma(v)^{\mathrm{ss}}$. As before, we conclude from $\mathrm{Sph}(v.\eta)^{\mathrm{ss}} \simeq \mathrm{Sph}(v)^{\mathrm{ss}} \otimes \eta$ that $L_{\zeta\eta^2}(v.\eta)$ corresponds to the block which contains $\pi_1 \otimes \eta$ and $\pi_2 \otimes \eta$ and that $L_{\zeta\eta^2}(v.\eta) = L_\zeta(v).\eta$. The commutativity of the diagram is proved.

Thus, we are reduced to prove that the map $L_\zeta$ comes from a morphism of $k$-schemes satisfying 4.11 and the remaining parts of 4.15 in the two basic cases of a character $\zeta$ such that $\zeta(p^{-1}) = 1$ and $\zeta|_{\mathbb{F}_p^\times} \in \{1, \omega^{-1}\}$. This is established in the next two subsections.

# 5 The morphism $L_\zeta$ in the basic even case

Let $\zeta : \mathbb{Q}_p^\times \to k^\times$ be the trivial character. Here we show that the map of sets $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) \to X_\zeta(k)$ that we have defined in 4.16 satisfies properties (ii) and (iiie) of 4.15, and we define a morphism of $k$-schemes $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides with the previous map of sets at the level of $k$-points. By construction, it will have the properties 4.12. This will complete the proof of 4.15, 4.12 and 4.11 in the case of an even character.

**5.1.** We verify the properties (ii) and (iiie). We work over an irreducible component $\mathbb{P}^1$ with label " $\mathrm{Sym}^r \otimes \det^a \mid \mathrm{Sym}^{p-3-r} \otimes \det^{r+1+a}$ " where $0 \leq r \leq p-3$ and $0 \leq a \leq p-2$, cf. 3.3. On this component, we choose an affine coordinate $x$ around the double point having $\mathrm{Sym}^r \otimes \det^a$ as one of its Serre weights. Away from this point, we have $x \neq 0$ and the corresponding Galois representation has the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{r+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^a$. By [Be11, 1.3] or [Br07, 4.11], we have

$$\pi(\rho_x) = \pi(r, x, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], x^{-1}, \omega^{r+1}\eta)^{\mathrm{ss}} =: \pi_1 \oplus \pi_2$$

where $[p-3-r]$ denotes the unique integer in $\{0, ..., p-2\}$ which is congruent to $p-3-r$ modulo $p-1$. Now suppose that $L_\zeta(v) = x$. We distinguish two cases.

1. *The generic case* $0 < r < p-3$. In this case, the point $x$ lies on one of the 'interior' components of the chain $X_\zeta$, which has no exceptional points. The length of $\pi(\rho_x)$ is 2. Indeed, $\pi_1 = \pi(r, x, \eta)$ and $\pi_2 = \pi(p-3-r, x^{-1}, \omega^{r+1}\eta)$ are two irreducible principal series representations [Br07, Thm. 4.4]. The block $b_x$ is of type 2 and contains only these two irreducible representations, cf. 4.9-4.10. We may write

$$\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$$

with $\chi = \mathrm{unr}(x) \otimes \omega^r \,\mathrm{unr}(x^{-1})$, according to [Br07, Rem. 4.4(ii)]. By our assumptions on $r$, the character $\chi|_{\mathbb{T}} = 1 \otimes \omega^r$ is regular (i.e. different from its $s$-conjugate). We conclude from 4.8 and 4.2 that $\pi_1^{I^{(1)}}$ is a simple 2-dimensional standard module in the regular component represented by the character $(1 \otimes \omega^r).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^r \in \mathbb{T}^\vee$. Similarly, we may write

$$\pi_2 = \mathrm{Ind}_B^G(\chi) \otimes \omega^{r+1}\eta$$

where now $\chi = \mathrm{unr}(x^{-1}) \otimes \omega^{p-3-r}\,\mathrm{unr}(x)$. By our assumptions on $r$, the character $\chi|_{\mathbb{T}} = 1 \otimes \omega^{p-3-r}$ is regular and we conclude, as above, that the $I^{(1)}$-invariants $\pi_2^{I^{(1)}}$ form a simple 2-dimensional standard module in the regular component represented by the character $(\eta|_{\mathbb{F}_p^\times})\omega^{r+1} \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{r+1}\omega^{p-3-r} \in \mathbb{T}^\vee$. Note that the component of $\pi_1^{I^{(1)}}$ is different from the component of $\pi_2^{I^{(1)}}$, by our assumptions on $r$.

We conclude from $L_\zeta(v) = x$ that either $\mathrm{Sph}(v) = \pi_1^{I^{(1)}}$ or $\mathrm{Sph}(v) = \pi_2^{I^{(1)}}$. Since for $\gamma$ regular, the map $\mathrm{Sph}^\gamma$ is a bijection onto all simple $\mathcal{H}^\gamma_{\mathbb{F}_p}$-modules, cf. [PS, Thm. 7.4.6], one finds that $L_\zeta^{-1}(x) = \{v_1, v_2\}$ has cardinality 2 and

$$\mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_x)^{I^{(1)}}.$$

This settles property (ii) of 4.15 in the generic case.

2. *The boundary cases* $r \in \{0, p-3\}$. In this case, the point $x$ lies on one of the two 'exterior' components of $X_\zeta$. On such a component, we will denote the variable $x$ rather by $z_1$, which is the notation[6] which we used already in 3.3.

(a) Suppose that $z_1 \neq \pm 1$. The length of $\pi(\rho_{z_1})$ is 2. Indeed, as in the generic case, $\pi_1 = \pi(r, z_1, \eta)$ and $\pi_2 = \pi(p-3-r, z_1^{-1}, \omega^{r+1}\eta)$ are two irreducible principal series representations. The

[6]The reason for this notation will become clear in the discussion of the non-regular case in 5.2.

block $b_{z_1}$ is of type 2 and contains only these two irreducible representations. It follows, as above, that their invariants $\pi_1^{I^{(1)}}$ and $\pi_2^{I^{(1)}}$ are simple 2-dimensional standard modules, in the components represented by $(\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^r \in \mathbb{T}^\vee$ and $(\eta|_{\mathbb{F}_p^\times})\omega^{r+1} \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{r+1}\omega^{p-3-r} \in \mathbb{T}^\vee$ respectively. Since $r \in \{0, p-3\}$, one of these components is regular, the other non-regular. In particular, the two components are different. We conclude from $L_\zeta(v) = z_1$ that either $\mathrm{Sph}(v) = \pi_1^{I^{(1)}}$ or $\mathrm{Sph}(v) = \pi_2^{I^{(1)}}$. Since for non-regular $\gamma$, the map $\mathrm{Sph}^\gamma(2)$ is a bijection from $D(2)_\gamma(k)$ onto all simple standard $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-modules, cf. [PS, Thm. 7.4.10], we may conclude as in the generic case: $L_\zeta^{-1}(z_1) = \{v_1, v_2\}$ has cardinality 2 and

$$\mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_{z_1})^{I^{(1)}}.$$

This settles property 4.15 (ii) in the remaining case $z_1 \neq \pm 1$.

(b) Suppose now that $z_1 = \pm 1$, i.e. we are at one of the four exceptional points. We will verify property (iiie). The length of $\pi(\rho_{z_1})$ is 3. Indeed, the representation $\pi(0, \pm 1, \eta)$ is a twist of the representation $\pi(0, 1, 1)$ (note that $\pi(r, z_1, \eta) \simeq \pi(r, -z_1, \mathrm{unr}(-1)\eta)$ according to [Br07, Rem. 4.4(v)]), which itself is an extension of $\mathbb{1}$ by St, cf. [Br07, Thm. 4.4(iii)]. As in the case (a), the representation $\pi_2 = \pi(p-3, \pm 1, \omega\eta)$ is an irreducible principal series representation. The block $b_{z_1}$ is of type 3 and contains only these three irreducible representations. The invariants $\pi_1^{I^{(1)}}$ form a direct sum of two spherical characters in a non-regular component $\gamma$, whereas the invariants $\pi_2^{I^{(1)}}$ form a simple standard module in a regular component, as before. Since for non-regular $\gamma$, the map $\mathrm{Sph}^\gamma(1)$ is a bijection from $D(1)_\gamma(k)$ onto all spherical pairs of characters of $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$, cf. [PS, Thm. 7.4.10], we may conclude that $L_\zeta^{-1}(z_1) = \{v_1, v_2\}$ has cardinality 2 with $v_1 \in D(1)_\gamma(k)$ and $\mathrm{Sph}^\gamma(1)(v_1)^{\mathrm{ss}} = \pi_1^{I^{(1)}}$. In particular,

$$\mathrm{Sph}(v_1)^{\mathrm{ss}} \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_x)^{I^{(1)}}.$$

This settles property 4.15 (iiie).

**5.2.** We define a morphism of $k$-schemes $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides on $k$-points with the map of sets $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) \longrightarrow X_\zeta(k)$. We work over a connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, indexed by some $\gamma \in \mathbb{T}^\vee/W_0$. Let $v$ be a $k$-point of this component.

Since $\zeta|_{\mathbb{F}_p^\times} = 1$, the connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ are indexed by the fibre $(\cdot)|^{-1}_{\mathbb{F}_p^\times}(1)$. This fibre consists of the $\frac{p-3}{2}$ regular components, represented by the characters of $\mathbb{T}$

$$\chi_k = \omega^k \otimes \omega^{-k}$$

for $k = 1, ..., \frac{p-3}{2}$, and of the two non-regular components, given by $\chi_0$ and $\chi_{\frac{p-1}{2}}$, cf. 2.2. We distinguish two cases. Note that $z_2 = \zeta(p^{-1}) = 1$.

1. *The regular case* $0 < k < \frac{p-1}{2}$. We fix the order $\gamma = (\chi_k, \chi_k^s)$ on the set $\gamma$ and choose the standard coordinates $x, y$. According to [PS, 7.4.3], our regular connected component identifies with two affine lines intersecting at the origin:

$$V_{\widehat{\mathbf{T}},0,1} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1.$$

Suppose that $v = (0, 0)$ is the origin, so that $\mathrm{Sph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation. It corresponds to the irreducible Galois representation $\rho(r, \eta) = \mathrm{ind}(\omega_2^{r+1}) \otimes \eta$, in the notation of [Be11, 1.3], whence $L_\zeta(v) = [\rho(r, \eta)]$. According to 4.2, the component of the Hecke module $\pi(r, 0, \eta)^{I^{(1)}}$ is given by $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times})$. Setting $\eta|_{\mathbb{F}_p^\times} = \omega^a$, this implies $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times}) = \omega^{r+a} \otimes \omega^a = \chi_k$ and hence $a = -k$ and $r = 2k$. Therefore the Serre weights of the irreducible representation $\rho(r, \eta)$ are $\{\mathrm{Sym}^{2k} \otimes \det^{-k}, \mathrm{Sym}^{p-1-2k} \otimes \det^k\}$, cf. [Br07, 1.9].

Comparing these pairs of Serre weights with the list 3.3 shows that the $\frac{p-3}{2}$ points

$$\{\text{origin } (0, 0) \text{ on the component } (\chi_k, \chi_k^s)\}$$

for $0 < k < \frac{p-1}{2}$ are mapped successively to the $\frac{p-3}{2}$ double points of the chain $X_\zeta$.

Fix $0 < k < \frac{p-1}{2}$ and consider the double point

$$Q = L_\zeta(\text{origin } (0,0) \text{ on the component } \gamma = (\chi_k, \chi_k^s)).$$

As we have just seen, $Q$ lies on the irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^{2k} \otimes \det^{-k}$ (i.e. on the component " $\mathrm{Sym}^{2k} \otimes \det^{-k}$ | $\mathrm{Sym}^{p-3-2k} \otimes \det^{k+1}$ "). We fix an affine coordinate on this $\mathbb{P}^1$ around $Q$, which we will also call $x$ (there will be no risk of confusion with the standard coordinate above!). Away from $Q$, the affine coordinate $x \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{2k+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{-k}$. As we have seen above, $\pi(\rho_x) = \pi(2k, x, \eta) \oplus \pi(p-3-2k, x^{-1}, \omega^{r+1}\eta) =: \pi_1 \oplus \pi_2$. Moreover, $\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(x) \otimes \omega^{2k}\,\mathrm{unr}(x^{-1})$. Since

$$(1 \otimes \omega^{2k}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{-k} \otimes \omega^k = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the regular case of 4.2 that

$$\pi_1^{I^{(1)}} = M(0, x, 1, \chi_k^s)$$

is a simple 2-dimensional standard module. Note that $M(0, x, 1, \chi_k^s) = M(x, 0, 1, \chi_k)$ according to [V04, Prop. 3.2].

Now suppose that $v = (x, 0), x \neq 0$, denotes a point on the $x$-line of $\mathbb{A}_k^1 \cup_0 \mathbb{A}_k^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(x, 0, 1, \chi_k)$. By our discussion, the point $L_\zeta((x,0))$ corresponds to the block which contains $\pi_1$. Since $\pi_1$ lies in the block parametrized by $[\rho_x]$, cf. 4.10, it follows that

$$L_\zeta((x,0)) = [\rho_x] = x \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$

Since $(0,0)$ maps to $Q$, i.e. to the point at $x = 0$, the map $L_\zeta$ identifies the whole affine $x$-line $\mathbb{A}^1 = \{(x,0) : x \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine $x$-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

On the other hand, the double point $Q$ lies also on the irreducible component $\mathbb{P}^1$ whose labelling includes the other weight of $Q$, i.e. the weight $\mathrm{Sym}^{p-1-2k} \otimes \det^k$. We fix an affine coordinate $y$ on this $\mathbb{P}^1$ around $Q$. Away from $Q$, the coordinate $y \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(y)\omega^{p-2k} & 0 \\ 0 & \mathrm{unr}(y^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^k$. As in the first case, $\pi(\rho_y)$ contains $\pi_1 := \pi(p-1-2k, y, \eta) = \mathrm{Ind}_B^G(\chi) \otimes \eta$ as a direct summand, where now $\chi = \mathrm{unr}(y) \otimes \omega^{p-1-2k}\,\mathrm{unr}(y^{-1})$. Since

$$(1 \otimes \omega^{p-1-2k}).(\eta|_{\mathbb{F}_p^\times}) = \omega^k \otimes \omega^{-k} = \chi_k \in \mathbb{T}^\vee,$$

we deduce, as above, that $\pi_1^{I^{(1)}} = M(0, y, 1, \chi_k)$ is a simple 2-dimensional standard module.

Now suppose that $v = (0, y), y \neq 0$, denotes a point on the $y$-line of $\mathbb{A}_k^1 \cup_0 \mathbb{A}_k^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(0, y, 1, \chi_k)$. By our discussion, the point $L_\zeta((0,y))$ corresponds to the block which contains $\pi_1$. Since $\pi_1$ lies in the block parametrized by $[\rho_y]$, cf. 4.10, it follows that

$$L_\zeta((0,y)) = [\rho_y] = y \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$

Since $(0,0)$ maps to $Q$, i.e. to the point at $y = 0$, the map $L_\zeta$ identifies the whole affine $y$-line $\mathbb{A}^1 = \{(0,y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine $y$-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each regular connected component of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$ in the scheme $X_\zeta$, which coincides on $k$-points with the restriction of the map of sets $L_\zeta$.

2. *The non-regular case* $k \in \{0, \frac{p-1}{2}\}$. We choose the Steinberg coordinate $z_1$. According to [PS, 7.4.4], our non-regular connected component identifies with an affine line :

$$V_{\widehat{\mathbf{T}},0,z_2}/W_0 \simeq \mathbb{A}^1.$$

Suppose that $v = (0)$ is the origin, so that $\mathrm{Sph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation so that $L_\zeta(v) = [\rho(r, \eta)]$. Exactly as in the regular case, we may conclude that the Serre weights of the irreducible representation $\rho(r, \eta)$ are $\{\mathrm{Sym}^{2k} \otimes \det^{-k}, \mathrm{Sym}^{p-1-2k} \otimes \det^k\}$. For the two values of $k = 0$ and $k = \frac{p-1}{2}$ we find $\{\mathrm{Sym}^0, \mathrm{Sym}^{p-1}\}$ and $\{\mathrm{Sym}^0 \otimes \det^{\frac{p-1}{2}}, \mathrm{Sym}^{p-1} \otimes \det^{\frac{p-1}{2}}\}$ respectively. Comparing with the list 3.3 shows that the 2 points

$$\{\text{origin (0) on the component } (\chi_k = \chi_k^s)\}$$

for $k \in \{0, \frac{p-1}{2}\}$ are mapped to the 2 smooth points in $X_\zeta^{\mathrm{irred}}$, which lie on the two 'exterior' components of $X_\zeta$, cf. 3.3.

Fix $k \in \{0, \frac{p-1}{2}\}$ and consider the point

$$Q = L_\zeta(\text{origin (0) on the component } \gamma = (\chi_k = \chi_k^s)).$$

As we have just seen, $Q$ lies on an 'exterior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^0 \otimes \det^k$. We fix an affine coordinate on this $\mathbb{P}^1$ around $Q$, which we call $z_1$ (there will be no risk of confusion with the Steinberg coordinate above!). Away from $Q$, the affine coordinate $z_1 \neq 0$ parametrizes Galois representations of the form

$$\rho_{z_1} = \begin{pmatrix} \mathrm{unr}(z_1)\omega & 0 \\ 0 & \mathrm{unr}(z_1^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^k$. As in the regular case, $\pi(\rho_{z_1}) = \pi(0, z_1, \eta)^{\mathrm{ss}} \oplus \pi(p-3, z_1^{-1}, \omega\eta)^{\mathrm{ss}}$. Moreover, $\pi(0, z_1, \eta) = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(z_1) \otimes \mathrm{unr}(z_1^{-1})$ [7]. Since

$$(1 \otimes 1).(\eta|_{\mathbb{F}_p^\times}) = \omega^k \otimes \omega^k = \chi_k = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the non-regular case of 4.2 that $\pi(0, z_1, \eta)^{I^{(1)}} = M(z_1, 1, \chi_k)$ is a 2-dimensional standard module. Moreover, the standard module is simple if and only if $\chi \neq \chi^s$, i.e. if and only if $z_1 \neq \pm 1$.

Now let $v = z_1 \neq 0$ denote a nonzero point on our connected component $\mathbb{A}^1 = V_{\widehat{\mathbf{T}},0,1}/W_0$. Suppose that $z_1 \neq \pm 1$, i.e. $v \in D(2)_\gamma$. In particular, $\mathrm{Sph}(v) = M(z_1, 1, \gamma)$ is irreducible. By our discussion, the point $L_\zeta(z_1)$ corresponds to the block (a block of type 2) which contains $\pi(0, z_1, \eta)$. Suppose that $z_1 = \pm 1$, i.e. $v \in D(1)_\gamma$. In particular, $\mathrm{Sph}^{\mathrm{ss}}(v) = M(z_1, 1, \chi_k)^{\mathrm{ss}}$ and again, $L_\zeta(z_1)$ corresponds to the block (now a block of type 3) which contains the simple constituents of $\pi(0, z_1, \eta)^{\mathrm{ss}}$. In both cases, we conclude

$$L_\zeta(z_1) = [\rho_{z_1}] = z_1 \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$

Since (0) maps to $Q$, i.e. to the point at $z_1 = 0$, the map $L_\zeta$ identifies the whole $z_1$-line $\mathbb{A}^1 = V_{\widehat{\mathbf{T}},0,1}/W_0$ with the $z_1$-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each non-regular connected component of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$ in the scheme $X_\zeta$, which coincides on $k$-points with the restriction of the map of sets $L_\zeta$.

# 6 The morphism $L_\zeta$ in the basic odd case

Let $\zeta := \omega^{-1} : \mathbb{Q}_p^\times \to k^\times$. Here we show that the map of sets $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta(k) \to X_\zeta(k)$ that we have defined in 4.16 satisfies properties (ii) and (iiio) of 4.15, and we define a morphism of $k$-schemes $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides with the previous map of sets at the level of $k$-points. By construction, it will have the properties 4.12. This will complete the proof of 4.15, 4.12 and 4.11 in the case of an odd character.

[7] The representations $\pi(0, z_1, \eta)$ constitute the *unramified* principal series of $G$.

**6.1.** We verify properties (ii) and (iiio). We work over an irreducible component $\mathbb{P}^1$ with label " $\mathrm{Sym}^r \otimes \det^a$ | $\mathrm{Sym}^{p-3-r} \otimes \det^{r+1+a}$ " where $1 \leq r \leq p-2$ and $0 \leq a \leq p-2$, cf. 3.4. We distinguish two cases.

1. *The generic case* $r \neq p-2$. In this case, the irreducible component of $X_\zeta$ we consider is an 'interior' component and has no exceptional points. On this component, we choose an affine coordinate $x$ around the double point having $\mathrm{Sym}^r \otimes \det^a$ as one of its Serre weights. Away from this point, we have $x \neq 0$ and the corresponding Galois representation has the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{r+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^a$. As before, we have

$$\pi(\rho_x) = \pi(r, x, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], x^{-1}, \omega^{r+1}\eta)^{\mathrm{ss}}.$$

The length of $\pi(\rho_x)$ is 2. Indeed, by our assumptions on $r$, the principal series representations $\pi(r, x, \eta)$ and $\pi(p-3-r, x^{-1}, \omega^{r+1}\eta)$ are irreducible and the block $b_x$ contains only these two irreducible representations. We may follow the argument of the generic case of 5.1 word for word and deduce property 4.15 (ii).

2. *The two boundary cases* $r = p-2$. In this case, the irreducible component is one of the two 'exterior' components with labels " $\mathrm{Sym}^{p-2}$ | " $\mathrm{Sym}^{-1}$ "" or "" $\mathrm{Sym}^{-1} \det^{\frac{p-1}{2}}$ " | $\mathrm{Sym}^{p-2} \det^{\frac{p-1}{2}}$ ". Points of the open locus $X_\zeta^{\mathrm{red}}$ lying on such a component correspond to twists of unramified Galois representations of the form

$$\rho_{x+x^{-1}} = \begin{pmatrix} \mathrm{unr}(x) & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = 1$ or $\eta = \omega^{\frac{p-1}{2}}$. Let us concentrate on one of the two components, i.e. let us fix $\eta$.

Mapping an unramified Galois representation $\rho_{x+x^{-1}}$ to $t := x + x^{-1} \in k$ identifies this open locus with the $t$-line $\mathbb{A}^1 \subset \mathbb{P}^1$. We have

$$\pi(\rho_t) = \pi(p-2, x, \eta)^{\mathrm{ss}} \oplus \pi(p-2, x^{-1}, \eta)^{\mathrm{ss}} =: \pi_1 \oplus \pi_2$$

since $[p-3-(p-2)] = p-2$ (indeed, $p-3-(p-2) = -1 \equiv p-2 \mod (p-1)$). The length of $\pi(\rho_t)$ is 2. Indeed, $\pi_1 = \pi(p-2, x, \eta)$ and $\pi_2 = \pi(p-2, x^{-1}, \eta)$ are two irreducible principal series representations and the block $b_t$ contains only these two irreducible representations. They are isomorphic if and only if $x = \pm 1$, i.e. if and only if $t = \pm 2$ is an exceptional point. In this case, $b_t$ contains only one irreducible representation and is of type 3, otherwise it is of type 2.

We may write

$$\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$$

with $\chi = \mathrm{unr}(x) \otimes \omega^{p-2}\,\mathrm{unr}(x^{-1})$. Similarly for $\pi_2$. The character $\chi|_{\mathbb{F}_p^\times} = 1 \otimes \omega^{p-2}$ is regular (i.e. different from its $s$-conjugate) and we are in the regular case of 4.2. We conclude that $\pi_1^{I(1)} = M(0, x, 1, (1 \otimes \omega^{p-2}).\eta)$ and $\pi_2^{I(1)} = M(0, x^{-1}, 1, (1 \otimes \omega^{p-2}).\eta)$ are both simple 2-dimensional standard modules in the regular component $\gamma$ represented by the character $(1 \otimes \omega^{p-2}).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} \in \mathbb{T}^\vee$. They are isomorphic if and only if $t = \pm 2$. We choose an order $\gamma = ((\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2}, (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} \otimes (\eta|_{\mathbb{F}_p^\times}))$ on the set $\gamma$. Then from $L_\zeta(v) = t$ we get that either $\mathrm{Sph}^\gamma(v) = \pi_1^{I(1)}$ or $\mathrm{Sph}^\gamma(v) = \pi_2^{I(1)}$. Since for regular $\gamma$, the map $\mathrm{Sph}^\gamma$ is a bijection onto all simple $\mathcal{H}_{\overline{\mathbb{F}}_p}^\gamma$-modules, one finds that $L_\zeta^{-1}(t) = \{v_1, v_2\}$ has cardinality 2 if $t \neq \pm 2$ and then

$$\mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_t)^{I(1)}.$$

This settles property 4.15 (ii). In turn, if $t = \pm 2$ is an exceptional point, then $L_\zeta^{-1}(t) = \{v\}$ has cardinality 1 and

$$\mathrm{Sph}(v) \oplus \mathrm{Sph}(v) \simeq \pi(\rho_t)^{I(1)}.$$

This settles property 4.15 (iiio).

**6.2.** We define a morphism of $k$-schemes $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides on $k$-points with the map of sets $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) \longrightarrow X_\zeta(k)$. We work over a connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, indexed by some $\gamma \in \mathbb{T}^\vee/W_0$. Let $v$ be a $k$-point of this component.

Since $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$, the connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ are indexed by the fibre $(\cdot)|^{-1}_{\mathbb{F}_p^\times}(\omega^{-1})$. This fibre consists of the $\frac{p-1}{2}$ regular components, represented by the characters

$$\chi_k = \omega^{k-1} \otimes \omega^{-k}$$

for $k = 1, ..., \frac{p-1}{2}$, cf. 2.2. Recall that $z_2 = \zeta(p) = 1$.

Fix an order $\gamma = (\chi_k, \chi_k^s)$ on the set $\gamma$ and choose standard coordinates $x, y$. According to [PS, 7.4.8], our regular connected component identifies with two affine lines intersecting at the origin:

$$V_{\widehat{\mathbf{T}},0,1} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1.$$

Suppose that $v = (0,0)$ is the origin, so that $\mathrm{Sph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation. It corresponds to the irreducible Galois representation $\rho(r, \eta)$, in the notation of [Be11, 1.3], whence $L_\zeta(v) = [\rho(r, \eta)]$. According to 4.2, the component of $\pi(r, 0, \eta)^{I^{(1)}}$ is given by $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times})$. Setting $\eta|_{\mathbb{F}_p^\times} = \omega^a$, this implies $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times}) = \omega^{r+a} \otimes \omega^a = \chi_k$ and hence $a = -k$ and $r = 2k - 1$. The Serre weights of the irreducible representation $\rho(r, \eta)$ are therefore $\{\mathrm{Sym}^{2k-1} \otimes \det^{-k}, \mathrm{Sym}^{p-2k} \otimes \det^{k-1}\}$, cf. [Br07, 1.9].

Comparing these pairs of Serre weights with the list 3.4 shows that the $\frac{p-1}{2}$ points

$$\{\text{origin } (0,0) \text{ on the component } (\chi_k, \chi_k^s)\}$$

for $k = 1, ..., \frac{p-1}{2}$ are mapped successively to the $\frac{p-1}{2}$ double points of the chain $X_\zeta$. We distinguish two cases.

*1. The generic case* $1 < k < \frac{p-1}{2}$. In this case, the argument proceeds as in the regular case of 5.2. Consider the double point

$$Q = L_\zeta(\text{origin } (0,0) \text{ on the component } \gamma = (\chi_k, \chi_k^s)).$$

As we have just seen, $Q$ lies on an 'interior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^{2k-1} \otimes \det^{-k}$. We fix an affine coordinate on this $\mathbb{P}^1$ around $Q$, which we will also call $x$. Away from $Q$, the affine coordinate $x \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{2k} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{-k}$. As we have seen above, $\pi(\rho_x) = \pi(2k-1, x, \eta) \oplus \pi(p-3-2k+1, x^{-1}, \omega^{2k}\eta) =: \pi_1 \oplus \pi_2$. Moreover, $\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(x) \otimes \omega^{2k-1}\,\mathrm{unr}(x^{-1})$. Since

$$(1 \otimes \omega^{2k-1}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{-k} \otimes \omega^{k-1} = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the regular case of 4.2 that $\pi_1^{I^{(1)}} = M(0, x, 1, \chi_k^s)$ is a simple 2-dimensional standard module. Note that $M(0, x, 1, \chi_k^s) = M(x, 0, 1, \chi_k)$ according to [V04, Prop. 3.2].

Now suppose that $v = (x, 0)$, $x \neq 0$, denotes a nonzero point on the $x$-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(x, 0, 1, \chi_k)$. Our discussion shows that the point $L_\zeta((x, 0))$ corresponds to the block which contains $\pi_1$. Since $\pi_1$ lies in the block parametrized by $[\rho_x]$, cf. 4.10, it follows that

$$L_\zeta((x, 0)) = [\rho_x] = x \in \mathbb{G}_m \subset \mathbb{P}^1.$$

Since $(0, 0)$ maps to $Q$, i.e. to the point at $x = 0$, the map $L_\zeta$ identifies the whole affine $x$-line $\mathbb{A}^1 = \{(x, 0) : x \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine $x$-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

On the other hand, the double point $Q$ also lies on the irreducible component whose labelling includes the other weight of $Q$, i.e. the weight $\mathrm{Sym}^{p-2k} \otimes \det^{k-1}$. We fix an affine coordinate $y$ on

this $\mathbb{P}^1$ around $Q$. Away from $Q$, the coordinate $y \neq 0$ parametrizes Galois representations of the form

$$\rho_y = \begin{pmatrix} \mathrm{unr}(y)\omega^{p-2k+1} & 0 \\ 0 & \mathrm{unr}(y^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{k-1}$. As in the first case, $\pi(\rho_y)$ contains $\pi_1 := \pi(p-2k, y, \eta) = \mathrm{Ind}_B^G(\chi) \otimes \eta$ as a direct summand, where now $\chi = \mathrm{unr}(y) \otimes \omega^{p-2k}\,\mathrm{unr}(y^{-1})$. Since

$$(1 \otimes \omega^{p-2k}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{k-1} \otimes \omega^{-k} = \chi_k \in \mathbb{T}^\vee,$$

we deduce from the regular case of 4.2 that $\pi_1^{I^{(1)}} = M(0, y, 1, \chi_k)$ is a simple 2-dimensional standard module.

Now suppose that $v = (0, y)$, $y \neq 0$, denotes a nonzero point on the $y$-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(0, y, 1, \chi_k)$. Our discussion shows that the point $L_\zeta((0, y))$ corresponds to the block which contains $\pi_1$, parametrized by $[\rho_y]$. Hence

$$L_\zeta((0, y)) = [\rho_y] = y \in \mathbb{G}_m \subset \mathbb{P}^1.$$

Since $(0, 0)$ maps to $Q$, i.e. to the point at $y = 0$, the map $L_\zeta$ identifies the whole $y$-line $\mathbb{A}^1 = \{(0, y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine $y$-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each connected component $(V^\gamma_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ such that $\gamma = (\chi_k, \chi_k^s)$ with $1 < k < \frac{p-1}{2}$, in the scheme $X_\zeta$, which coincides on $k$-points with the restriction of the map of sets $L_\zeta$.

*2. The two boundary cases $k \in \{1, \frac{p-1}{2}\}$.* Consider the double point

$$Q = L_\zeta(\text{origin } (0, 0) \text{ on the component } \gamma = (\chi_k, \chi_k^s)).$$

As we have just seen, $Q$ lies on an 'interior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^1 \otimes \det^{-1}$ (for $k = 1$) or the weight $\mathrm{Sym}^1 \otimes \det^{\frac{p-3}{2}}$ (for $k = \frac{p-1}{2}$). We fix an affine coordinate on this $\mathbb{P}^1$ around $Q$, which we will call $z$. Away from $Q$, the coordinate $z \neq 0$ parametrizes Galois representations of the form

$$\rho_z = \begin{pmatrix} \mathrm{unr}(z)\omega^2 & 0 \\ 0 & \mathrm{unr}(z^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^{-1}$ or $\eta = \omega^{\frac{p-3}{2}}$.

Let $k = 1$, i.e. $\eta = \omega^{-1}$. Following the argument in the generic case word for word, we may conclude that $L_\zeta$ identifies the $x$-line $\mathbb{A}^1 = \{(x, 0) : x \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the $z$-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

Let $k = \frac{p-1}{2}$, i.e. $\eta = \omega^{\frac{p-3}{2}}$. As in the generic case, we may conclude that $L_\zeta$ identifies the $y$-line $\mathbb{A}^1 = \{(0, y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the $z$-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

On the other hand, the double point $Q$ lies also on the irreducible component $\mathbb{P}^1$ whose labelling includes the other weight of $Q$, i.e. the weight $\mathrm{Sym}^{p-2}$ (for $k = 1$) or the weight $\mathrm{Sym}^{p-2} \otimes \det^{\frac{p-1}{2}}$ (for $k = \frac{p-1}{2}$). These are the two 'exterior' components. Points of the open locus $X_\zeta^{\mathrm{red}}$ lying on such a component correspond to unramified (up to twist) Galois representations of the form

$$\rho_t = \begin{pmatrix} \mathrm{unr}(z) & 0 \\ 0 & \mathrm{unr}(z^{-1}) \end{pmatrix} \otimes \eta$$

where $\eta = 1$ (for $k = 1$) or $\eta = \omega^{\frac{p-1}{2}}$ (for $k = \frac{p-1}{2}$) and with $t = z + z^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1$. As in the boundary case of 6.1, we have $\pi(\rho_t) = \pi(p-2, z, \eta) \oplus \pi(p-2, z^{-1}, \eta) =: \pi_1 \oplus \pi_2$ and these are irreducible principal series representations. We may write $\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(z) \otimes \omega^{p-2}\,\mathrm{unr}(z^{-1})$. The character $\chi|_{\mathbb{F}_p^\times} = 1 \otimes \omega^{p-2}$ is regular (i.e. different from its $s$-conjugate) and we are in the regular case of 4.2. We conclude that

$$\pi_1^{I^{(1)}} = M(0, z, 1, (1 \otimes \omega^{p-2}).\eta)$$

is a simple 2-dimensional standard module in the regular component represented by the character

$$(1 \otimes \omega^{p-2}).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{-1} \in \mathbb{T}^\vee.$$

This latter character equals $\chi_1$ for $\eta = 1$ and $(\chi_{\frac{p-1}{2}})^s$ for $\eta = \omega^{\frac{p-1}{2}}$ (indeed, note that $\frac{p-1}{2} \equiv -\frac{p-1}{2}$ mod $p-1$).

Now suppose that $k = 1$, i.e. $\eta = 1$. Let $v = (0, y)$, $y \neq 0$, be a nonzero point on the $y$-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(0, y, 1, \chi_1)$. Our discussion shows that the point $L_\zeta((0, y))$ corresponds to the block which contains $\pi_1$, i.e. which is parametrized by $[\rho_t]$. It follows that

$$L_\zeta((0, y)) = [\rho_t] = t = y + y^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1.$$

Since $(0, 0)$ maps to $Q$, i.e. to the point at $t = \infty$, the map of sets $L_\zeta$ maps the $k$-points of the whole affine $y$-line $\mathbb{A}^1 = \{(0, y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ to the $k$-points of the whole 'left exterior' component $\mathbb{P}^1 \subset X_\zeta$ via the formula

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \\ y & \longmapsto & \begin{cases} y + y^{-1} & \text{if } y \neq 0 \\ \infty = Q & \text{if } y = 0. \end{cases} \end{array}$$

This formula is algebraic: indeed, for $y \in \mathbb{A}^1 \setminus \{\pm i\}$ (where $\pm i$ are the roots of the polynomial $f(y) = y^2 + 1$), we have $y + y^{-1} \neq 0$ and $(y + y^{-1})^{-1} = y/(y^2 + 1)$, which is equal to 0 at $y = 0$. Moreover, it glues at the origin $(0, 0)$ with the open immersion of the $x$-line of $V_{\widehat{\mathbf{T}},0,1} = \mathbb{A}^1 \cup_0 \mathbb{A}^1$ in $X_\zeta$ defined above, since both map $(0, 0)$ to $Q$. We take the resulting morphism of $k$-schemes $\mathbb{A}^1 \cup_0 \mathbb{A}^1 \to X_\zeta$ as the definition of $L_\zeta$ on the connected component $(V^{(\chi_1, \chi_1^s)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$. Note that its restriction to the open subset $\{y \neq 0\}$ in the $y$-line $\mathbb{A}^1$ is the morphism $\mathbb{G}_m \to \mathbb{A}^1$ corresponding to the ring extension

$$k[t] \longrightarrow k[y, y^{-1}] = k[t][y]/(y^2 - ty + 1),$$

and that the discriminant $t^2 - 4$ of $y^2 - ty + 1 \in k[t][y]$ vanishes precisely at the two exceptional points $t = \pm 2$.

Suppose $k = \frac{p-1}{2}$, i.e. $\eta = \omega^{\frac{p-1}{2}}$. Let $v = (x, 0)$, $x \neq 0$, denote a nonzero point on the $x$-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular,

$$\mathrm{Sph}^\gamma(v) = M(0, x, 1, (\chi_{\frac{p-1}{2}})^s) = M(x, 0, 1, \chi_{\frac{p-1}{2}}).$$

Our discussion shows that the point $L_\zeta((x, 0))$ corresponds to the block which contains $\pi_1$, i.e. which is parametrized by $[\rho_t]$. It follows that $L_\zeta((x, 0)) = [\rho_t] = t = x + x^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1$. Since $(0, 0)$ maps to the point $Q$ at $t = \infty$, the map of sets $L_\zeta$ maps the $k$-points of the whole affine $x$-line $\mathbb{A}^1 = \{(x, 0) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ to the $k$-points of the whole 'right exterior' component $\mathbb{P}^1 \subset X_\zeta$ via the formula

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \\ x & \longmapsto & \begin{cases} x + x^{-1} & \text{if } x \neq 0 \\ \infty = Q & \text{if } x = 0. \end{cases} \end{array}$$

This formula is algebraic. Moreover, it glues at the origin $(0, 0)$ with the open immersion of the $y$-line of $V_{\widehat{\mathbf{T}},0,1} = \mathbb{A}^1 \cup_0 \mathbb{A}^1$ in $X_\zeta$ defined above, since both map $(0, 0)$ to $Q$. We take the resulting morphism of $k$-schemes $\mathbb{A}^1 \cup_0 \mathbb{A}^1 \to X_\zeta$ as the definition of $L_\zeta$ on the connected component $(V^{(\chi_{\frac{p-1}{2}}, (\chi_{\frac{p-1}{2}})^s)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$.

# 7 An interpolation of the semisimple mod $p$ correspondence

In this subsection we continue to assume $p \geq 5$.

**7.1.** Recall the mod $p$ parametrization functor $P$ from 4.6. For $\zeta \in \mathcal{Z}^\vee(k)$, let $\mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ be the full subcategory of $\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ whose objets are the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules $M$ whose Satake parameter $S(M)$ is supported on the closed subscheme $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. A $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module $M$ lies in the category $\mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ if and only if: $M$ is only supported in $\gamma$-components where $\gamma|_{\mathbb{F}_p^\times} = \zeta|_{\mathbb{F}_p^\times}$ and the operator $U^2$ acts on $M$ via the $\mathbb{G}_m$-part of $\zeta$. Then $P$ induces a *mod $p$ $\zeta$-parametrization functor*

$$P_\zeta : \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \longrightarrow \mathrm{QCoh}((V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta).$$

Let $\zeta \in \mathcal{Z}^\vee(k)$. We have the functor

$$L_{\zeta *} : \mathrm{QCoh}((V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta) \longrightarrow \mathrm{QCoh}(X_\zeta)$$

*push-forward along the $k$-morphism* $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \to X_\zeta$ from 4.11. Finally recall that for $\zeta \in \mathcal{Z}^\vee(k)$, the functor of $I^{(1)}$-invariants $(\cdot)^{I^{(1)}} : \mathrm{Mod}^{\mathrm{sm}}(k[G]) \to \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ induces a functor

$$(\cdot)^{I^{(1)}}_\zeta : \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \to \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}),$$

by 4.7.

**7.2. Definition.** *Let $\zeta \in \mathcal{Z}^\vee(k)$. The* mod $p$ $\zeta$-Langlands parametrization functor *is the functor* $\mathrm{L}_\zeta\mathrm{P}_\zeta := L_{\zeta *} \circ P_\zeta$ :

$$\begin{array}{c} \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \\ \Big\downarrow \\ \mathrm{QCoh}(X_\zeta) \end{array}$$

*Identifying $\zeta$ with a central character of $G$, the functor $\mathrm{L}_\zeta\mathrm{P}_\zeta$ extends to the category $\mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G])$ by precomposing with the functor $(\cdot)^{I^{(1)}}_\zeta : \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \to \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$. This gives the functor $\mathrm{L}_\zeta\mathrm{P}_\zeta \circ (\cdot)^{I^{(1)}}_\zeta$ :*

$$\begin{array}{c} \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \\ \Big\downarrow \\ \mathrm{QCoh}(X_\zeta). \end{array}$$

**7.3. Theorem.** *Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. Fix a character $\zeta : Z(G) = \mathbb{Q}_p^\times \to k^\times$, corresponding to a point $(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) \in \mathcal{Z}^\vee(k)$ under the identification $\mathcal{Z}(G)^\vee \cong \mathcal{Z}^\vee(k)$ from 4.6.*

*The mod $p$ $\zeta$-Langlands parametrization functor $\mathrm{L}_\zeta\mathrm{P}_\zeta$ interpolates the Langlands parametrization of the blocks of the category $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$, cf. 4.9 : for all $x \in X_\zeta(k)$ and for all $\pi \in b_{[\rho_x]}$,*

$$\mathrm{L}_\zeta\mathrm{P}_\zeta(\pi^{I^{(1)}}) = \begin{cases} i_{x*}(\pi^{I^{(1)}}) & \text{if } x \text{ is not an exceptional point in the odd case} \\ i_{x*}(\pi^{I^{(1)}})^{\oplus 2} & \text{otherwise} \end{cases} \in \mathrm{QCoh}(X_\zeta)$$

*where $i_x : \mathrm{Spec}(k) \to X_\zeta$ is the $k$-point $x$.*

*Proof.* By definition of a block of a category as a certain equivalence class of simple objects [Pas13], if $\pi \in b_{[\rho_x]}$ then in particular $\pi$ is simple. Then $\pi^{I^{(1)}}$ is simple too, and hence has a central character. Therefore $P_\zeta(\pi^{I^{(1)}})$ is the underlying $k$-vector space of $\pi^{I^{(1)}}$ supported at the $k$-point $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ corresponding to its central character under the isomorphism $\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p}$, which lies on some connected component $\gamma$. Suppose $\dim_k(\pi^{I^{(1)}}) = 2$. Then $\pi^{I^{(1)}}$ is isomorphic to the

simple standard module of $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_p}$ with central character $v$, i.e. to $\mathrm{Sph}^{\gamma}(v)$, and hence $L_{\zeta}(v) = x$ by definition of the map of sets $L_{\zeta}(k)$. Suppose $\dim_k(\pi^{I^{(1)}}) = 1$. Then $\pi^{I^{(1)}}$ is one of the two spherical characters of $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_p}$ whose restriction to the center $Z(\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_p})$ is equal to $v$, i.e. it is one of the simple constituents of $(\mathrm{Sph}^{\gamma}(v))^{\mathrm{ss}}$, and hence again $L_{\zeta}(v) = x$ by definition of the map of sets $L_{\zeta}(k)$. Now if $x$ is not an exceptional point in an odd case, then $L_{\zeta}$ is an open immersion at $v$, and otherwise it has ramification index 2 at $v$. The theorem follows. □

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