Semisimple Langlands for $GL_2(\mathbb{Q}_p)$ and mod p Hecke modules

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Abstract

Let $p \geq 5$ and let $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ be the center of the mod p pro-p-Iwahori Hecke algebra of $\mathbf{GL}_2(\mathbb{Q}_p)$. Let X be the projective curve parametrizing 2-dimensional mod p semi-simple representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We construct a quotient morphism of schemes $\mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \to X$. We then show that the correspondence between the specialization $\mathcal{M}^{(1)}_z$ of the spherical $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module $\mathcal{M}^{(1)}$ from [PS] in closed points $z \in \mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ and the Galois representation $\rho_{x(z)}$ *is* the semi-simple mod p local Langlands correspondence for the group $\mathbf{GL}_2(\mathbb{Q}_p)$.

Contents

1 Introduction

Background. The mod p (and the p-adic) Langlands correspondence for $\mathbf{GL}_2(\mathbb{Q}_p)$ was conjectured by Breuil, and has been fully established by Colmez-Dospinescu-Paškūnas [CDP14], building on work of Breuil, Colmez, Emerton, Kisin, Paškūnas and many others. Its *semisimple* version was established by Breuil in [Br03]. It is an explicit map $\rho \mapsto \pi(\rho)$, from the set of semisimple continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on 2-dimensional $\overline{\mathbb{F}}_p$-vector spaces, to the set of semisimple smooth representations of $\mathbf{GL}_2(\mathbb{Q}_p)$ on $\overline{\mathbb{F}}_p$-vector spaces.

Set $G := \mathbf{GL}_2(\mathbb{Q}_p)$, let $Z(G) = \mathbb{Q}_p^\times$ be the center of G and $\zeta : Z(G) \to \overline{\mathbb{F}}_p^\times$ be a central character. Assume $p \geq 5$. In [DEG22], Dotto-Emerton-Gee introduce a curve X_ζ over $\mathbb{F}_p$ (denoted by X in loc.cit.), which is a chain of projective lines with ordinary double points and of length $(p \pm 1)/2$, where the sign is equal to $-\zeta(-1)$. The definition of X_ζ is motivated by the Galois side of Breuil's semisimple correspondence: the closed $\overline{\mathbb{F}}_p$-points of X_ζ parametrize isomorphism classes of semisimple 2-dimensional continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$ with determinant $\omega\zeta$:

$$X_\zeta(\overline{\mathbb{F}}_p) \cong \{\text{semisimple continuous } \rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{GL}_2}(\overline{\mathbb{F}}_p) \ \text{ with } \det \rho = \omega\zeta\}/\sim;$$

here ω is the mod p cyclotomic character. See [DEG22, 1.4] for further discussion on the curve X_ζ. In the the sequel, we let X be the disjoint union over all X_ζ, base changed to $\overline{\mathbb{F}}_p$.

Let $I^{(1)} \subset G$ be the standard pro-p Iwahori subgroup consisting of integral matrices which are upper unipotent mod p, and let $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$ is the pro-p-Iwahori Hecke algebra of G with coefficients in $\overline{\mathbb{F}}_p$. By work of Ollivier [O09], the functor of $I^{(1)}$-invariants $\pi \mapsto \pi^{I^{(1)}}$ is an equivalence from the category of mod p smooth representations of G which are generated by their $I^{(1)}$-invariants, to the category of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules. Thus the composed map $\rho \mapsto \pi(\rho)^{I^{(1)}}$ is a correspondence from the set of semisimple mod p 2-dimensional continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to the set of semisimple $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules.

Statement of the result. Let $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ be the center of the algebra $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$. In [PS, 7.4.1], we constructed the mod p spherical module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$. It is a distinguished $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-action on a maximal commutative subring of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$, which is a mod p analogue (plus extension to the pro-p Iwahori level) of the classical (anti)spherical module appearing in complex Kazhdan-Lusztig theory [KL87, 3.9]. The quasi-coherent module associated to $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ on $\operatorname{Spec} Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$, when specialized at closed points, gives rise to a parametrization of *all* irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules [PS, 7.4.9/7.4.15].

Here we prove the following (cf. Theorem 4.9):

Theorem. *Let $G = \mathbf{GL}_2(\mathbb{Q}_p)$ with $p \geq 5$. There exists a quotient morphism of $\overline{\mathbb{F}}_p$-schemes*

$$\mathscr{L} : \operatorname{Spec} Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \longrightarrow X,$$

with the following property: given a closed point $z \in \operatorname{Spec} Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$, the correspondence between the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module $\mathcal{M}^{(1)}_z$, equal to the specialization of $\mathcal{M}^{(1)}$ in the central character z, and the Galois representation $\rho_{x(z)}$, is the semisimple mod p local Langlands correspondence.

Thus, the quasi-coherent $\mathcal{O}_X$-module $\mathscr{L}_*\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$, equal to the push-forward of $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ along $\mathscr{L}$, interpolates the semisimple Langlands correspondence: for all $x \in X(\overline{\mathbb{F}}_p)$, one has an isomorphism of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules

$$\left(\mathscr{L}_*\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{\mathcal{O}_X} k(x)\right)^{\mathrm{ss}} = \left(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})} \mathcal{O}_{\mathscr{L}^{-1}(x)}\right)^{\mathrm{ss}} \cong \pi(\rho_x)^{I^{(1)}}.$$

As a byproduct of our constructions, we also obtain an interpolation of Paškūnas' parametrization of the blocks of the category $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(\overline{\mathbb{F}}_p[G])$ of locally admissible smooth G-representations over $\overline{\mathbb{F}}_p$ with central character ζ [Pas13]. See 7.3 for the precise statement.

More details on the construction. The construction of the morphism $\mathscr{L}$ is a consequence of our results from [PS] on the geometry of the *generic* pro-p-Iwahori-Hecke algebra (with coefficients in the ring $\mathbb{Z}[\mathbf{q}]$ where $\mathbf{q}$ is a formal variable) for $\mathbf{GL}_2(\mathbb{Q}_p)$, specialized at $\mathbf{q} = p = 0 \in \overline{\mathbb{F}}_p$. To give more details, let $\widehat{\mathbf{G}}$ be the Langlands dual group of $\mathbf{GL}_2$ over $\overline{\mathbb{F}}_p$, with maximal torus $\widehat{\mathbf{T}}$. We consider the special fibre at $\mathbf{q} = 0$ of the Vinberg fibration $V_{\widehat{\mathbf{T}}} \xrightarrow{\mathbf{q}} \mathbb{A}^1$ associated to $\widehat{\mathbf{T}} \subset \widehat{\mathbf{G}}$ followed by base change to $\overline{\mathbb{F}}_p$. This yields the $\overline{\mathbb{F}}_p$-semigroup scheme

$$V_{\widehat{\mathbf{T}},0} := \mathrm{SingDiag}_{2\times 2} \times_{\overline{\mathbb{F}}_p} \mathbb{G}_m,$$

where $\mathrm{SingDiag}_{2\times 2}$ represents the semigroup of singular diagonal 2×2-matrices over $\overline{\mathbb{F}}_p$, cf. [PS, 7.1]. Let $\mathbb{T}^\vee$ be the finite abelian group dual to $\mathbb{T} = \mathbf{T}(\mathbb{F}_p)$, and consider the extended semigroup

$$V^{(1)}_{\widehat{\mathbf{T}},0} := \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}},0}.$$

It has a natural diagonal W_0-action. In [PS, 7.2.2] we established the mod p pro-p-Iwahori Satake isomorphism

$$\operatorname{Spec} \mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p} : V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \xrightarrow{\ \sim\ } \operatorname{Spec} Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$$

identifying the center $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ with the ring of regular functions on the quotient $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. It encodes the duality between $\mathbf{GL}_2$ and the dual group $\widehat{\mathbf{G}}$. The morphism $\mathscr{L}$ is then a composition of the inverse of $\operatorname{Spec}\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p}$ with a certain morphism L (see below) from $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ to X:

$$\mathscr{L} := L \circ (\operatorname{Spec}\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p})^{-1}.$$

Organization of the article. In section 2 we recall some results from [PS], notably that the quotient $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ is naturally fibered over the central characters ζ of $\mathbf{GL}_2(\mathbb{Q}_p)$. Any fibre $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ is a naturally ordered union of connected components, which generically[1] are equal to two affine lines $\mathbb{A}^1 \cup_0 \mathbb{A}^1$ intersecting at the origin. In section 3 we recall some properties of X_ζ. Whereas in [DEG22] the irreducible components of X_ζ are labeled by certain cuspidal types, we choose a labelling of irreducible components by certain pairs of Serre weights, which is inspired from [Em19][2] and which is more suitable for our purposes. In section 4 we state the existence and properties of a distinguished morphism

$$L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$$

and set $L := \coprod_\zeta L_\zeta$. We first define the morphism L_ζ on the level of $\overline{\mathbb{F}}_p$-points. This uses Paškūnas' parametrization of the blocks of the category $\operatorname{Mod}^{\mathrm{ladm}}_\zeta(\overline{\mathbb{F}}_p[G])$ from [Pas13]. The morphism L_ζ is locally given by the toric construction of the projective line: it identifies the open subset $\mathbb{G}_m$ in the "first" irreducible component $\mathbb{A}^1$ of the connected component $\mathbb{A}^1 \cup_0 \mathbb{A}^1$ with the open subset $\mathbb{G}_m$ in "second" irreducible component $\mathbb{A}^1$ of the "next" connected component $\mathbb{A}^1 \cup_0 \mathbb{A}^1$ via the map $z \mapsto z^{-1}$, thus forming a $\mathbb{P}^1$. We reduce the case of a general central character ζ to two basic cases according to a certain parity of ζ. In sections $5, 6$ we prove all stated properties of the morphism L_ζ in the basic cases. Finally, in section 7 we explain the interpolation of the semisimple mod p correspondence.

Notation. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let k be its residue field, an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. We let $G = \mathbf{GL}_2(\mathbb{Q}_p)$. We let $\mathbf{T}$ denote the diagonal torus in $\mathbf{GL}_2$ and W_0 its Weyl group. Let $\mathbb{T} = \mathbf{T}(\mathbb{F}_p)$. If H is a finite group, then $H^\vee := \operatorname{Hom}(H, k^\times)$. Finally, $\widehat{\mathbf{G}}$ denotes the dual group of $\mathbf{GL}_2$ over k, with maximal torus $\widehat{\mathbf{T}}$.

2 Mod p Satake parameters with fixed central character

We recall some results from [PS, 7.5] in the special case $F = \mathbb{Q}_p$.

2.1. Let $\omega : \mathbb{F}_p^\times \to k^\times$ be given by the embedding $\mathbb{F}_p \subset k$. The group $(\mathbb{F}_p^\times)^\vee = \langle\omega\rangle$ is cyclic of order $p-1$. Any element ω^r gives rise to a non-regular character of $\mathbb{T}$ via $\omega^r(t_1,t_2) := \omega^r(t_1)\omega^r(t_2)$ for all $(t_1,t_2) \in \mathbb{T} = \mathbb{F}_p^\times \times \mathbb{F}_p^\times$. Composition with multiplication in $\mathbb{T}^\vee$ produces an action of $(\mathbb{F}_p^\times)^\vee$ on $\mathbb{T}^\vee$, which factors through the quotient $\mathbb{T}^\vee/W_0$:

$$\mathbb{T}^\vee/W_0 \times (\mathbb{F}_q^\times)^\vee \longrightarrow \mathbb{T}^\vee/W_0,\ (\gamma, \omega^r) \mapsto \gamma\omega^r.$$

If $\gamma \in \mathbb{T}^\vee/W_0$ is regular (non-regular), then $\gamma\omega^r$ is regular (non-regular).

2.2. We may restrict characters to the subgroup $\mathbb{F}_p^\times \simeq \{\operatorname{diag}(a,a) : a \in \mathbb{F}_p^\times\} \subset \mathbb{T}$ and this gives a homomorphism $\mathbb{T}^\vee \to (\mathbb{F}_p^\times)^\vee$ which factors into a restriction map

$$\mathbb{T}^\vee/W_0 \to (\mathbb{F}_p^\times)^\vee,\ \gamma \mapsto \gamma|_{\mathbb{F}_p^\times}.$$

The relation to the $(\mathbb{F}_p^\times)^\vee$-action on the source $\mathbb{T}^\vee/W_0$ is $(\gamma\omega^r)|_{\mathbb{F}_p^\times} = \gamma|_{\mathbb{F}_p^\times}\,\omega^{2r}$. We recall the fibers of the restriction map $\gamma \mapsto \gamma|_{\mathbb{F}_q^\times}$. Let $(\cdot)|^{-1}_{\mathbb{F}_p^\times}(\omega^{2r})$ be the fibre at a square element ω^{2r}. The action

[1] There occur also connected components equal to $\mathbb{A}^1$ corresponding to *non-regular* components of $\operatorname{Spec} Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$.

[2] The idea of relating the curve X and the spherical module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ came to the authors when listening to the talk [Em19], and led to the first preprint [PS2] in 2020. We thank M. Emerton for this enlightening talk. This article is a revised version of [PS2].

of ω^{-r} on $\mathbb{T}^\vee/W_0$ induces a bijection with the fibre $(\cdot)|_{\mathbb{F}_q^\times}^{-1}(1)$. The fibre

$$(\cdot)|_{\mathbb{F}_q^\times}^{-1}(1)=\{1\otimes 1\}\coprod\{\omega\otimes\omega^{-1},\omega^2\otimes\omega^{-2},...,\omega^{\frac{q-3}{2}}\otimes\omega^{-\frac{q-3}{2}}\}\coprod\{\omega^{\frac{q-1}{2}}\otimes\omega^{-\frac{q-1}{2}}\}$$

has cardinality $\frac{p+1}{2}$ and, in the above list, we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. The W_0-orbits represented by the characters $\omega^r\otimes\omega^{-r}$ for $r=1,...,\frac{p-3}{2}$, are all regular W_0-orbits. The two orbits at the two ends of the list are non-regular orbits. Since the action of ω^{-r} preserves regular (non-regular) orbits, any fibre at a square element (there are $\frac{p-1}{2}$ such fibres) has the same structure. On the other hand, let $(\cdot)|_{\mathbb{F}_p^\times}^{-1}(\omega^{2r-1})$ be the fibre at a non-square element ω^{2r-1}. The action of ω^{-r} induces a bijection with the fibre $(\cdot)|_{\mathbb{F}_p^\times}^{-1}(\omega^{-1})$. The fibre

$$(\cdot)|_{\mathbb{F}_p^\times}^{-1}(\omega^{-1})=\{1\otimes\omega^{-1},\omega\otimes\omega^{-2},...,\omega^{\frac{p-1}{2}-1}\otimes\omega^{-\frac{p-1}{2}}\}$$

has cardinality $\frac{p-1}{2}$ and we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. All elements of the fibre are regular W_0-orbits. Since the action of ω^{-r} preserves regular (non-regular) orbits, any fibre at a non-square element (there are $\frac{p-1}{2}$ such fibres) has the same structure.

2.3. We have the commutative k-semigroup scheme

$$V_{\widehat{\mathbf{T}},0}^{(1)}=\mathbb{T}^\vee\times V_{\widehat{\mathbf{T}},0}=\mathbb{T}^\vee\times\mathrm{SingDiag}_{2\times 2}\times\mathbb{G}_m.$$

cf. [PS, 7.5.3]. It has a natural W_0-action: the natural action of W_0 on the factors $\mathbb{T}^\vee$ and $\mathrm{SingDiag}_{2\times 2}$ and the trivial one on $\mathbb{G}_m$. In addition to this, there is a commuting action of the k-group scheme

$$\mathcal{Z}^\vee:=(\mathbb{F}_p^\times)^\vee\times\mathbb{G}_m$$

on $V_{\widehat{\mathbf{T}},0}^{(1)}$: the (constant finite diagonalizable) group $(\mathbb{F}_p^\times)^\vee$ acts only on the factor $\mathbb{T}^\vee$ and in the way described in 2.1; an element $z_0\in\mathbb{G}_m$ acts trivially on $\mathbb{T}^\vee$, by multiplication with the diagonal matrix $\mathrm{diag}(z_0,z_0)$ on $\mathrm{SingDiag}_{2\times 2}$ and by multiplication with the square z_0^2 on $\mathbb{G}_m$. Therefore the quotient $V_{\widehat{\mathbf{T}},0}^{(1)}/W_0$ inherits a $\mathcal{Z}^\vee$-action. We have a decomposition

$$V_{\widehat{\mathbf{T}},0}^{(1)}/W_0=\coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\mathrm{reg}}}V_{\widehat{\mathbf{T}},0}\coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\text{non-reg}}}V_{\widehat{\mathbf{T}},0}/W_0.$$

In this optic, the $(\mathbb{F}_p^\times)^\vee$-action is by permutations on the index set $\mathbb{T}^\vee/W_0$. It preserves the subsets of regular and non-regular components. The $\mathbb{G}_m$-action on $V_{\widehat{\mathbf{T}},0}^{(1)}/W_0$ preserves connected components.

2.4. Recall from [PS, 7.5.6] the spherical map

$$\mathrm{Sph}:(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)(k)\longrightarrow\{\text{left }\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}\text{-modules}\}/\sim$$

and the twisting action of $\mathcal{Z}^\vee(k)$ on semisimple $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$-modules. Let $\mathrm{Sph}^{\mathrm{ss}}$ be the map Sph followed by semisimplification.

2.5. Lemma. *The map* $\mathrm{Sph}^{\mathrm{ss}}$ *is* $\mathcal{Z}^\vee(k)$*-equivariant.*

Proof. This is [PS, 7.5.2]. □

2.6. According to [PS, 7.5.4], we have two projection morphisms

$$\begin{array}{ccc} & V_{\widehat{\mathbf{T}},0}^{(1)}/W_0 & \\ {}^{\mathrm{pr}_{\mathbb{T}^\vee/W_0}}\swarrow & & \searrow^{\mathrm{pr}_{\mathbb{G}_m}} \\ \mathbb{T}^\vee/W_0 & & \mathbb{G}_m. \end{array}$$

Composing $\mathrm{pr}_{\mathbb{T}^\vee/W_0}$ with the restriction map $(\cdot)|_{\mathbb{F}_p^\times}:\mathbb{T}^\vee/W_0\to(\mathbb{F}_p^\times)^\vee$, setting

$$\theta := \big((\cdot)|_{\mathbb{F}_p^\times}\circ \mathrm{pr}_{\mathbb{T}^\vee/W_0}\big)\times \mathrm{pr}_{\mathbb{G}_m}$$

yields

$$\begin{array}{c} V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \\ \big\downarrow \theta \\ \mathscr{Z}^\vee. \end{array}$$

The relation to the $\mathscr{Z}^\vee$-action on the source $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ is given by the formula

$$\theta(x.(\omega^r,z_0))=\theta(x)(\omega^{2r},z_0^2)=\theta(x)(\omega^r,z_0)^2$$

for $x\in V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ and $(\omega^r,z_0)\in\mathscr{Z}^\vee$. The following definition is [PS, 7.5.1].

2.7. Definition. *Let $\zeta\in\mathscr{Z}^\vee$. The* space of mod p Satake parameters with central character ζ *is the k-scheme*

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta := \theta^{-1}(\zeta).$$

2.8. Let $\zeta=(\zeta|_{\mathbb{F}_p^\times},z_2)\in\mathscr{Z}^\vee(k)=(\mathbb{F}_p^\times)^\vee\times k^\times$. Denote by $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{z_2}$ the fibre of $\mathrm{pr}_{\mathbb{G}_m}$ at $z_2\in k^\times$. Recall from [PS, 7.5.5] that

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta=\coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\mathrm{reg}},\gamma|_{\mathbb{F}_p^\times}=\zeta|_{\mathbb{F}_p^\times}} V_{\widehat{\mathbf{T}},0,z_2}\quad\coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\text{non-reg}},\gamma|_{\mathbb{F}_p^\times}=\zeta|_{\mathbb{F}_p^\times}} V_{\widehat{\mathbf{T}},0,z_2}/W_0.$$

There are standard coordinates x,y such that $V_{\widehat{\mathbf{T}},0,z_2}\simeq\mathbb{A}^1\cup_0\mathbb{A}^1$, two affine lines crossing at the origin. There is a Steinberg coordinate z_1 such that

$$V_{\widehat{\mathbf{T}},0,z_2}/W_0\simeq\mathbb{A}^1.$$

2.9. Lemma. *Let $\zeta,\eta\in\mathscr{Z}^\vee$. The action of η on $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ induces an isomorphism of k-schemes $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta\simeq(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta\eta^2}$.*

Proof. This is [PS, 7.5.2]. □

3 Mod p Langlands parameters with fixed determinant

3.1. We normalize local class field theory $\mathbb{Q}_p^\times\to\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}}$ by sending p to a geometric Frobenius. In this way, we identify the k-valued smooth characters of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and of $\mathbb{Q}_p^\times$. Finally, $\omega:\mathbb{Q}_p^\times\to k^\times$ denotes the extension of the character $\omega:\mathbb{F}_p^\times\to k^\times$ to $\mathbb{Q}_p^\times$ satisfying $\omega(p)=1$, and $\mathrm{unr}(x):\mathbb{Q}_p^\times\to k^\times$ denotes the character trivial on $\mathbb{F}_p^\times$ and sending p to x.

3.2. Let $\zeta:\mathbb{Q}_p^\times\to k^\times$ be a character. Recall from [DEG22] the projective curve X_ζ over $\mathbb{F}_p$ whose $\overline{\mathbb{F}}_p$-points parametrize (isomorphism classes of) two-dimensional semisimple continuous Galois representations over k with determinant $\omega\zeta$:

$$X_\zeta(k)\cong\big\{\text{semisimple continuous }\rho:\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\to\widehat{\mathbf{G}}(k)\ \text{ with }\det\rho=\omega\zeta\big\}/\sim.$$

The curve X_ζ is a chain of projective lines over k of length $\frac{p\pm1}{2}$, whose irreducible components intersect at ordinary double points. The sign ±1 is equal to $-\zeta(-1)$. We refer to ζ in the case $-\zeta(-1)=-1$ resp. $-\zeta(-1)=+1$ as an *even character* resp. *odd character*. From now on, we let X_ζ denote its base change to k. There is a finite set of closed points $X_\zeta^{\mathrm{irred}}\subset X_\zeta$ which correspond to the classes of irreducible representations. Its open complement $X_\zeta^{\mathrm{red}}=X_\zeta\setminus X_\zeta^{\mathrm{irred}}$ parametrizes the reducible representations (i.e. direct sums of characters). Let $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to k^\times$ be a character. Since $\det(\rho \otimes \eta) = (\det \rho)\eta^2$, twisting representations with η induces an isomorphism

$$(\cdot) \otimes \eta : X_\zeta \xrightarrow{\sim} X_{\zeta\eta^2}.$$

Hence one is reduced to consider only two 'basic' cases: the even case where $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = 1$ and the odd case where $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$. Indeed, if $\zeta|_{\mathbb{F}_p^\times} = \omega^r$ for some even r, then choosing η with $\eta(p)^2 = \zeta(p)^{-1}$ and $\eta|_{\mathbb{F}_p^\times} = \omega^{-\frac{r}{2}}$, one finds that $(\zeta\eta^2)(p) = 1$ and $(\zeta\eta^2)|_{\mathbb{F}_p^\times} = 1$; if $\zeta|_{\mathbb{F}_p^\times} = \omega^r$ for some odd r, then choosing η with $\eta(p)^2 = \zeta(p)^{-1}$ and $\eta|_{\mathbb{F}_p^\times} = \omega^{-\frac{r+1}{2}}$, one finds that $(\zeta\eta^2)(p) = 1$ and $(\zeta\eta^2)|_{\mathbb{F}_p^\times} = \omega^{-1}$.

3.3. We make explicit some structure elements of X_ζ in the even case $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = 1$. Every irreducible component of X_ζ is isomorphic to $\mathbb{P}^1$ and there are $\frac{p-1}{2}$ components. They are labelled by pairs of Serre weights of the following form:

$$\begin{array}{ccc} \mathrm{Sym}^0 & | & \mathrm{Sym}^{p-3} \otimes \det \\ \mathrm{Sym}^2 \otimes \det^{-1} & | & \mathrm{Sym}^{p-5} \otimes \det^2 \\ \mathrm{Sym}^4 \otimes \det^{-2} & | & \mathrm{Sym}^{p-7} \otimes \det^3 \\ \vdots & \vdots & \vdots \\ \mathrm{Sym}^{p-3} \otimes \det^{\frac{p+1}{2}} & | & \mathrm{Sym}^0 \otimes \det^{\frac{p-1}{2}} . \end{array}$$

The component with label " $\mathrm{Sym}^0 \mid \mathrm{Sym}^{p-3} \otimes \det$" intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^2 \otimes \det^{-1}, \mathrm{Sym}^{p-3} \otimes \det\}$. The component with label " $\mathrm{Sym}^2 \otimes \det^{-1} \mid \mathrm{Sym}^{p-5} \otimes \det^2$" intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^4 \otimes \det^{-2}, \mathrm{Sym}^{p-5} \otimes \det^2\}$. Continuing in this way, one finds $\frac{p-3}{2}$ points of X_ζ^{irred}, which correspond to the $\frac{p-3}{2}$ double points of the chain X_ζ. There are two more points in X_ζ^{irred}: they are smooth points, each one lies on one of the two 'exterior' components and corresponds there to the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^0, \mathrm{Sym}^{p-1}\}$ and $\{\mathrm{Sym}^0 \otimes \det^{\frac{p-1}{2}}, \mathrm{Sym}^{p-1} \otimes \det^{\frac{p-1}{2}}\}$ respectively. So X_ζ^{irred} has cardinality $\frac{p+1}{2}$. Suppose we are on one of the two exterior components $\mathbb{P}^1$. There is a canonical affine coordinate z_1 on the open complement of the double point, identifying this open complement with $\mathbb{A}^1$. We call the four points where $z_1 = \pm 1$ *the four exceptional points* of X_ζ.

3.4. We make explicit some structure elements of X_ζ in the odd case $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$. Every irreducible component of X_ζ is isomorphic to $\mathbb{P}^1$ and there are $\frac{p+1}{2}$ components. They are labelled by pairs of Serre weights of the following form:

$$\begin{array}{ccc} \mathrm{Sym}^{p-2} & | & \text{"}\,\mathrm{Sym}^{-1}\,\text{"} \\ \mathrm{Sym}^{p-4} \otimes \det & | & \mathrm{Sym}^1 \otimes \det^{-1} \\ \mathrm{Sym}^{p-6} \otimes \det^2 & | & \mathrm{Sym}^3 \otimes \det^{-2} \\ \vdots & \vdots & \vdots \\ \mathrm{Sym}^1 \otimes \det^{\frac{p-3}{2}} & | & \mathrm{Sym}^{p-4} \otimes \det^{\frac{p+1}{2}} \\ \text{"}\,\mathrm{Sym}^{-1} \otimes \det^{\frac{p-1}{2}}\,\text{"} & | & \mathrm{Sym}^{p-2} \otimes \det^{\frac{p-1}{2}} . \end{array}$$

The component with label " $\mathrm{Sym}^{p-2} \mid$ " Sym^{-1}"" intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^1 \otimes \det^{-1}, \mathrm{Sym}^{p-2}\}$. The component with label " $\mathrm{Sym}^{p-4} \otimes \det \mid \mathrm{Sym}^1 \otimes \det^{-1}$" intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^3 \otimes \det^{-2}, \mathrm{Sym}^{p-4} \otimes \det\}$. Continuing in this way, one finds $\frac{p-1}{2}$ points of X_ζ^{irred}, which correspond to the $\frac{p-1}{2}$ double points of the chain X_ζ. There are no more points in X_ζ^{irred} and X_ζ^{irred} has cardinality $\frac{p-1}{2}$. Suppose we are on one of the two exterior components $\mathbb{P}^1$. There is a canonical affine coordinate t on the open complement of the double point, identifying this open complement with $\mathbb{A}^1$. We call the four points where $t = \pm 2$ *the four exceptional points* of X_ζ. [3]

4 A morphism from Hecke to Galois

4.1. We let $I \subset G$ be the standard Iwahori subgroup of G consisting of integral matrices which are upper triangular mod p. Let $I^{(1)} \subset I$ be its p-Sylow subgroup, i.e. matrices which are upper unipotent mod p. We identify W_0 with the subgroup of G generated by the matrix $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We also put

$$u = \begin{pmatrix} 0 & p^{-1} \\ 1 & 0 \end{pmatrix}, \quad u^{-1} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \quad us = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad su = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}.$$

Moreover, $u^2 = \mathrm{diag}(p^{-1}, p^{-1})$.[4] Since

$$\begin{pmatrix} 0 & p^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} = \begin{pmatrix} d & p^{-1}c \\ pb & a \end{pmatrix}$$

the element $u \in G$ normalizes the group $I^{(1)}$.

4.2. Let $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$ be the pro-p Iwahori-Hecke algebra of G relativ to $I^{(1)}$ with coefficients in $k = \overline{\mathbb{F}}_p$. We denote by $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ the category of smooth G-representations over k. We have the functor of $I^{(1)}$-invariants$\pi \mapsto \pi^{I^{(1)}}$ from $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ to the category $\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$. It gives a bijection between the irreducible G-representations and the irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules. Thereby, supersingular representations correspond to supersingular Hecke modules [V04].

We recall the $I^{(1)}$-invariants for some classes of representations. If $\pi = \mathrm{Ind}_B^G(\chi)$ is a principal series representation with $\chi = \chi_1 \otimes \chi_2$, then $\pi^{I^{(1)}}$ is a standard module in the component $\gamma := \{\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}}\}$. In the regular case, one chooses the ordering $(\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}})$ on the set γ and standard coordinates x, y. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(0, \chi(su), \chi(u^2), \chi|_{\mathbb{T}}) = M(0, \chi_2(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1}), \chi|_{\mathbb{T}}).$$

In the non-regular case we obtain

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(\chi(su), \chi(u^2), \chi|_{\mathbb{T}}) = M(\chi_2(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1}), \chi|_{\mathbb{T}}).$$

These standard modules are irreducible if and only if $\chi \neq \chi^s$ [V04, 4.2/4.3].[5]

If $\pi = \pi(r, 0, \eta)$ is a standard supersingular representation with parameter $r = 0, ..., p-1$ and a character $\eta : \mathbb{Q}_p^\times \to k^\times$, then $\pi^{I^{(1)}}$ is a supersingular module in the component $\gamma = \{\chi, \chi^s\}$ represented by the character $\chi := (\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times})$, cf. [Br07, 5.1/5.3]. If π is the trivial representation $\mathbb{1}$ or the Steinberg representation St, then $\gamma = 1$ and $\pi^{I^{(1)}}$ is the character $(0,1)$ or $(-1,-1)$ respectively.

4.3. Let $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$. Since $u \in G$ normalizes the group $I^{(1)}$, one has $I^{(1)}uI^{(1)} = uI^{(1)}$. It follows that the convolution action of the Hecke operator U (resp. U^2) on $\pi^{I^{(1)}}$ is therefore induced by the action of u (resp. u^2 on π). Similarly, the group $I^{(1)}$ is normalized by the Iwahori subgroup I and $I/I^{(1)} \simeq \mathbb{T}$. It follows that the convolution action of the operators $T_t, t \in \mathbb{T}$ on $\pi^{I^{(1)}}$ is the factorization of the $\mathbf{T}(\mathbb{Z}_p)$-action on π.

[3] The Galois representations living on the two exterior components in the odd case are *unramified* (up to twist), i.e. of type $\rho = \begin{pmatrix} \mathrm{unr}(x) & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$ and t equals the 'trace of Frobenius' $x + x^{-1}$. Hence $t = \pm 2$ if and only if $x = \pm 1$.

[4] Note that our element u equals the element u^{-1} in [Be11],[Br07] and [V04].

[5] Our formulas differ from [V04, 4.2/4.3] by $\chi(\cdot) \leftrightarrow \chi(\cdot)^{-1}$, since we are working with left modules; also compare with the explicit calculation with right convolution given in [V04, Appendix A.5].

4.4. We identify $\mathbb{Q}_p^\times$ with the center $Z(G)$ in the usual way. A (smooth) character $\zeta : Z(G) = \mathbb{Q}_p^\times \to k^\times$ is determined by its value $\zeta(p^{-1}) \in k^\times$ and its restriction $\zeta|_{\mathbb{Z}_p^\times}$. Since the latter is trivial on the subgroup $1+p\mathbb{Z}_p$, we may view it as a character of $\mathbb{F}_p^\times$; we will write $\zeta|_{\mathbb{F}_p^\times}$ for this restriction in the following. Thus the group of characters of $Z(G)$ gets identified with the group of k-points of the group scheme $\mathscr{Z}^\vee = (\mathbb{F}_p^\times)^\vee \times \mathbb{G}_m$:

$$Z(G)^\vee \xrightarrow{\sim} \mathscr{Z}^\vee(k),\ \zeta \mapsto (\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})).$$

Recall from [PS, 7.2.2] the mod p pro-p-Iwahori Satake isomorphism

$$\operatorname{Spec} \mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p} : V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \xrightarrow{\sim} \operatorname{Spec} Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$$

It allows us to view $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules M as quasi-coheren sheaves $S(M)$ on $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. The rule $M \mapsto S(M)$ is the mod p parametrization functor $P : \operatorname{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \to \operatorname{QCoh}(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)$ from [PS, 7.3.6] in the special case $F = \mathbb{Q}_p$.

4.5. Lemma. *Suppose that $\pi \in \operatorname{Mod}^{\mathrm{sm}}(k[G])$ has a central character $\zeta : Z(G) \to k^\times$. Then the Satake parameter $S(\pi^{I^{(1)}})$ of $\pi^{I^{(1)}} \in \operatorname{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ has central character ζ, i.e. it is supported on the closed subscheme*

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1}))} \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0.$$

Proof. This is [PS, 7.5.4] in the case $F = \mathbb{Q}_p$. □

Next, recall the twisting action of the group $\mathscr{Z}^\vee(k)$ on the standard $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules and their simple constituents from 2.4.

4.6. Proposition. *Let $\pi \in \operatorname{Mod}^{\mathrm{ladm}}(k[G])$ be irreducible or a reducible principal series representation. Let $\eta : \mathbb{Q}_p^\times \to k^\times$ be a character. Then*

$$(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p^{-1}))$$

as $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules.

Proof. For future reference, we remark that the statement holds true, mutatis mutandis, with $\mathbb{Q}_p$ replaced by a finite extension. We therefore give a proof and references that work in this generality. An irreducible locally admissible representation, being a finitely generated $k[G]$-module, is admissible [Em10, 2.2.19]. A principal series representation (irreducible or not) is always admissible [Em10, 4.1.7]. The list of irreducible admissible smooth G-representations is given in [H11b, Thm. 1.1]. There are four families: principal series representations, supersingular representations, characters and twists of the Steinberg representation.

We first suppose that π is a principal series representation (irreducible or not), i.e. of the form $\operatorname{Ind}_B^G(\chi)$ with a character $\chi = \chi_1 \otimes \chi_2$. Then $\pi \otimes \eta \simeq \operatorname{Ind}_B^G(\chi_1\eta \otimes \chi_2\eta)$. We use the results from 4.2 (which hold for general F, cf. [PS, 7.5.8]. The modules $\pi^{I^{(1)}}$ and $(\pi \otimes \eta)^{I^{(1)}}$ are standard modules in the components $\gamma := \{\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}}\}$ and $\gamma(\eta|_{\mathbb{F}_p^\times})$ respectively. Suppose that γ is regular. We choose the ordering $(\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}})$ and standard coordinates x, y. Then

$$\operatorname{Ind}_B^G(\chi)^{I^{(1)}} = M(0, \chi_2(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1}), \chi|_{\mathbb{T}})$$

and

$$\operatorname{Ind}_B^G(\chi_1\eta \otimes \chi_2\eta)^{I^{(1)}} = M(0, \chi_2(p^{-1})\eta(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1})\eta(p^{-2}), (\chi|_{\mathbb{T}}).(\eta|_{\mathbb{F}_p^\times})).$$

This shows $(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p^{-1}))$ in the regular case. Suppose that γ is non-regular. Then

$$\operatorname{Ind}_B^G(\chi)^{I^{(1)}} = M(\chi_2(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1}), \chi|_{\mathbb{T}})$$

and

$$\mathrm{Ind}_B^G(\chi_1\eta\otimes\chi_2\eta)^{I^{(1)}} = M(\chi_2(p^{-1})\eta(p^{-1}), \chi_1(p^{-1})\chi_2(p^{-1})\eta(p^{-2}), (\chi|_{\mathbb{T}}).(\eta|_{\mathbb{F}_p^\times})).$$

This shows $(\pi\otimes\eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p^{-1}))$ in the non-regular case.

We now treat the case where π is a character or a twist of the Steinberg representation. Consider the exact sequence

$$1 \to \mathbb{1} \to \mathrm{Ind}_B^G(1) \to \mathrm{St} \to 1.$$

According to [V04, 4.4] the sequence of invariants

$$(S): 1 \to \mathbb{1}^{I^{(1)}} \to \mathrm{Ind}_B^G(1)^{I^{(1)}} \to \mathrm{St}^{I^{(1)}} \to 1$$

is still exact and $\mathbb{1}^{I^{(1)}}$ resp. $\mathrm{St}^{I^{(1)}}$ is the trivial character $(0,1)$ resp. sign character $(-1,-1)$ in the Iwahori component $\gamma = 1$. Tensoring the first exact sequence with η produces the exact sequence

$$1 \to \eta \to \mathrm{Ind}_B^G(1)\otimes\eta \to \mathrm{St}\otimes\eta \to 1.$$

Since the restriction $\eta|_{\mathbb{Z}_p^\times}$ is trivial on $1+p\mathbb{Z}_p$, one has $(\eta\circ\det)|_{I^{(1)}} = 1$ and so, as a sequence of k-vector spaces with k-linear maps, the sequence of invariants

$$1 \to \eta^{I^{(1)}} \to (\mathrm{Ind}_B^G(1)\otimes\eta)^{I^{(1)}} \to (\mathrm{St}\otimes\eta)^{I^{(1)}} \to 1$$

coincides with the sequence (S). It is therefore an exact sequence of $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$-modules, with outer terms being characters of $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$. From the discussion above, we deduce

$$(\mathrm{Ind}_B^G(1)\otimes\eta)^{I^{(1)}} = \mathrm{Ind}_B^G(1)^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p)^{-1}) = M(\eta(p^{-1}), \eta(p^{-2}), 1.(\eta|_{\mathbb{F}_p^\times})).$$

It follows then from [V04, 1.1] that $\eta^{I^{(1)}}$ must be the trivial character $(0, \eta(p^{-1}))$ in the component $1.(\eta|_{\mathbb{F}_p^\times})$ and $(\mathrm{St}\otimes\eta)^{I^{(1)}}$ must be the sign character $(-1, -\eta(p^{-1}))$ in the component $1.(\eta|_{\mathbb{F}_p^\times})$. This implies

$$\eta^{I^{(1)}} = \mathbb{1}^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p)^{-1}) \quad\text{and}\quad (\mathrm{St}\otimes\eta)^{I^{(1)}} = \mathrm{St}^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(\varpi)^{-1}).$$

This proves the claim in the cases $\pi = \mathbb{1}$ or $\pi = \mathrm{St}$. If, more generally, $\pi = \eta'$ is a general character of G, then

$$(\pi\otimes\eta)^{I^{(1)}} = (\eta'\eta)^{I^{(1)}} = \mathbb{1}^{I^{(1)}}.((\eta'\eta)|_{\mathbb{F}_q^\times}, (\eta'\eta)(p)^{-1}) = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p)^{-1}).$$

On the other hand, if $\pi = \mathrm{St}\otimes\eta'$ is a twist of Steinberg, then

$$(\pi\otimes\eta)^{I^{(1)}} = (\mathrm{St}\otimes(\eta'\eta))^{I^{(1)}} = \mathrm{St}^{I^{(1)}}.((\eta'\eta)|_{\mathbb{F}_p^\times}, (\eta'\eta)(p)^{-1}) = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p)^{-1}).$$

It remains to treat the case where π is a supersingular representation. In this case $\pi\otimes\eta$ is also supersingular and the two modules $\pi^{I^{(1)}}$ and $(\pi\otimes\eta)^{I^{(1)}}$ are supersingular $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$-modules [V04, 4.9]. Let γ be the component of the module $\pi^{I^{(1)}}$. By 4.3, the component of $(\pi\otimes\eta)^{I^{(1)}}$ equals $\gamma(\eta|_{\mathbb{F}_p^\times})$. Moreover, if U^2 acts on $\pi^{I^{(1)}}$ via the scalar $z_2 \in k^\times$, then U^2 acts on $(\pi\otimes\eta)^{I^{(1)}}$ via $z_2(\eta\circ\det)(u^2) = z_2\eta(p)^{-2}$, cf. 4.3. Since the supersingular modules are uniquely characterized by their component and their U^2-action, we obtain $(\pi\otimes\eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_p^\times}, \eta(p)^{-1})$, as claimed. □

4.7. Let $p \geq 5$. We let $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$ be the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ consisting of locally admissible representations having central character ζ. By work of Paškūnas [Pas13], the blocks b of the category $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$, defined as certain equivalence classes of simple objects, can be parametrized by the set of isomorphism classes $[\rho]$ of semisimple continuous Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$ having determinant $\det\rho = \omega\zeta$, i.e. by the k-points of X_ζ. There are three types of blocks. Blocks of type 1 are supersingular blocks. Each such block contains only one irreducible G-representation, which is supersingular. Blocks of type 2 contain only two irreducible representations. These two representations are two generic principal series representations of the form $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2\omega^{-1})$ and $\mathrm{Ind}_B^G(\chi_2 \otimes \chi_1\omega^{-1})$ (where $\chi_1\chi_2 \neq 1, \omega^{\pm 1}$). There are four blocks of type 3 which correspond to the four exceptional points. In the even case, each such block contains only three irreducible representations. These representations are of the form $\eta, \mathrm{St} \otimes \eta$ and $\mathrm{Ind}_B^G(\omega \otimes \omega^{-1}) \otimes \eta$. In the odd case, each block of type 3 contains only one irreducible representation. It is of the form $\mathrm{Ind}_B^G(\chi \otimes \chi\omega^{-1})$.

4.8. Let $p \geq 5$. Paškūnas' parametrization $[\rho] \mapsto b_{[\rho]}$ is compatible with Breuil's semisimple mod p local Langlands correspondence

$$\rho \mapsto \pi(\rho)$$

for the group G [Br07, Be11], in the sense that if ρ has determinant $\omega\zeta$, then the simple constituents of the G-representation $\pi(\rho)$ lie in the block $b_{[\rho]}$ of $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$. The correspondence and the parametrizations (for varying ζ) commute with twists: for a character $\eta : \mathbb{Q}_p^\times \to k^\times$, $\pi(\rho \otimes \eta) = \pi(\rho) \otimes \eta$ and $b_{[\rho]} \otimes \eta = b_{[\rho\otimes\eta]}$.

4.9. Theorem. *Suppose $p \geq 5$. Fix a character $\zeta : Z(G) = \mathbb{Q}_p^\times \to k^\times$, corresponding to a point $(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) \in \mathscr{Z}^\vee(k)$ under the identification $\mathcal{Z}(G)^\vee \cong \mathscr{Z}^\vee(k)$ from 4.4. There exists a finite morphism of k-schemes*

$$L_\zeta : (V_{\hat{\mathbf{T}},0}^{(1)}/W_0)_\zeta \longrightarrow X_\zeta$$

such that the quasi-coherent $\mathcal{O}_{X_\zeta}$-module

$$L_{\zeta*}S(\mathcal{M}_{\overline{\mathbb{F}}_p}^{(1)})|_{(V_{\hat{\mathbf{T}},0}^{(1)}/W_0)_\zeta}$$

equal to the push-forward along L_ζ of the restriction to $(V_{\hat{\mathbf{T}},0}^{(1)}/W_0)_\zeta \subset V_{\hat{\mathbf{T}},0}^{(1)}/W_0$ of the Satake parameter $S(\mathcal{M}_{\overline{\mathbb{F}}_p}^{(1)})$ interpolates the $I^{(1)}$-invariants of the semisimple mod p Langlands correspondence

$$\begin{array}{ccccc} X_\zeta(k) & \longrightarrow & \mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G]) & \longrightarrow & \mathrm{Mod}(\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}) \\ x & \longmapsto & \pi(\rho_x) & \longmapsto & \pi(\rho_x)^{I^{(1)}}, \end{array}$$

in the sense that for all $x \in X_\zeta(k)$,

$$\Big(\big(L_{\zeta*}S(\mathcal{M}_{\overline{\mathbb{F}}_p}^{(1)})|_{(V_{\hat{\mathbf{T}},0}^{(1)}/W_0)_\zeta}\big) \otimes_{\mathcal{O}_{X_\zeta}} k(x)\Big)^{\mathrm{ss}} = \Big(\mathcal{M}_{\overline{\mathbb{F}}_p}^{(1)} \otimes_{Z(\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)})} (\mathscr{S}_{\overline{\mathbb{F}}_p}^{(1)})^{-1}(\mathcal{O}_{L_\zeta^{-1}(x)})\Big)^{\mathrm{ss}} \cong \pi(\rho_x)^{I^{(1)}}$$

in $\mathrm{Mod}(\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)})$.

4.10. The connected components of $(V_{\hat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$ are either regular and then of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$, or non-regular and then of type $\mathbb{A}^1$. The morphism L_ζ appearing in the theorem depends on the choice of an order of the two affine lines in each regular component. It is surjective and quasi-finite. Moreover, writing L_ζ^γ for its restriction to the connected component $(V_{\hat{\mathbf{T}},0}^\gamma/W_0)_\zeta \subset (V_{\hat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$, one has:

- (e) *Even case.* All connected components are of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$, except for the two 'exterior' components which are of type $\mathbb{A}^1$. L_ζ^γ is an open immersion for any γ.
- (o) *Odd case.* All connected components are of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. L_ζ is an open immersion on all connected components, except for the two 'exterior' ones. On an 'exterior' component γ, the restriction of L_ζ^γ to one irreducible component $\mathbb{A}^1$ is an open immersion, and its restriction to the open complement $\mathbb{G}_m$ is a degree 2 finite flat covering of its image, with branched locus equal to the intersection of this image with the exceptional locus of X_ζ.

4.11. We set $L := \coprod_\zeta L_\zeta$. This is the morphism

$$L : V_{\hat{\mathbf{T}},0}^{(1)}/W_0 \longrightarrow X$$

referred to in the introduction.

4.12. Note that the semisimple mod p Langlands correspondence associates with any semisimple $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$ a semisimple smooth G-representation $\pi(\rho)$ of length $1, 2$ or 3, hence whose semisimple $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module of $I^{(1)}$-invariants $\pi(\rho)^{I^{(1)}}$ has length $1, 2$ or 3. On the other hand, the antispherical map

$$\mathrm{Sph} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k) \longrightarrow \{\text{left } \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}\text{-modules}\}$$

has an image consisting of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules are of length 1 or 2, cf. [PS, 7.4.9] and [PS, 7.4.15]. Theorem 4.9 combined with the properties 4.10 of the morphism L_ζ provide the following case-by-case elucidation of the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules $\pi(\rho)^{I^{(1)}}$.

4.13. Corollary. *Let $x \in X_\zeta(k)$, corresponding to $\rho_x : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$. Then the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module $\pi(\rho)^{I^{(1)}}$ admits the following explicit description.*

(i) If $x \in X^{irred}_\zeta(k)$, then the fibre $L^{-1}_\zeta(x) = \{v\}$ has cardinality 1 and

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v).$$

It is irreducible and supersingular.

(ii) If $x \in X^{red}_\zeta(k) \setminus \{$the four exceptional points$\}$, then $L^{-1}_\zeta(x) = \{v_1, v_2\}$ has cardinality 2 and

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2).$$

It has length 2.

(iiie) If $x \in X^{red}_\zeta(k)$ is exceptional in the even case, then $L^{-1}_\zeta(x) = \{v_1, v_2\}$ has cardinality 2 and

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v_1)^{\mathrm{ss}} \oplus \mathrm{Sph}(v_2).$$

It has length 3.

(iiio) If $x \in X^{red}_\zeta(k)$ is exceptional in the odd case, then $L^{-1}_\zeta(x) = \{v\}$ has cardinality 1 and

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v) \oplus \mathrm{Sph}(v).$$

It has length 2.

4.14. Now we proceed to the proof of 4.9, 4.10 and 4.13.

We start by defining the morphism L_ζ at the level of k-points. Let $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^\vee/W_0$.

1. Suppose that γ is regular. Then $\mathrm{Sph}(v) = \mathrm{Sph}^\gamma(v)$ is a simple two-dimensional $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-module, cf. [PS, 7.4.9]. Let $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$ be the simple module, unique up to isomorphism, such that $\pi^{I^{(1)}} \simeq \mathrm{Sph}^\gamma(v)$, cf. 4.2. Then $\pi \in \mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$ with

$$\zeta = (\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) = (\gamma|_{\mathbb{F}_p^\times}, z_2)$$

by 4.5. Let b be the block of $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$ which contains π. We define $L_\zeta(v)$ to be the point of $X_\zeta(k)$ which corresponds to b.

2. Suppose that γ is non-regular.

(a) If $v \in D(2)_\gamma(k)$, then $\mathrm{Sph}(v) = \mathrm{Sph}^\gamma(2)(v)$ is a simple two-dimensional $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-module, cf. [PS, 7.4.15]. As in the regular case, there is a simple module π, unique up to isomorphism, such that $\pi^{I^{(1)}} \simeq \mathrm{Sph}^\gamma(2)(v)$. It has central character $\zeta = (\gamma|_{\mathbb{F}_p^\times}, z_2)$ and there is a block b of $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$ which contains π. We define $L_\zeta(v)$ to be the point of $X_\zeta(k)$ which corresponds to b.

(b) If $v \in D(1)_\gamma(k)$, then $\mathrm{Sph}(v)^{\mathrm{ss}}$ is the direct sum of the two characters forming the antispherical pair $\mathrm{Sph}^\gamma(1)(v) = \{(0, z_1), (-1, -z_1)\}$ where $z_2 = z_1^2$, cf. [PS, 7.4.15]. As in the regular case, there are two simple modules π_1 and π_2, unique up to isomorphism, such that $\pi_1^{I^{(1)}} \simeq (0, z_1)$ and $\pi_2^{I^{(1)}} \simeq (-1, -z_1)$ and π_1, π_2 have central character $\zeta = (\gamma|_{\mathbb{F}_p^\times}, z_2)$. Moreover, we claim that there is a unique block b of $\mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$ which contains both π_1 and π_2. Indeed, if $\gamma = \{1 \otimes 1\}$ and $z_1 = 1$, then $\pi_1 = \mathbb{1}$ and $\pi_2 = \mathrm{St}$, cf. 4.2. Then by 4.6 it follows more generally that if $\gamma = \{\omega^r \otimes \omega^r\}$, then $\pi_1 = \eta$ and $\pi_2 = \mathrm{St} \otimes \eta$ with $\eta = (\eta|_{\mathbb{F}_p^\times}, \eta(p^{-1})) := (\omega^r, z_1)$. Consequently π_1, π_2 are contained in a unique block b of type 3, cf. 4.7. We define $L_\zeta(v)$ to be the point of $X_\zeta(k)$ which corresponds to b.

Thus we have a well-defined map of sets $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta(k) \longrightarrow X_\zeta(k)$.

We show property (i) of 4.13. Let $x \in X_\zeta^{\mathrm{irred}}(k)$ and suppose $L_\zeta(v) = x$. Then b_x is a supersingular block, contains a unique irreducible representation π, which is supersingular, and $\pi = \pi(\rho_x)$, cf. 4.7-4.8. By definition of L_ζ, one has $\mathrm{Sph}(v) \simeq \pi^{I^{(1)}}$. Since the spherical map Sph is $1:1$ over supersingular modules, cf. [PS, 7.4.9] and [PS, 7.4.15], such a preimage v of x exists and is uniquely determined by x. Summarizing, we have $L_\zeta^{-1}(x) = \{v\}$ and $\mathrm{Sph}(v) \simeq \pi(\rho_x)^{I^{(1)}}$. This is property (i).

As a next step, we take a second character $\eta : \mathbb{Q}_p^\times \to k^\times$ and show that the diagram

$$\begin{array}{ccc} (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta(k) & \xrightarrow{L_\zeta} & X_\zeta(k) \\ {\scriptstyle \cdot\eta}\downarrow \simeq & & \simeq \downarrow {\scriptstyle (\cdot)\otimes\eta} \\ (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_{\zeta\eta^2}(k) & \xrightarrow{L_{\zeta\eta^2}} & X_{\zeta\eta^2}(k) \end{array}$$

commutes. Here, the vertical arrows are the bijections coming from 2.9 and 3.2. To verify the commutativity, let $v \in (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^\vee/W_0$. Suppose that γ is regular or that γ is non-regular with $v \in D(2)_\gamma(k)$. Let π be the simple G-module with $\pi^{I^{(1)}} \simeq \mathrm{Sph}(v)$ and let $b_{[\rho]}$ be the block corresponding to the point $L_\zeta(v)$. By the equivariance property 2.5, one has $\mathrm{Sph}(v.\eta) \simeq \mathrm{Sph}(v).\eta$. Taking $I^{(1)}$-invariants is compatible with twist, cf. 4.6, and so $L_{\zeta\eta^2}(v.\eta)$ corresponds to the block which contains the representation $\pi \otimes \eta$, i.e. to $b_{[\rho]} \otimes \eta = b_{[\rho\otimes\eta]}$, cf. 4.8, and so $L_{\zeta\eta^2}(v.\eta) = [\rho \otimes \eta] = L_\zeta(v).\eta$.

If $v \in D(1)_\gamma(k)$, let π_1 and π_2 be the simple modules such that $(\pi_1 \oplus \pi_2)^{I^{(1)}} \simeq \mathrm{Sph}^\gamma(v)^{\mathrm{ss}}$. As before, we conclude from $\mathrm{Sph}(v.\eta)^{\mathrm{ss}} \simeq \mathrm{Sph}(v)^{\mathrm{ss}} \otimes \eta$ that $L_{\zeta\eta^2}(v.\eta)$ corresponds to the block which contains $\pi_1 \otimes \eta$ and $\pi_2 \otimes \eta$ and that $L_{\zeta\eta^2}(v.\eta) = L_\zeta(v).\eta$. The commutativity of the diagram is proved.

Thus, we are reduced to prove that the map L_ζ comes from a morphism of k-schemes satisfying 4.9 and the remaining parts of 4.13 in the two basic cases of a character ζ such that $\zeta(p^{-1}) = 1$ and $\zeta|_{\mathbb{F}_p^\times} \in \{1, \omega^{-1}\}$. This is established in the next two subsections.

5 The morphism L_ζ in the basic even case

Let $\zeta : \mathbb{Q}_p^\times \to k^\times$ be the trivial character. Here we show that the map of sets $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta(k) \to X_\zeta(k)$ that we have defined in 4.14 satisfies properties (ii) and (iiie) of 4.13, and we define a morphism of k-schemes $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides with the previous map of sets at the level of k-points. By construction, it will have the properties 4.10. This will complete the proof of 4.13, 4.10 and 4.9 in the case of an even character.

5.1. We verify the properties (ii) and (iiie). We work over an irreducible component $\mathbb{P}^1$ with label " $\mathrm{Sym}^r \otimes \det^a \mid \mathrm{Sym}^{p-3-r} \otimes \det^{r+1+a}$ " where $0 \le r \le p-3$ and $0 \le a \le p-2$, cf. 3.3. On this component, we choose an affine coordinate x around the double point having $\mathrm{Sym}^r \otimes \det^a$

as one of its Serre weights. Away from this point, we have $x \neq 0$ and the corresponding Galois representation has the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{r+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^a$. By [Be11, 1.3] or [Br07, 4.11], we have

$$\pi(\rho_x) = \pi(r, x, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], x^{-1}, \omega^{r+1}\eta)^{\mathrm{ss}} =: \pi_1 \oplus \pi_2$$

where $[p-3-r]$ denotes the unique integer in $\{0, ..., p-2\}$ which is congruent to $p-3-r$ modulo $p-1$. Now suppose that $L_\zeta(v) = x$. We distinguish two cases.

1. *The generic case* $0 < r < p-3$. In this case, the point x lies on one of the 'interior' components of the chain X_ζ, which has no exceptional points. The length of $\pi(\rho_x)$ is 2. Indeed, $\pi_1 = \pi(r, x, \eta)$ and $\pi_2 = \pi(p-3-r, x^{-1}, \omega^{r+1}\eta)$ are two irreducible principal series representations [Br07, Thm. 4.4]. The block b_x is of type 2 and contains only these two irreducible representations, cf. 4.7-4.8. We may write

$$\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$$

with $\chi = \mathrm{unr}(x) \otimes \omega^r \,\mathrm{unr}(x^{-1})$, according to [Br07, Rem. 4.4(ii)]. By our assumptions on r, the character $\chi|_{\mathbb{T}} = 1 \otimes \omega^r$ is regular (i.e. different from its s-conjugate). We conclude from 4.6 and 4.2 that $\pi_1^{I^{(1)}}$ is a simple 2-dimensional standard module in the regular component represented by the character $(1 \otimes \omega^r).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^r \in \mathbb{T}^\vee$. Similarly, we may write

$$\pi_2 = \mathrm{Ind}_B^G(\chi) \otimes \omega^{r+1}\eta$$

where now $\chi = \mathrm{unr}(x^{-1}) \otimes \omega^{p-3-r}\,\mathrm{unr}(x)$. By our assumptions on r, the character $\chi|_{\mathbb{T}} = 1 \otimes \omega^{p-3-r}$ is regular and we conclude, as above, that the $I^{(1)}$-invariants $\pi_2^{I^{(1)}}$ form a simple 2-dimensional standard module in the regular component represented by the character $(\eta|_{\mathbb{F}_p^\times})\omega^{r+1} \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{r+1}\omega^{p-3-r} \in \mathbb{T}^\vee$. Note that the component of $\pi_1^{I^{(1)}}$ is different from the component of $\pi_2^{I^{(1)}}$, by our assumptions on r.

We conclude from $L_\zeta(v) = x$ that either $\mathrm{Sph}(v) = \pi_1^{I^{(1)}}$ or $\mathrm{Sph}(v) = \pi_2^{I^{(1)}}$. Since for γ regular, the map Sph^γ is a bijection onto all simple $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-modules, cf. [PS, 7.4.9], one finds that $L_\zeta^{-1}(x) = \{v_1, v_2\}$ has cardinality 2 and

$$\mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_x)^{I^{(1)}}.$$

This settles property (ii) of 4.13 in the generic case.

2. *The boundary cases* $r \in \{0, p-3\}$. In this case, the point x lies on one of the two 'exterior' components of X_ζ. On such a component, we will denote the variable x rather by z_1, which is the notation[6] which we used already in 3.3.

(a) Suppose that $z_1 \neq \pm 1$. The length of $\pi(\rho_{z_1})$ is 2. Indeed, as in the generic case, $\pi_1 = \pi(r, z_1, \eta)$ and $\pi_2 = \pi(p-3-r, z_1^{-1}, \omega^{r+1}\eta)$ are two irreducible principal series representations. The block b_{z_1} is of type 2 and contains only these two irreducible representations. It follows, as above, that their invariants $\pi_1^{I^{(1)}}$ and $\pi_2^{I^{(1)}}$ are simple 2-dimensional standard modules, in the components represented by $(\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^r \in \mathbb{T}^\vee$ and $(\eta|_{\mathbb{F}_p^\times})\omega^{r+1} \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{r+1}\omega^{p-3-r} \in \mathbb{T}^\vee$ respectively. Since $r \in \{0, p-3\}$, one of these components is regular, the other non-regular. In particular, the two components are different. We conclude from $L_\zeta(v) = z_1$ that either $\mathrm{Sph}(v) = \pi_1^{I^{(1)}}$ or $\mathrm{Sph}(v) = \pi_2^{I^{(1)}}$. Since for non-regular γ, the map $\mathrm{Sph}^\gamma(2)$ is a bijection from $D(2)_\gamma(k)$ onto all simple standard $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-modules, cf. [PS, 7.4.15], we may conclude as in the generic case: $L_\zeta^{-1}(z_1) = \{v_1, v_2\}$ has cardinality 2 and

$$\mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_{z_1})^{I^{(1)}}.$$

[6] The reason for this notation will become clear in the discussion of the non-regular case in 5.2.

This settles property 4.13 (ii) in the remaining case $z_1 \neq \pm 1$.

(b) Suppose now that $z_1 = \pm 1$, i.e. we are at one of the four exceptional points. We will verify property (iiie). The length of $\pi(\rho_{z_1})$ is 3. Indeed, the representation $\pi(0, \pm 1, \eta)$ is a twist of the representation $\pi(0,1,1)$ (note that $\pi(r, z_1, \eta) \simeq \pi(r, -z_1, \mathrm{unr}(-1)\eta)$ according to [Br07, Rem. 4.4(v)]), which itself is an extension of $\mathbb{1}$ by St, cf. [Br07, Thm. 4.4(iii)]. As in the case (a), the representation $\pi_2 = \pi(p-3, \pm 1, \omega\eta)$ is an irreducible principal series representation. The block b_{z_1} is of type 3 and contains only these three irreducible representations. The invariants $\pi_1^{I^{(1)}}$ form a direct sum of two spherical characters in a non-regular component γ, whereas the invariants $\pi_2^{I^{(1)}}$ form a simple standard module in a regular component, as before. Since for non-regular γ, the map $\mathrm{Sph}^{\gamma}(1)$ is a bijection from $D(1)_{\gamma}(k)$ onto all spherical pairs of characters of $\mathcal{H}^{\gamma}_{\mathbb{F}_p}$, cf. [PS, 7.4.15], we may conclude that $L_{\zeta}^{-1}(z_1) = \{v_1, v_2\}$ has cardinality 2 with $v_1 \in D(1)_{\gamma}(k)$ and $\mathrm{Sph}^{\gamma}(1)(v_1)^{\mathrm{ss}} = \pi_1^{I^{(1)}}$. In particular,

$$\mathrm{Sph}(v_1)^{\mathrm{ss}} \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_x)^{I^{(1)}}.$$

This settles property 4.13 (iiie).

5.2. We define a morphism of k-schemes $L_{\zeta} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta} \longrightarrow X_{\zeta}$ which coincides on k-points with the map of sets $L_{\zeta} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta}(k) \longrightarrow X_{\zeta}(k)$. We work over a connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta}$, indexed by some $\gamma \in \mathbb{T}^{\vee}/W_0$. Let v be a k-point of this component.

Since $\zeta|_{\mathbb{F}_p^{\times}} = 1$, the connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta}$ are indexed by the fibre $(\cdot)|^{-1}_{\mathbb{F}_p^{\times}}(1)$. This fibre consists of the $\frac{p-3}{2}$ regular components, represented by the characters of $\mathbb{T}$

$$\chi_k = \omega^k \otimes \omega^{-k}$$

for $k = 1, ..., \frac{p-3}{2}$, and of the two non-regular components, given by χ_0 and $\chi_{\frac{p-1}{2}}$, cf. 2.2. We distinguish two cases. Note that $z_2 = \zeta(p^{-1}) = 1$.

1. *The regular case* $0 < k < \frac{p-1}{2}$. We fix the order $\gamma = (\chi_k, \chi_k^s)$ on the set γ and choose the standard coordinates x, y. According to [PS, 7.4.8], our regular connected component identifies with two affine lines intersecting at the origin:

$$V_{\widehat{\mathbf{T}},0,1} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1.$$

Suppose that $v = (0,0)$ is the origin, so that $\mathrm{Sph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation. It corresponds to the irreducible Galois representation $\rho(r, \eta) = \mathrm{ind}(\omega_2^{r+1}) \otimes \eta$, in the notation of [Be11, 1.3], whence $L_{\zeta}(v) = [\rho(r, \eta)]$. According to 4.2, the component of the Hecke module $\pi(r, 0, \eta)^{I^{(1)}}$ is given by $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^{\times}})$. Setting $\eta|_{\mathbb{F}_p^{\times}} = \omega^a$, this implies $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^{\times}}) = \omega^{r+a} \otimes \omega^a = \chi_k$ and hence $a = -k$ and $r = 2k$. Therefore the Serre weights of the irreducible representation $\rho(r, \eta)$ are $\{\mathrm{Sym}^{2k} \otimes \det^{-k}, \mathrm{Sym}^{p-1-2k} \otimes \det^{k}\}$, cf. [Br07, 1.9].

Comparing these pairs of Serre weights with the list 3.3 shows that the $\frac{p-3}{2}$ points

$$\{\text{origin } (0,0) \text{ on the component } (\chi_k, \chi_k^s)\}$$

for $0 < k < \frac{p-1}{2}$ are mapped successively to the $\frac{p-3}{2}$ double points of the chain X_{ζ}.

Fix $0 < k < \frac{p-1}{2}$ and consider the double point

$$Q = L_{\zeta}(\text{origin } (0,0) \text{ on the component } \gamma = (\chi_k, \chi_k^s)).$$

As we have just seen, Q lies on the irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^{2k} \otimes \det^{-k}$ (i.e. on the component " $\mathrm{Sym}^{2k} \otimes \det^{-k} \mid \mathrm{Sym}^{p-3-2k} \otimes \det^{k+1}$ "). We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we will also call x (there will be no risk of confusion with the standard coordinate above!). Away from Q, the affine coordinate $x \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{2k+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{-k}$. As we have seen above, $\pi(\rho_x) = \pi(2k, x, \eta) \oplus \pi(p-3-2k, x^{-1}, \omega^{r+1}\eta) =: \pi_1 \oplus \pi_2$. Moreover, $\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(x) \otimes \omega^{2k}\,\mathrm{unr}(x^{-1})$. Since

$$(1 \otimes \omega^{2k}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{-k} \otimes \omega^k = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the regular case of 4.2 that

$$\pi_1^{I^{(1)}} = M(0, x, 1, \chi_k^s)$$

is a simple 2-dimensional standard module. Note that $M(0, x, 1, \chi_k^s) = M(x, 0, 1, \chi_k)$ according to [V04, Prop. 3.2].

Now suppose that $v = (x, 0), x \neq 0$, denotes a point on the x-line of $\mathbb{A}_k^1 \cup_0 \mathbb{A}_k^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(x, 0, 1, \chi_k)$. By our discussion, the point $L_\zeta((x,0))$ corresponds to the block which contains π_1. Since π_1 lies in the block parametrized by $[\rho_x]$, cf. 4.8, it follows that

$$L_\zeta((x,0)) = [\rho_x] = x \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$

Since $(0,0)$ maps to Q, i.e. to the point at $x = 0$, the map L_ζ identifies the whole affine x-line $\mathbb{A}^1 = \{(x,0) : x \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine x-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

On the other hand, the double point Q lies also on the irreducible component $\mathbb{P}^1$ whose labelling includes the other weight of Q, i.e. the weight $\mathrm{Sym}^{p-1-2k} \otimes \det^k$. We fix an affine coordinate y on this $\mathbb{P}^1$ around Q. Away from Q, the coordinate $y \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(y)\omega^{p-2k} & 0 \\ 0 & \mathrm{unr}(y^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^k$. As in the first case, $\pi(\rho_y)$ contains $\pi_1 := \pi(p-1-2k, y, \eta) = \mathrm{Ind}_B^G(\chi) \otimes \eta$ as a direct summand, where now $\chi = \mathrm{unr}(y) \otimes \omega^{p-1-2k}\,\mathrm{unr}(y^{-1})$. Since

$$(1 \otimes \omega^{p-1-2k}).(\eta|_{\mathbb{F}_p^\times}) = \omega^k \otimes \omega^{-k} = \chi_k \in \mathbb{T}^\vee,$$

we deduce, as above, that $\pi_1^{I^{(1)}} = M(0, y, 1, \chi_k)$ is a simple 2-dimensional standard module.

Now suppose that $v = (0, y), y \neq 0$, denotes a point on the y-line of $\mathbb{A}_k^1 \cup_0 \mathbb{A}_k^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(0, y, 1, \chi_k)$. By our discussion, the point $L_\zeta((0,y))$ corresponds to the block which contains π_1. Since π_1 lies in the block parametrized by $[\rho_y]$, cf. 4.8, it follows that

$$L_\zeta((0,y)) = [\rho_y] = y \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$

Since $(0,0)$ maps to Q, i.e. to the point at $y = 0$, the map L_ζ identifies the whole affine y-line $\mathbb{A}^1 = \{(0,y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine y-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each regular connected component of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$ in the scheme X_ζ, which coincides on k-points with the restriction of the map of sets L_ζ.

2. *The non-regular case* $k \in \{0, \frac{p-1}{2}\}$. We choose the Steinberg coordinate z_1. According to [PS, 7.4.10], our non-regular connected component identifies with an affine line :

$$V_{\widehat{\mathbf{T}},0,z_2}/W_0 \simeq \mathbb{A}^1.$$

Suppose that $v = (0)$ is the origin, so that $\mathrm{Sph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation so that $L_\zeta(v) = [\rho(r, \eta)]$. Exactly as in the regular case, we may conclude that the Serre weights of the irreducible representation $\rho(r, \eta)$ are $\{\mathrm{Sym}^{2k} \otimes \det^{-k}, \mathrm{Sym}^{p-1-2k} \otimes \det^k\}$. For the two values of $k = 0$ and $k = \frac{p-1}{2}$ we find

$\{\mathrm{Sym}^0, \mathrm{Sym}^{p-1}\}$ and $\{\mathrm{Sym}^0 \otimes \det^{\frac{p-1}{2}}, \mathrm{Sym}^{p-1} \otimes \det^{\frac{p-1}{2}}\}$ respectively. Comparing with the list 3.3 shows that the 2 points

$$\{\text{origin } (0) \text{ on the component } (\chi_k = \chi_k^s)\}$$

for $k \in \{0, \frac{p-1}{2}\}$ are mapped to the 2 smooth points in X_ζ^{irred}, which lie on the two 'exterior' components of X_ζ, cf. 3.3.

Fix $k \in \{0, \frac{p-1}{2}\}$ and consider the point

$$Q = L_\zeta(\text{origin } (0) \text{ on the component } \gamma = (\chi_k = \chi_k^s)).$$

As we have just seen, Q lies on an 'exterior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^0 \otimes \det^k$. We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we call z_1 (there will be no risk of confusion with the Steinberg coordinate above!). Away from Q, the affine coordinate $z_1 \neq 0$ parametrizes Galois representations of the form

$$\rho_{z_1} = \begin{pmatrix} \mathrm{unr}(z_1)\omega & 0 \\ 0 & \mathrm{unr}(z_1^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^k$. As in the regular case, $\pi(\rho_{z_1}) = \pi(0, z_1, \eta)^{\mathrm{ss}} \oplus \pi(p-3, z_1^{-1}, \omega\eta)^{\mathrm{ss}}$. Moreover, $\pi(0, z_1, \eta) = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(z_1) \otimes \mathrm{unr}(z_1^{-1})$ [7]. Since

$$(1 \otimes 1).(\eta|_{\mathbb{F}_p^\times}) = \omega^k \otimes \omega^k = \chi_k = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the non-regular case of 4.2 that $\pi(0, z_1, \eta)^{I^{(1)}} = M(z_1, 1, \chi_k)$ is a 2-dimensional standard module. Moreover, the standard module is simple if and only if $\chi \neq \chi^s$, i.e. if and only if $z_1 \neq \pm 1$.

Now let $v = z_1 \neq 0$ denote a nonzero point on our connected component $\mathbb{A}^1 = V_{\widehat{\mathbf{T}},0,1}/W_0$. Suppose that $z_1 \neq \pm 1$, i.e. $v \in D(2)_\gamma$. In particular, $\mathrm{Sph}(v) = M(z_1, 1, \gamma)$ is irreducible. By our discussion, the point $L_\zeta(z_1)$ corresponds to the block (a block of type 2) which contains $\pi(0, z_1, \eta)$. Suppose that $z_1 = \pm 1$, i.e. $v \in D(1)_\gamma$. In particular, $\mathrm{Sph}^{\mathrm{ss}}(v) = M(z_1, 1, \chi_k)^{\mathrm{ss}}$ and again, $L_\zeta(z_1)$ corresponds to the block (now a block of type 3) which contains the simple constituents of $\pi(0, z_1, \eta)^{\mathrm{ss}}$. In both cases, we conclude

$$L_\zeta(z_1) = [\rho_{z_1}] = z_1 \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$

Since (0) maps to Q, i.e. to the point at $z_1 = 0$, the map L_ζ identifies the whole z_1-line $\mathbb{A}^1 = V_{\widehat{\mathbf{T}},0,1}/W_0$ with the z_1-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each non-regular connected component of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$ in the scheme X_ζ, which coincides on k-points with the restriction of the map of sets L_ζ.

6 The morphism L_ζ in the basic odd case

Let $\zeta := \omega^{-1} : \mathbb{Q}_p^\times \to k^\times$. Here we show that the map of sets $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta(k) \to X_\zeta(k)$ that we have defined in 4.14 satisfies properties (ii) and (iiio) of 4.13, and we define a morphism of k-schemes $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides with the previous map of sets at the level of k-points. By construction, it will have the properties 4.10. This will complete the proof of 4.13, 4.10 and 4.9 in the case of an odd character.

6.1. We verify properties (ii) and (iiio). We work over an irreducible component $\mathbb{P}^1$ with label "$\mathrm{Sym}^r \otimes \det^a \mid \mathrm{Sym}^{p-3-r} \otimes \det^{r+1+a}$" where $1 \leq r \leq p-2$ and $0 \leq a \leq p-2$, cf. 3.4. We distinguish two cases.

1. *The generic case $r \neq p-2$.* In this case, the irreducible component of X_ζ we consider is an 'interior' component and has no exceptional points. On this component, we choose an affine

[7] The representations $\pi(0, z_1, \eta)$ constitute the *unramified* principal series of G.

coordinate x around the double point having $\mathrm{Sym}^r \otimes \det^a$ as one of its Serre weights. Away from this point, we have $x \neq 0$ and the corresponding Galois representation has the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{r+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^a$. As before, we have

$$\pi(\rho_x) = \pi(r, x, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], x^{-1}, \omega^{r+1}\eta)^{\mathrm{ss}}.$$

The length of $\pi(\rho_x)$ is 2. Indeed, by our assumptions on r, the principal series representations $\pi(r, x, \eta)$ and $\pi(p-3-r, x^{-1}, \omega^{r+1}\eta)$ are irreducible and the block b_x contains only these two irreducible representations. We may follow the argument of the generic case of 5.1 word for word and deduce property 4.13 (ii).

2. *The two boundary cases* $r = p-2$. In this case, the irreducible component is one of the two 'exterior' components with labels " Sym^{p-2} | " Sym^{-1} "" or "" $\mathrm{Sym}^{-1} \det^{\frac{p-1}{2}}$ " | $\mathrm{Sym}^{p-2} \det^{\frac{p-1}{2}}$ ". Points of the open locus X_ζ^{red} lying on such a component correspond to twists of unramified Galois representations of the form

$$\rho_{x+x^{-1}} = \begin{pmatrix} \mathrm{unr}(x) & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = 1$ or $\eta = \omega^{\frac{p-1}{2}}$. Let us concentrate on one of the two components, i.e. let us fix η.

Mapping an unramified Galois representation $\rho_{x+x^{-1}}$ to $t := x + x^{-1} \in k$ identifies this open locus with the t-line $\mathbb{A}^1 \subset \mathbb{P}^1$. We have

$$\pi(\rho_t) = \pi(p-2, x, \eta)^{\mathrm{ss}} \oplus \pi(p-2, x^{-1}, \eta)^{\mathrm{ss}} =: \pi_1 \oplus \pi_2$$

since $[p-3-(p-2)] = p-2$ (indeed, $p-3-(p-2) = -1 \equiv p-2 \mod (p-1)$). The length of $\pi(\rho_t)$ is 2. Indeed, $\pi_1 = \pi(p-2, x, \eta)$ and $\pi_2 = \pi(p-2, x^{-1}, \eta)$ are two irreducible principal series representations and the block b_t contains only these two irreducible representations. They are isomorphic if and only if $x = \pm 1$, i.e. if and only if $t = \pm 2$ is an exceptional point. In this case, b_t contains only one irreducible representation and is of type 3, otherwise it is of type 2.

We may write

$$\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$$

with $\chi = \mathrm{unr}(x) \otimes \omega^{p-2}\,\mathrm{unr}(x^{-1})$. Similarly for π_2. The character $\chi|_{\mathbb{F}_p^\times} = 1 \otimes \omega^{p-2}$ is regular (i.e. different from its s-conjugate) and we are in the regular case of 4.2. We conclude that $\pi_1^{I^{(1)}} = M(0, x, 1, (1 \otimes \omega^{p-2}).\eta)$ and $\pi_2^{I^{(1)}} = M(0, x^{-1}, 1, (1 \otimes \omega^{p-2}).\eta)$ are both simple 2-dimensional standard modules in the regular component γ represented by the character $(1 \otimes \omega^{p-2}).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} \in \mathbb{T}^\vee$. They are isomorphic if and only if $t = \pm 2$. We choose an order $\gamma = ((\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2}, (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} \otimes (\eta|_{\mathbb{F}_p^\times}))$ on the set γ. Then from $L_\zeta(v) = t$ we get that either $\mathrm{Sph}^\gamma(v) = \pi_1^{I^{(1)}}$ or $\mathrm{Sph}^\gamma(v) = \pi_2^{I^{(1)}}$. Since for regular γ, the map Sph^γ is a bijection onto all simple $\mathcal{H}^\gamma_{\mathbb{F}_p}$-modules, cf. [PS, 7.4.9], one finds that $L_\zeta^{-1}(t) = \{v_1, v_2\}$ has cardinality 2 if $t \neq \pm 2$ and then

$$\mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_t)^{I^{(1)}}.$$

This settles property 4.13 (ii). In turn, if $t = \pm 2$ is an exceptional point, then $L_\zeta^{-1}(t) = \{v\}$ has cardinality 1 and

$$\mathrm{Sph}(v) \oplus \mathrm{Sph}(v) \simeq \pi(\rho_t)^{I^{(1)}}.$$

This settles property 4.13 (iiio).

6.2. We define a morphism of k-schemes $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides on k-points with the map of sets $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) \longrightarrow X_\zeta(k)$. We work over a connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, indexed by some $\gamma \in \mathbb{T}^\vee/W_0$. Let v be a k-point of this component.

Since $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$, the connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ are indexed by the fibre $(\cdot)|^{-1}_{\mathbb{F}_p^\times}(\omega^{-1})$. This fibre consists of the $\frac{p-1}{2}$ regular components, represented by the characters

$$\chi_k = \omega^{k-1} \otimes \omega^{-k}$$

for $k = 1, ..., \frac{p-1}{2}$, cf. 2.2. Recall that $z_2 = \zeta(p) = 1$.

Fix an order $\gamma = (\chi_k, \chi_k^s)$ on the set γ and choose standard coordinates x, y. According to [PS, 7.4.8], our regular connected component identifies with two affine lines intersecting at the origin:

$$V_{\widehat{\mathbf{T}},0,1} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1.$$

Suppose that $v = (0,0)$ is the origin, so that $\mathrm{Sph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation. It corresponds to the irreducible Galois representation $\rho(r, \eta)$, in the notation of [Be11, 1.3], whence $L_\zeta(v) = [\rho(r, \eta)]$. According to 4.2, the component of $\pi(r, 0, \eta)^{I^{(1)}}$ is given by $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times})$. Setting $\eta|_{\mathbb{F}_p^\times} = \omega^a$, this implies $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times}) = \omega^{r+a} \otimes \omega^a = \chi_k$ and hence $a = -k$ and $r = 2k - 1$. The Serre weights of the irreducible representation $\rho(r, \eta)$ are therefore $\{\mathrm{Sym}^{2k-1} \otimes \det^{-k}, \mathrm{Sym}^{p-2k} \otimes \det^{k-1}\}$, cf. [Br07, 1.9].

Comparing these pairs of Serre weights with the list 3.4 shows that the $\frac{p-1}{2}$ points

$$\{\text{origin } (0,0) \text{ on the component } (\chi_k, \chi_k^s)\}$$

for $k = 1, ..., \frac{p-1}{2}$ are mapped successively to the $\frac{p-1}{2}$ double points of the chain X_ζ. We distinguish two cases.

1. The generic case $1 < k < \frac{p-1}{2}$. In this case, the argument proceeds as in the regular case of 5.2. Consider the double point

$$Q = L_\zeta(\text{origin } (0,0) \text{ on the component } \gamma = (\chi_k, \chi_k^s)).$$

As we have just seen, Q lies on an 'interior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^{2k-1} \otimes \det^{-k}$. We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we will also call x. Away from Q, the affine coordinate $x \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{2k} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{-k}$. As we have seen above, $\pi(\rho_x) = \pi(2k-1, x, \eta) \oplus \pi(p-3-2k+1, x^{-1}, \omega^{2k}\eta) =: \pi_1 \oplus \pi_2$. Moreover, $\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(x) \otimes \omega^{2k-1}\,\mathrm{unr}(x^{-1})$. Since

$$(1 \otimes \omega^{2k-1}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{-k} \otimes \omega^{k-1} = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the regular case of 4.2 that $\pi_1^{I^{(1)}} = M(0, x, 1, \chi_k^s)$ is a simple 2-dimensional standard module. Note that $M(0, x, 1, \chi_k^s) = M(x, 0, 1, \chi_k)$ according to [V04, Prop. 3.2].

Now suppose that $v = (x, 0)$, $x \neq 0$, denotes a nonzero point on the x-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(x, 0, 1, \chi_k)$. Our discussion shows that the point $L_\zeta((x, 0))$ corresponds to the block which contains π_1. Since π_1 lies in the block parametrized by $[\rho_x]$, cf. 4.8, it follows that

$$L_\zeta((x, 0)) = [\rho_x] = x \in \mathbb{G}_m \subset \mathbb{P}^1.$$

Since $(0,0)$ maps to Q, i.e. to the point at $x = 0$, the map L_ζ identifies the whole affine x-line $\mathbb{A}^1 = \{(x, 0) : x \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine x-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

On the other hand, the double point Q also lies on the irreducible component whose labelling includes the other weight of Q, i.e. the weight $\mathrm{Sym}^{p-2k} \otimes \det^{k-1}$. We fix an affine coordinate y on this $\mathbb{P}^1$ around Q. Away from Q, the coordinate $y \neq 0$ parametrizes Galois representations of the form

$$\rho_y = \begin{pmatrix} \mathrm{unr}(y)\omega^{p-2k+1} & 0 \\ 0 & \mathrm{unr}(y^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{k-1}$. As in the first case, $\pi(\rho_y)$ contains $\pi_1 := \pi(p-2k, y, \eta) = \mathrm{Ind}_B^G(\chi)\otimes\eta$ as a direct summand, where now $\chi = \mathrm{unr}(y)\otimes\omega^{p-2k}\,\mathrm{unr}(y^{-1})$. Since

$$(1\otimes\omega^{p-2k}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{k-1}\otimes\omega^{-k} = \chi_k \in \mathbb{T}^\vee,$$

we deduce from the regular case of 4.2 that $\pi_1^{I^{(1)}} = M(0,y,1,\chi_k)$ is a simple 2-dimensional standard module.

Now suppose that $v=(0,y)$, $y\neq 0$, denotes a nonzero point on the y-line of $\mathbb{A}^1\cup_0\mathbb{A}^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(0,y,1,\chi_k)$. Our discussion shows that the point $L_\zeta((0,y))$ corresponds to the block which contains π_1, parametrized by $[\rho_y]$. Hence

$$L_\zeta((0,y)) = [\rho_y] = y\in\mathbb{G}_m\subset\mathbb{P}^1.$$

Since $(0,0)$ maps to Q, i.e. to the point at $y=0$, the map L_ζ identifies the whole y-line $\mathbb{A}^1 = \{(0,y): y\in k\}\subset V_{\widehat{\mathbf{T}},0,1}$ with the affine y-line $\mathbb{A}^1\subset\mathbb{P}^1\subset X_\zeta$.

In this way, we get an open immersion of each connected component $(V_{\widehat{\mathbf{T}},0}^\gamma/W_0)_\zeta$ of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$ such that $\gamma=(\chi_k,\chi_k^s)$ with $1<k<\frac{p-1}{2}$, in the scheme X_ζ, which coincides on k-points with the restriction of the map of sets L_ζ.

2. The two boundary cases $k\in\{1,\frac{p-1}{2}\}$. Consider the double point

$$Q = L_\zeta(\text{origin } (0,0) \text{ on the component } \gamma=(\chi_k,\chi_k^s)).$$

As we have just seen, Q lies on an 'interior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^1\otimes\det^{-1}$ (for $k=1$) or the weight $\mathrm{Sym}^1\otimes\det^{\frac{p-3}{2}}$ (for $k=\frac{p-1}{2}$). We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we will call z. Away from Q, the coordinate $z\neq 0$ parametrizes Galois representations of the form

$$\rho_z = \begin{pmatrix} \mathrm{unr}(z)\omega^2 & 0\\ 0 & \mathrm{unr}(z^{-1})\end{pmatrix}\otimes\eta$$

with $\eta=\omega^{-1}$ or $\eta=\omega^{\frac{p-3}{2}}$.

Let $k=1$, i.e. $\eta=\omega^{-1}$. Following the argument in the generic case word for word, we may conclude that L_ζ identifies the x-line $\mathbb{A}^1=\{(x,0): x\in k\}\subset V_{\widehat{\mathbf{T}},0,1}$ with the z-line $\mathbb{A}^1\subset\mathbb{P}^1\subset X_\zeta$.

Let $k=\frac{p-1}{2}$, i.e. $\eta=\omega^{\frac{p-3}{2}}$. As in the generic case, we may conclude that L_ζ identifies the y-line $\mathbb{A}^1=\{(0,y): y\in k\}\subset V_{\widehat{\mathbf{T}},0,1}$ with the z-line $\mathbb{A}^1\subset\mathbb{P}^1\subset X_\zeta$.

On the other hand, the double point Q lies also on the irreducible component $\mathbb{P}^1$ whose labelling includes the other weight of Q, i.e. the weight Sym^{p-2} (for $k=1$) or the weight $\mathrm{Sym}^{p-2}\otimes\det^{\frac{p-1}{2}}$ (for $k=\frac{p-1}{2}$). These are the two 'exterior' components. Points of the open locus X_ζ^{red} lying on such a component correspond to unramified (up to twist) Galois representations of the form

$$\rho_t = \begin{pmatrix} \mathrm{unr}(z) & 0\\ 0 & \mathrm{unr}(z^{-1})\end{pmatrix}\otimes\eta$$

where $\eta=1$ (for $k=1$) or $\eta=\omega^{\frac{p-1}{2}}$ (for $k=\frac{p-1}{2}$) and with $t=z+z^{-1}\in\mathbb{A}^1\subset\mathbb{P}^1$. As in the boundary case of 6.1, we have $\pi(\rho_t)=\pi(p-2,z,\eta)\oplus\pi(p-2,z^{-1},\eta) =: \pi_1\oplus\pi_2$ and these are irreducible principal series representations. We may write $\pi_1=\mathrm{Ind}_B^G(\chi)\otimes\eta$ with $\chi = \mathrm{unr}(z)\otimes\omega^{p-2}\,\mathrm{unr}(z^{-1})$. The character $\chi|_{\mathbb{F}_p^\times}=1\otimes\omega^{p-2}$ is regular (i.e. different from its s-conjugate) and we are in the regular case of 4.2. We conclude that

$$\pi_1^{I^{(1)}} = M(0,z,1,(1\otimes\omega^{p-2}).\eta)$$

is a simple 2-dimensional standard module in the regular component represented by the character

$$(1\otimes\omega^{p-2}).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times})\otimes(\eta|_{\mathbb{F}_p^\times})\omega^{p-2} = (\eta|_{\mathbb{F}_p^\times})\otimes(\eta|_{\mathbb{F}_p^\times})\omega^{-1}\in\mathbb{T}^\vee.$$

This latter character equals χ_1 for $\eta = 1$ and $(\chi_{\frac{p-1}{2}})^s$ for $\eta = \omega^{\frac{p-1}{2}}$ (indeed, note that $\frac{p-1}{2} \equiv -\frac{p-1}{2}$ mod $p-1$).

Now suppose that $k = 1$, i.e. $\eta = 1$. Let $v = (0, y)$, $y \neq 0$, be a nonzero point on the y-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(0, y, 1, \chi_1)$. Our discussion shows that the point $L_\zeta((0, y))$ corresponds to the block which contains π_1, i.e. which is parametrized by $[\rho_t]$. It follows that

$$L_\zeta((0,y)) = [\rho_t] = t = y + y^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1.$$

Since $(0, 0)$ maps to Q, i.e. to the point at $t = \infty$, the map of sets L_ζ maps the k-points of the whole affine y-line $\mathbb{A}^1 = \{(0, y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ to the k-points of the whole 'left exterior' component $\mathbb{P}^1 \subset X_\zeta$ via the formula

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \\ y & \longmapsto & \begin{cases} y + y^{-1} & \text{if } y \neq 0 \\ \infty = Q & \text{if } y = 0. \end{cases} \end{array}$$

This formula is algebraic: indeed, for $y \in \mathbb{A}^1 \setminus \{\pm i\}$ (where $\pm i$ are the roots of the polynomial $f(y) = y^2 + 1$), we have $y + y^{-1} \neq 0$ and $(y + y^{-1})^{-1} = y/(y^2 + 1)$, which is equal to 0 at $y = 0$. Moreover, it glues at the origin $(0, 0)$ with the open immersion of the x-line of $V_{\widehat{\mathbf{T}},0,1} = \mathbb{A}^1 \cup_0 \mathbb{A}^1$ in X_ζ defined above, since both map $(0, 0)$ to Q. We take the resulting morphism of k-schemes $\mathbb{A}^1 \cup_0 \mathbb{A}^1 \to X_\zeta$ as the definition of L_ζ on the connected component $(V_{\widehat{\mathbf{T}},0}^{(\chi_1,\chi_1^s)}/W_0)_\zeta$ of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$. Note that its restriction to the open subset $\{y \neq 0\}$ in the y-line $\mathbb{A}^1$ is the morphism $\mathbb{G}_m \to \mathbb{A}^1$ corresponding to the ring extension

$$k[t] \longrightarrow k[y, y^{-1}] = k[t][y]/(y^2 - ty + 1),$$

and that the discriminant $t^2 - 4$ of $y^2 - ty + 1 \in k[t][y]$ vanishes precisely at the two exceptional points $t = \pm 2$.

Suppose $k = \frac{p-1}{2}$, i.e. $\eta = \omega^{\frac{p-1}{2}}$. Let $v = (x, 0)$, $x \neq 0$, denote a nonzero point on the x-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular,

$$\mathrm{Sph}^\gamma(v) = M(0, x, 1, (\chi_{\frac{p-1}{2}})^s) = M(x, 0, 1, \chi_{\frac{p-1}{2}}).$$

Our discussion shows that the point $L_\zeta((x, 0))$ corresponds to the block which contains π_1, i.e. which is parametrized by $[\rho_t]$. It follows that $L_\zeta((x, 0)) = [\rho_t] = t = x + x^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1$. Since $(0, 0)$ maps to the point Q at $t = \infty$, the map of sets L_ζ maps the k-points of the whole affine x-line $\mathbb{A}^1 = \{(x, 0) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ to the k-points of the whole 'right exterior' component $\mathbb{P}^1 \subset X_\zeta$ via the formula

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \\ x & \longmapsto & \begin{cases} x + x^{-1} & \text{if } x \neq 0 \\ \infty = Q & \text{if } x = 0. \end{cases} \end{array}$$

This formula is algebraic. Moreover, it glues at the origin $(0, 0)$ with the open immersion of the y-line of $V_{\widehat{\mathbf{T}},0,1} = \mathbb{A}^1 \cup_0 \mathbb{A}^1$ in X_ζ defined above, since both map $(0, 0)$ to Q. We take the resulting morphism of k-schemes $\mathbb{A}^1 \cup_0 \mathbb{A}^1 \to X_\zeta$ as the definition of L_ζ on the connected component $(V_{\widehat{\mathbf{T}},0}^{(\chi_{\frac{p-1}{2}},(\chi_{\frac{p-1}{2}})^s)}/W_0)_\zeta$ of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$.

7 An interpolation of the semisimple mod p correspondence

In this subsection we continue to assume $p \geq 5$.

7.1. Recall the mod p parametrization functor P from 4.4. For $\zeta \in \mathcal{Z}^\vee(k)$, let $\mathrm{Mod}_\zeta(\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)})$ be the full subcategory of $\mathrm{Mod}(\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)})$ whose objets are the $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$-modules M whose Satake parameter $S(M)$ is supported on the closed subscheme $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta \subset V_{\widehat{\mathbf{T}},0}^{(1)}/W_0$. A $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$-module M lies in the category $\mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ if and only if: M is only supported in γ-components where $\gamma|_{\mathbb{F}_p^\times} = \zeta|_{\mathbb{F}_p^\times}$ and the operator U^2 acts on M via the $\mathbb{G}_m$-part of ζ. Then P induces a *mod p ζ-parametrization functor*

$$P_\zeta : \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \longrightarrow \mathrm{QCoh}((V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta).$$

Let $\zeta \in \mathscr{Z}^\vee(k)$. We have the functor

$$L_{\zeta *} : \mathrm{QCoh}((V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta) \longrightarrow \mathrm{QCoh}(X_\zeta)$$

push-forward along the k-morphism $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \to X_\zeta$ from 4.9. Finally recall that for $\zeta \in \mathscr{Z}^\vee(k)$, the functor of $I^{(1)}$-invariants $(\cdot)^{I^{(1)}} : \mathrm{Mod}^{\mathrm{sm}}(k[G]) \to \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ induces a functor

$$(\cdot)^{I^{(1)}}_\zeta : \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \to \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}),$$

by 4.5.

7.2. Definition. *Let $\zeta \in \mathscr{Z}^\vee(k)$. The* mod p ζ-Langlands parametrization functor *is the functor* $\mathrm{L}_\zeta\mathrm{P}_\zeta := L_{\zeta *} \circ P_\zeta$:

$$\begin{array}{c} \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \\ \downarrow \\ \mathrm{QCoh}(X_\zeta) \end{array}$$

Identifying ζ with a central character of G, the functor $\mathrm{L}_\zeta\mathrm{P}_\zeta$ extends to the category $\mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G])$ by precomposing with the functor $(\cdot)^{I^{(1)}}_\zeta : \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \to \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$. This gives the functor $\mathrm{L}_\zeta\mathrm{P}_\zeta \circ (\cdot)^{I^{(1)}}_\zeta$:

$$\begin{array}{c} \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \\ \downarrow \\ \mathrm{QCoh}(X_\zeta). \end{array}$$

7.3. Theorem. *Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. Fix a character $\zeta : Z(G) = \mathbb{Q}_p^\times \to k^\times$, corresponding to a point $(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) \in \mathscr{Z}^\vee(k)$ under the identification $\mathscr{Z}(G)^\vee \cong \mathscr{Z}^\vee(k)$ from 4.4.*

The mod p ζ-Langlands parametrization functor $\mathrm{L}_\zeta\mathrm{P}_\zeta$ interpolates the Langlands parametrization of the blocks of the category $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$, cf. 4.7 : for all $x \in X_\zeta(k)$ and for all $\pi \in b_{[\rho_x]}$,

$$\mathrm{L}_\zeta\mathrm{P}_\zeta(\pi^{I^{(1)}}) = \begin{cases} i_{x*}(\pi^{I^{(1)}}) & \text{if } x \text{ is not an exceptional point in the odd case} \\ i_{x*}(\pi^{I^{(1)}})^{\oplus 2} & \text{otherwise} \end{cases} \in \mathrm{QCoh}(X_\zeta)$$

where $i_x : \mathrm{Spec}(k) \to X_\zeta$ is the k-point x.

Proof. By definition of a block of a category as a certain equivalence class of simple objects [Pas13], if $\pi \in b_{[\rho_x]}$ then in particular π is simple. Then $\pi^{I^{(1)}}$ is simple too, and hence has a central character. Therefore $P_\zeta(\pi^{I^{(1)}})$ is the underlying k-vector space of $\pi^{I^{(1)}}$ supported at the k-point $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ corresponding to its central character under the isomorphism $\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p}$, which lies on some connected component γ. Suppose $\dim_k(\pi^{I^{(1)}}) = 2$. Then $\pi^{I^{(1)}}$ is isomorphic to the simple standard module of $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$ with central character v, i.e. to $\mathrm{Sph}^\gamma(v)$, and hence $L_\zeta(v) = x$ by definition of the map of sets $L_\zeta(k)$. Suppose $\dim_k(\pi^{I^{(1)}}) = 1$. Then $\pi^{I^{(1)}}$ is one of the two spherical characters of $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$ whose restriction to the center $Z(\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p})$ is equal to v, i.e. it is one of the simple constituents of $(\mathrm{Sph}^\gamma(v))^{\mathrm{ss}}$, and hence again $L_\zeta(v) = x$ by definition of the map of sets $L_\zeta(k)$. Now if x is not an exceptional point in an odd case, then L_ζ is an open immersion at v, and otherwise it has ramification index 2 at v. The theorem follows. □

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