ON UNITARY DEFORMATIONS OF SMOOTH MODULAR REPRESENTATIONS

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Abstract. Let G be a locally $\mathbb{Q}_p$-analytic group and K a finite extension of $\mathbb{Q}_p$ with residue field k. Adapting a strategy of B. Mazur (cf. [Maz89]) we use deformation theory to study the possible liftings of a given smooth G-representation ρ over k to unitary G-Banach space representations over K. The main result proves the existence of a universal deformation space in case ρ admits only scalar endomorphisms. As an application we let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and compute the fibers of the reduction map in principal series representations.

1. Introduction

Let G a locally $\mathbb{Q}_p$-analytic group and K a finite extension of $\mathbb{Q}_p$ with ring of integers o and residue field k. The aim of the present note is to study the set of possible liftings of a given smooth G-representation ρ over k to unitary G-Banach space representations over K. To do this we adapt the techniques of deformation theory for representations of profinite groups as developed by B. Mazur (cf. [Maz89]) to our present situation. We prove, in case ρ admits only scalar endomorphisms (e.g. admissible and absolutely irreducible), the existence of a formal scheme Spf $R(G, K, \rho)$ over o which depends functorially on the datum (G, K, ρ) and respects elementary operations on ρ such as tensor product. Moreover, its o-rational points biject canonically with the isomorphism classes of unitary liftings of ρ. The ring $R(G, K, \rho)$ is a local profinite o-algebra with residue field k which is noetherian if and only if the k-vector space of extensions $\mathrm{Ext}^1_G(\rho, \rho)$ is of finite dimension. Basic features of the geometry of *Spec* $R(G, K, \rho)$ such as dimensions or the number of irreducible components remain unclear at this point.

The ring $R(G, K, \rho)$ represents a deformation problem for Iwasawa modules which is based on the simple observation (due to V. Paskunas, cf. [Pas10])) that the duality functor introduced by P. Schneider and J. Teitelbaum (cf. [ST02]) on the category of unitary representations is compatible with Pontryagin duality over the profinite ring o. By work of M. Emerton (cf. [Eme10a]) the dual categories admit generalizations to complete local noetherian o-algebras which provide a natural framework to study deformations of (the Pontryagin dual of) ρ.

We emphasize two cases in which our main result is well-known. If G is a compact group and ρ is admissible absolutely irreducible our result follows directly from work of B. Mazur. On the other hand, in the important case of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ our result was essentially established by M. Kisin building on work of P. Colmez (cf. [Col10]) and M. Schlessinger (cf. [Sch68]). In [Kis10] this result is proved to show the essential surjectivity of Colmez's Montreal functor, an important result in the p-adic Langlands programme for $\mathrm{GL}_2(\mathbb{Q}_p)$. In this light we hope our result will prove important in extending the p-adic Langlands programme to other groups than $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let us briefly outline the paper. We begin by recalling and establishing some basic results on pseudocompact rings (sect. 2). For sake of clarity and to have a greater flexibility in future applications we then proceed axiomatically and work in a quite general setting as follows. Let k be a field (finite or not) of characteristic $p > 0$ and ρ be a smooth G-representation over k. Let o be an arbitrary complete local noetherian ring with residue field k. We introduce a category $\hat{C}$ of coefficient rings consisting of commutative local pseudocompact o-algebras A such that the structure map $o \to A$ gives an isomorphism on residue fields. Analogous to work of M. Emerton (loc.cit.) we introduce a suitable category of Iwasawa modules over such an A and study its basic properties (sect. 3). By results of A. Brumer (cf. [Bru66]) this category depends "naturally" on A which results in a deformation functor D_ρ for ρ. It is straightforward (sect. 3) to prove the representability of D_ρ for characters ρ and to explain the notion of deformation conditions in our setting. Section 4 contains the proof of the representability of D_ρ in case ρ has only scalar endomorphisms and k is finite and establishes the Ext^1-criterion. We remark straightaway that Schlessinger theory (loc.cit.) is not applicable in our situation since we refrain from any finiteness assumption on the tangent space of D_ρ. Instead, we proceed directly from A. Grothendieck's fundamental representability theorem (cf. [Gro]) using ideas of M. Dickinson (cf. [Gou01], Appendix 1). In section 5 we show the usual functorial properties of the universal deformation ring. This is almost a formality.

In the final section we turn to Banach space representations and specialize the deformation theory to the situation where o is given as ring of integers in K. This yields the relation between unitary lifts of ρ and rational points of Spf $R(G, K, \rho)$. We illustrate this method with an application to the group $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and compute the set of unitary deformations in case ρ equals a principal series representation. This heavily builds upon results of M. Emerton concerning his functor of 'ordinary parts' ([Eme10b]). We also remark that the structure of such principal series representations ρ over k has been made explicit by the work of Barthel/Livné ([BL94]).

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2. Pseudocompact rings

For any unital ring A we let $\mathfrak{M}(A)$ be the abelian category of left unital A-modules. If A is left noetherian then the finitely generated left A-modules form a full subcategory $\mathfrak{M}_{fg}(A)$ of $\mathfrak{M}(A)$. A left and right noetherian ring will be called noetherian.

A *left pseudocompact ring* is a complete Hausdorff topological unital ring A which admits a system of open neighbourhoods of zero consisting of left ideals $\mathfrak{a}$ such that $A/\mathfrak{a}$ has finite length as a left A-module (cf. [Gab70]). In particular, A equals the topological inverse limit of the artinian quotients $A/\mathfrak{a}$ each endowed with the discrete topology. A morphism of left pseudocompact rings is by definition a continuous unital ring homomorphism. A left artinian ring with the discrete topology is evidently left pseudocompact. More generally, the topology on a left pseudocompact ring A which is left noetherian is uniquely determined and coincides with the adic topology defined by the Jacobson radical of A.

Let A be a left pseudocompact ring. A complete Hausdorff topological left unital A-module M is called *left pseudocompact* if it has a system of open neighbourhoods of zero consisting of submodules M' such that M/M' has finite length. A morphism between two left pseudocompact modules is by definition a continuous A-linear map. It necessarily has closed image. Borrowing notation from [SV06] we denote the category of left pseudocompact A-modules by $\mathfrak{PM}(A)$. It is abelian with exact projective limits and the forgetful functor $\mathfrak{PM}(A) \to \mathfrak{M}(A)$ is faithful and exact and commutes with projective limits (cf. [Gab62], IV.3. Thm. 3, [vGvdB97], Prop. 3.3). An arbitrary direct product of left pseudocompact modules is left pseudocompact in the product topology. A left pseudocompact module M is called *topologically free* if it is topologically isomorphic to a product $\prod_I A$ with some index set I. The set of images in M of the "unit vectors" $(...,0,1,0,...) \in \prod_I A$ under such an isomorphism is called a *left pseudobasis* of M. If A is left noetherian then any finitely generated abstract left A-module has a unique left pseudocompact topology. We thus have a natural fully faithful and exact embedding $\mathfrak{M}_{fg}(A) \to \mathfrak{PM}(A)$.

There are obvious "right" versions of the statements above.

Remark: For sake of clarity we point out the following. In [Bru66], A. Brumer defines a pseudocompact ring to be a complete Hausdorff topological unital ring A that admits a system of open neighborhoods of zero consisting of two sided ideals I for which A/I is an Artin ring (i.e. satisfies the descending chain condition for chains of two-sided ideals). If A is commutative then this is equivalent to A being left and right pseudocompact.

For the rest of this section let us fix a left pseudocompact ring A which is *commutative*. Evidently, it is then right pseudocompact and we will not distinguish between left and right A-modules.

Given a pseudocompact module M over such an A write $M^* := \mathrm{Hom}(M, A)$ for the A-module of morphisms $M \to A$ in $\mathfrak{PM}(A)$. We obtain a functor

$$M \mapsto M^* \tag{1}$$

between $\mathfrak{PM}(A)$ and $\mathfrak{M}(A)$ that changes direct products into direct sums. If A is artinian this establishes an anti-equivalence of categories between projective objects in $\mathfrak{PM}(A)$ and $\mathfrak{M}(A)$ respectively (cf. [Gab70], 0.2.2). In general, if $A^I := \prod_I A$ denotes a topologically free module on a pseudobasis indexed by I and

$$M_I(A) := \mathrm{End}_{\mathfrak{PM}(A)}(A^I)$$

its endomorphism ring we obtain an isomorphism of abstract A-modules

$$M_I(A) \xrightarrow{\cong} \prod_I (A^I)^* = \prod_I \oplus_I A, \tag{2}$$

natural in A. In the light of (2) we sometimes view elements of $M_I(A)$ as infinite '$I \times I$-matrices' with entries from A. We always equip $M_I(A)$ with the compact-open topology. If A^I is locally compact then $M_I(A)$ is a topological ring ([Bou89], X.§3.4 Prop.9).

If M, N denote two pseudocompact A-modules define the A-module

$$M \hat{\otimes}_A N := \varprojlim_{M',N'} M/M' \otimes_A N/N' \tag{3}$$

where M' and N' run through the open submodules of M and N respectively. If each $M/M' \otimes_A N/N'$ is endowed with the discrete topology the projective limit topology makes $M \hat{\otimes}_A N$ a pseudocompact A-module. Indeed, given M', N' as above there exists an open ideal $\mathfrak{a} \subseteq A$ such that $\mathfrak{a}M \subseteq M'$ and $\mathfrak{a}N \subseteq N'$ so that $M/M' \otimes_A N/N'$ is a finitely generated module over the artinian ring $A/\mathfrak{a}$ and therefore of finite A-length. The binary operation $\hat{\otimes}_A$ on $\mathfrak{PM}(A)$ is associative and commutative with A as a unit object and functorial in both variables. It commutes with projective limits and direct products (cf. [Gab70], 0.3.5/6). Now let $\phi : A \to B$ be a morphism between commutative pseudocompact rings and $M \in \mathfrak{PM}(A)$. Define the B-module $M \hat{\otimes}_A B$ by the exact analogue of formula (3). Arguing similarly as above shows $M \hat{\otimes}_A B$ to be a pseudocompact B-module. We obtain a "base change" functor

$$\phi^* : \mathfrak{PM}(A) \longrightarrow \mathfrak{PM}(B)$$

which commutes with tensor products, projective limits and direct products (cf. [Gab70], 0.5).

A *pseudocompact algebra over* A is a topological unital A-algebra B (commutative or not) which admits a system of open neighbourhoods of zero consisting of twosided ideals $\mathfrak{b}$ such that $B/\mathfrak{b}$ has finite length as an A-module (cf. [Bru66]). For example, an A-algebra which is of finite length as A-module is evidently a pseudocompact A-algebra in the discrete topology. The following simple fact will prove useful in the sequel.

Lemma 2.1. *Let B be a topological unital A-algebra. Then B is a pseudocompact A-algebra if and only if the underlying topological A-module of B is pseudocompact.*

Proof: If suffices to show the "if" part. Let $P \subseteq B$ be a left ideal such that B/P is of finite A-length and consider the multiplication map followed by the natural projection $B \to B/P$

$$\varphi : B \times B \longrightarrow B \longrightarrow B/P.$$

Arguing similarly as in [Gab70], Lem. 0.3.1 we find a left open submodule M and a right open submodule N of B of finite colength such that $\varphi(B, N) = \varphi(M, B) = 0$. In other words, P contains the two-sided ideal of B generated by $M \cap N$ which is open. ☐

A *morphism* between two pseudocompact algebras over A is by definition a continuous unital A-algebra homomorphism. As a consequence of the above lemma the category of pseudocompact A-algebras has projective limits. Moreover, if $B \to C$ and $B \to D$ are morphisms between pseudocompact A-algebras then the completed tensor product $C \hat{\otimes}_B D$ is a pseudocompact A-algebra in the obvious way. Finally, if B is a pseudocompact A-algebra and C is a pseudocompact B-algebra then evidently C is a pseudocompact A-algebra.

We finally point out the following simple construction. Evidently, a pseudocompact A-algebra B is a left and right pseudocompact ring. Furthermore, a discrete (topological) B-module has finite length as a B-module if and only if it has finite length as an A-module. Letting $\phi : A \to B$ denote the structure map we therefore have a natural faithful and exact forgetful functor

$$\phi_* : \mathfrak{PM}(B) \longrightarrow \mathfrak{PM}(A). \tag{4}$$

3. The deformation functor

3.1. **Completed group algebras.** Let A be a commutative pseudocompact ring and H be a profinite group. Writing $\mathcal{N}$ for the system of open normal subgroups of H we denote by

$$A[[H]] := \varprojlim_{N \in \mathcal{N}} A[H/N]$$

the completed group algebra of H over A. It is a pseudocompact A-algebra with respect to the projective limit topology and the correspondance

$$H \mapsto A[[H]]$$

is a covariant functor from profinite groups to pseudocompact A-algebras (cf. [Bru66], Sect. 2). The anti-involution $h \mapsto h^{-1}$ on H identifies $A[[H]]$ and $A[[H]]^{opp}$ as pseudocompact A-algebras making it unnecessary to distinguish between left and right modules. Let $\mathfrak{PM}(A[[H]])$ be the abelian category of (left) pseudocompact $A[[H]]$-modules.

Remark: Suppose A is noetherian and H is locally $\mathbb{Q}_p$-analytic. Since the pseudocompact topology on A is defined by the Jacobson radical (cf. sect. 2) a mild generalization of [Emeb], Thm. 2.1.1 shows that $A[[H]]$ is noetherian. We shall make no use of this fact.

Recall that a topological A-module M is called a (topological) *left H-module* over A if it has a H-action by A-linear maps such that the map

$$H \times M \longrightarrow M$$

giving the action is continuous. It follows that a discrete A-module of finite length is an H-module if and only if it is a (discrete) $A[[H]]$-module. A projective limit argument shows that a pseudocompact A-module M is an H-module if and only if it is a (pseudocompact) $A[[H]]$-module (cf. [Bru66], p. 454/455).

After these preliminaries let $M, N \in \mathfrak{PM}(A[[H]])$ be given. The diagonal H-action on the pseudocompact A-module $M \hat{\otimes}_A N$ is then continuous and, by our initial remarks, extends therefore to a pseudocompact $A[[H]]$-module structure. The resulting binary operation $\hat{\otimes}_A$ on $\mathfrak{PM}(A[[H]])$ is associative, commutative and functorial in both variables. The usual augmentation homomorphism $A[[H]] \to A$ provides a unit object.

Now consider a pseudocompact A-algebra B and let $\phi : A \to B$ be the structure map. The base change $\phi^* : \mathfrak{PM}(A) \to \mathfrak{PM}(B)$ commutes with projective limits (cf. sect. 2). Thus, the compatible system of natural isomorphisms

$$B \otimes_A A[H/N] \xrightarrow{\cong} B[H/N], \; N \in \mathcal{N}$$

induces a natural isomorphism $\phi^*(A[[H]]) \xrightarrow{\cong} B[[H]]$ which is multiplicative. This discussion yields a functor

$$\phi^*_H : \mathfrak{PM}(A[[H]]) \to \mathfrak{PM}(B[[H]]) \tag{5}$$

compatible with ϕ^* via the appropriate forgetful functors. Given another pseudocompact B-algebra B' with structure map ψ we evidently have

$$(\psi \circ \phi)^*_H = \psi^*_H \circ \phi^*_H. \tag{6}$$

Let additionally, $\mathfrak{PM}(A[[H]])^{\mathrm{fl}}$ denote the full subcategory of $\mathfrak{PM}(A[[H]])$ consisting of modules on underlying topologically free A-modules (similarly for B). Since the tensor product commutes with direct products (cf. sect. 2) the functor ϕ^*_H is seen to respect these subcategories.

3.2. **Augmented representations.** Let o be a fixed commutative complete local noetherian ring and A a commutative pseudocompact o-algebra. We now bring in a locally $\mathbb{Q}_p$-analytic group G and introduce a certain category of G-representations over A.

Let $A[G]$ be the group algebra of G over A. A *G-representation over A* is simply a (left) $A[G]$-module. Following [Eme10a] such a representation M is called *augmented* if the induced $A[H]$-action extends to an $A[[H]]$-action on M for every compact open subgroup H of G. Let M be an augmented G-representation on a pseudocompact A-module. It is called a *pseudocompact augmented* G-representation over A if the topology on M makes it a pseudocompact $A[[H]]$-module for every compact open subgroup H of G. By the above this is equivalent to requiring that the induced H-action makes M into an H-module for every compact open subgroup H of G.

Lemma 3.1. *On a pseudocompact augmented G-representation M over A the group G acts by continuous automorphisms.*

Proof: Pick a compact open subgroup H of G. Invoking the duality $M \mapsto M^\vee$ between $\mathfrak{PM}(A[[H]])$ and the category of discrete topological $A[[H]]$-modules (cf. [Bru66]) it suffices to consider $M^\vee$ with the G-action induced by functoriality. But $M^\vee$ is a discrete H-module and so the G-action is even jointly continuous on $M^\vee$. $\square$

We define a morphism between two such representations to be a G-equivariant morphism in $\mathfrak{PM}(A)$. Since $A[H] \subseteq A[[H]]$ is dense it is plain that any such morphism extends to a morphism in $\mathfrak{PM}(A[[H]])$ for every compact open subgroup H of G. Borrowing notation from [Eme10a] we denote the resulting category by $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$.

Lemma 3.2. *The category $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ is an A-linear abelian tensor category.*

Proof: Let H be a compact open subgroup of G. Since $\mathfrak{PM}(A)$ and $\mathfrak{PM}(A[[H]])$ are abelian categories $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ is abelian as well. Furthermore, given $M, N \in \mathfrak{PM}(A[[H]])$ the diagonal G-action on

$$M \hat{\otimes}_A N \in \mathfrak{PM}(A[[H]])$$

for any compact open subgroup $H \subseteq G$ makes the latter module a pseudocompact augmented G-representation over A. This yields the desired tensor product on $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$. $\square$

Remark: In case of a compact discrete valuation ring o and a complete local noetherian ring A with finite residue field the category $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ was introduced by M. Emerton. It is Pontryagin dual to certain smooth G-representations over A and plays a central role in M. Emerton's theory of ordinary parts of admissible representations. For more details we refer to [Eme10a] and [Eme10b].

3.3. **Functors on coefficient algebras.** Let as before o be a commutative complete local noetherian ring. We now define two subcategories of commutative pseudocompact o-algebras which will serve as coefficient algebras within the upcoming deformation theory.

Let $\mathfrak{m}$ be the maximal ideal of o with residue field k. Let $\hat{C}$ be the full subcategory of pseudocompact algebras A over o that are commutative local rings and such that the structure map $o \to A$ is local and induces an isomorphism on residue fields. Let C denote the full subcategory of $\hat{C}$ consisting of discrete algebras having finite length as o-module. Without recalling the precise definition of a *pro-object* (cf. [Gro], A.2) we have the following simple observation.

Lemma 3.3. *The category of pro-objects of C is equivalent to $\hat{C}$.*

Proof: Let us denote by $Pro(.)$ the passage from a category to the its category of pro-objects and pro-morphisms. Let C' be the category of all discrete o-algebras having finite length as o-module. Mapping a pseudocompact o-algebra to the system of all its artinian quotients induces an equivalence between pseudocompact o-algebras and $Pro(C')$ (cf. [Gro], A.5). Restricting this functor to $\hat{C}$ yields a fully faithful functor into $Pro(C)$. Given an element $(R_i)_i$ in $Pro(C)$ the projective limit $\varprojlim_i R_i$ lies in $\hat{C}$ and hence, this functor is essentially surjective. $\square$

We now bring in a set-valued covariant functor on C

$$D : C \longrightarrow Sets.$$

The category C contains k as a terminal object and admits finite products and finite fiber products (cf. [Maz97], Lem. IV.§14). As to the latter, recall that if $\phi_i : A_i \to A_0$ are two morphisms in C their fiber product is given as the equalizer

$$A_1 \times_{A_0} A_2 = \{(a_1, a_2) \in A_1 \times A_2 : \phi_1(a_1) = \phi_2(a_2)\}$$

with ring structure induced from $A_1 \times A_2$. In this situation D is called *left exact* if it respects finite products and finite fiber products. Furthermore, since $\hat{C}$ identifies with the pro-objects of C, the functor D being *pro-representable* is tantamount to being of the form $\mathrm{Hom}_{\hat{C}}(R, .)$ with some $R \in \hat{C}$ (cf. [Gro], A.2).

Let $k[\epsilon] = k[x]/x^2$ be the ring of dual numbers viewed as an object in C. If D is pro-representable the set $D(k[\epsilon])$ evidently has a natural k-vector space structure (the "tangent space" of D).

Theorem 3.4. *The functor $D : C \to Sets$ is pro-representable if and only if it is left exact. In this situation the representing ring R is noetherian if and only if the k-vector space $D(k[\epsilon])$ has finite dimension d. In this case, R equals a quotient of the formal power series ring $o[[x_1, ..., x_d]]$.*

Proof: This follows directly from A. Grothendieck's fundamental representability theorem (cf. [Gro], Prop. A.3.1/A.5.1). □

Suppose we now have a functor $D : \hat{C} \to Sets$ on the larger category $\hat{C}$. By the above lemma $\hat{C}$ is stable under arbitrary projective limits. The above discussion therefore shows that D is representable as a functor on $\hat{C}$ if and only if it commutes with projective limits and the restriction of D to C is pro-representable.

3.4. **Deformations.** We define the deformation problem and state the main representability result. We keep the assumptions of the previous subsection but **assume** additionally that the residue field of o has characteristic $p > 0$.

Given a pseudocompact A-algebra B let $\phi : A \to B$ be the structure map. For every $H \in \mathcal{N}$ we have the base change ϕ_H^* commuting with forgetful functors (cf. (5)) and, hence, a functor

$$\phi_G^* : \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A) \longrightarrow \mathrm{Mod}_G^{\mathrm{pro\,aug}}(B) \tag{7}$$

respecting the full subcategories $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)^{\mathrm{fl}}$ and $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(B)^{\mathrm{fl}}$ of modules which are topologically free over A and B respectively. Given another pseudocompact B-algebra B' with structure map ψ one has

$$(\psi \circ \phi)_G^* = \psi_G^* \circ \phi_G^* \tag{8}$$

according to (6). After these preliminaries we fix once and for all an element

$$N \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k).$$

Let I be an index set of a pseudobasis for the topologically free k-module underlying N. Invoking the duality $N \mapsto N^*$ (cf. (1)) we see that the cardinality $|I|$ does not depend on the choice of pseudobasis. Given a local pseudocompact o-algebra $A \in \hat{C}$ with residue homomorphism $\phi : A \to k$ we consider couples (M, α) such that $M \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)^{\mathrm{fl}}$ and

$$\alpha : k \hat{\otimes}_A M = \phi_G^*(M) \xrightarrow{\cong} N$$

is an isomorphism in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)^{\mathrm{fl}}$.

Lemma 3.5. *Given $M \in \mathfrak{PM}(A)$ the natural map*

$$k \hat{\otimes}_A M \longrightarrow M/\overline{\mathfrak{m}M}$$

is an isomorphism in $\mathfrak{PM}(k)$. If $\mathfrak{m}$ is finitely generated, the subset $\mathfrak{m}M \subseteq M$ is closed.

Proof: The first statement is [Gab70], 0.3.2. Suppose $\mathfrak{m}$ is finitely generated. The A-submodule $\mathfrak{m}M \subseteq M$ equals the image of a morphism in $\mathfrak{PM}(A)$ of the form $\prod M \to M$ where the product is indexed by finitely many generators of the ideal $\mathfrak{m}$. It is therefore closed. □

A morphism of couples

$$(M, \alpha) \longrightarrow (M', \alpha')$$

is by definition a morphism $M \to M'$ in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)^{\mathrm{fl}}$ such that the resulting diagram

$$\begin{array}{ccc} M/\overline{\mathfrak{m}M} & \xrightarrow[\alpha]{\sim} & N \\ \downarrow & & \downarrow = \\ M'/\overline{\mathfrak{m}M'} & \xrightarrow[\alpha']{\sim} & N \end{array} \tag{9}$$

is commutative. We denote the set of isomorphism classes of such couples by $D_N(A)$. As usual elements in $D_N(A)$ will be called *deformations* of N to A and we will often abbreviate M for $[M, \alpha]$ when no confusion can arise. Since base change to k commutes with arbitrary direct products any pseudobasis of $M \in D_N(A)$ must have cardinality I, too. This shows

Lemma 3.6. *If $[M, \alpha] \in D_N(A)$ then $M \simeq A^I$ in $\mathfrak{PM}(A)$.*

By associativity (8) of the base change ϕ_G^* we obtain a covariant set-valued functor

$$D_N : \hat{C} \longrightarrow Sets$$

such that $D_N(k)$ is a singleton. By Lem. 3.2 the category $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ is abelian whence we have the k-vector space $\mathrm{Ext}^1(N, N)$ of *Yoneda extensions* of N by itself.

Proposition 3.7. *Let k be finite. There is a natural bijection*

$$D_N(k[\epsilon]) \xrightarrow{\cong} \mathrm{Ext}^1(N, N)$$

which is k-linear in case D_N is representable.

Proof: This is a standard phenomenon in deformation theory (cf. [Maz97], Prop. V.§22) and most probably true without the restriction on the field k. In any case, let $A = k[\epsilon]$ with $\pi : A \to k$ and $\iota : k \to A$ the canonical maps. Let $[M, \alpha] \in D_N(A)$. Invoking an isomorphism $M \simeq A^I$ in $\mathfrak{PM}(A)$ (cf. Lem. 3.6) we have $\epsilon M \simeq \prod_I k \simeq N$ in $\mathfrak{PM}(k)$. Since multiplication by ϵ is a homeomorphism onto its image the restriction map induced by $\epsilon M \subseteq M$ and ι

$$res : \mathrm{End}_A^{cont}(M) \longrightarrow \mathrm{End}_k^{cont}(\epsilon M)$$

is continuous for compact-open topologies. The diagram

$$\begin{array}{ccccc} A[[H]] & \xrightarrow{cont} & \mathrm{End}_A^{cont}(M) & \xrightarrow{res} & \mathrm{End}_k^{cont}(\epsilon M) \\ \subseteq\uparrow & & \uparrow & \nearrow & \\ A[H] & \xrightarrow{\subseteq} & A[G] & & \end{array} .$$

is commutative and factores in an obvious sense through the projection π applied to coefficients. By the discussion at the beginning of section 4 below we therefore see that $\epsilon M \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ and that the isomorphism $\epsilon M \simeq N$ in $\mathfrak{PM}(k)$ lifts to an isomorphism in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$. In other words

$$0 \to \epsilon M \to M \to M/\epsilon M \to 0$$

yields an element in $\mathrm{Ext}^1(N,N)$. We obtain a map $D_N(A) \to \mathrm{Ext}^1(N,N)$. To construct the inverse let

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\pi} N \longrightarrow 0 \tag{10}$$

be an extension in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$. By topological freeness it splits in $\mathfrak{PM}(k)$ and we may assume that M admits a neighbourhood basis of zero consisting of k-vector spaces preserved under the endomorphism $\iota\circ\pi$. Setting $\epsilon.m := (\iota\circ\pi)(m)$ makes M a linearly topologized A-module whence $M \in \mathfrak{PM}(A)$. An easy argument shows that the A-module on the dual k-vector space M^* is free. Applying the quasi-inverse to (1) we find M to be topologically free over A. Finally, since ι, π are linear with respect to $A[[H]]$ and $A[G]$ it follows that $M \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$. This construction gives the inverse. The assertion about linearity follows immediately from the construction. □

We come to the main result of this section which will be proved in section 4.

Theorem 3.8. *Let k be finite. If $\mathrm{End}_{\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)}(N) = k$ then D_N is representable.*

Remarks: 1. Suppose N is finitely generated over $k[[H]]$ for some (equivalently any) compact open $H \subseteq G$. Then any $M \in D_N(A)$ is finitely generated over $A[[H]]$. Indeed, lifting finitely many $k[[H]]$-module generators of $M/\overline{\mathfrak{m}M} = N$ to M we obtain a map $\prod A[[H]] \to M$ whose cokernel Q satisfies $Q/\overline{\mathfrak{m}Q} = 0$. Since A is local $Q = 0$ by the Nakayama lemma (cf. [Gab70], 0.3.3).

2. The theorem is most probably true without the restriction on k. We impose this restriction since our proof makes crucial use of the compact-open topology. This topology is only well-behaved for locally compact spaces ([Bou89], X.§3.4).

3. Let k be finite. A pseudocompact o-algebra A is then a profinite o-algebra and a pseudocompact A-module is a profinite A-module. In the light of the main result of this work we could therefore have worked from the beginning on with profinite algebras and profinite modules. However, to have more flexibility in future applications it seemed advantageous to us to produce as much as possible of this theory in the more general 'pseudocompact' language.

Corollary 3.9. *Let k be finite (so that D_N is representable). The representing o-algebra R is noetherian if and only if $d := \dim_k \mathrm{Ext}^1(N,N) < \infty$. In this situation R equals a quotient of the formal power series ring $o[[x_1, ..., x_d]]$.*

Proof: This follows from prop. 3.7 and thm. 3.4. □

We give an important example in which the theorem applies. Assume o equals the integers in a finite extension $K/\mathbb{Q}_p$. In particular, k is a *finite* field of characteristic $p > 0$. As usual, a *smooth* G-representation (V, ρ) over k is a k-vector space V with a G-action such that the stabilizer of each vector $v \in V$ is open in G. With G-equivariant linear maps such representations form an abelian category $\mathrm{Mod}_G^{\mathrm{sm}}(k)$. A full subcategory is formed by *admissible* G-representations V having the property that the k-vector space of H-fixed vectors V^H is finite dimensional for every compact open subgroup $H \subseteq G$. Regarding a smooth G-representation as a discrete torsion o-module with a continuous G-action Pontryagin duality

$$V \mapsto V^\vee := \mathrm{Hom}_o^{cont}(V, K/o)$$

over the profinite ring o induces an anti-equivalence of k-linear categories

$$(.)^\vee : \mathrm{Mod}_G^{\mathrm{sm}}(k) \xrightarrow{\cong} \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k). \tag{11}$$

(cf. [Eme10a], Lem. 2.2.6). It is compatible with base extension relative to finite field extensions $k \subseteq k'$.

Lemma 3.10. *Let $(V, \rho) \in \mathrm{Mod}_G^{\mathrm{sm}}(k)$ be admissible and absolutely irreducible. Up to a finite extension of k we then have $\mathrm{End}_{\mathrm{Mod}_G^{\mathrm{sm}}(k)}(\rho) = k$ and therefore $N = \rho^\vee$ satisfies the conditions of the theorem.*

Proof: This is Schur's lemma for smooth mod p representations (e.g. [Bre], Lem. 4.1 in the case $G = GL_2(\mathbb{Q}_p)$). Let f be an endomophism of ρ. For a fixed open pro-p subgroup H of G the space V^H is finite dimensional, nonzero (cf. [Wil98], Lem. 11.1.1) and stabilized by f. After possibly extending scalars any nonzero eigenvector of f in V^H generates the G-representation V. □

Remark: Keeping the assumptions on o let $A \in \hat{C}$ be noetherian. In [Eme10a], 2.2 M. Emerton defines a new notion of *smooth* G-representation over A (which coincides with the usual one if A is artinian). With morphisms being G-equivariant A-linear maps such representations form an abelian category which is Pontryagin dual to $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$. In this situation, the deformation theory above translates therefore completely to such smooth representations. We will not use this point of view in this work and therefore refrain from formulating a precise picture.

3.5. **The compact case.** Suppose G is compact and k is a *finite* field of characteristic $p > 0$. We indicate how our deformation theory reduces to the situation studied by B. Mazur in the seminal article [Maz89].

Given an absolutely irreducible object $N \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ let I be an indexing set for a pseudobasis of N. Since any open normal pro-p subgroup of G acts trivially on the smooth representation $\rho = N^\vee$ ([Wil98], Lem. 11.1.1) we have $n := |I| < \infty$ and ρ is evidently admissible. Furthermore, any deformation of N to $A \in \hat{C}$ has a finite free underlying A-module (cf. Lem. 3.6) and thus, D_N describes the equivalence classes of continuous lifts

$$G \longrightarrow \mathrm{GL}_n(A)$$

of N to A.

Let $M_n(k)$ be the group of $n \times n$-matrices with entries from k. Letting $g.M := \rho(g)M\rho(g)^{-1}$ for $M \in M_n(k), g \in G$ makes $M_n(k)$ a (discrete) G-module which we denote by $\mathrm{Ad}(\rho)$ (the so-called *adjoint representation* of ρ). The group $\mathrm{GL}_n(k[\epsilon])$ is the semidirect product of the groups $1 + \epsilon M_n(k)$ and $\mathrm{GL}_n(k)$. Let ρ_1 be a deformation of ρ to $k[\epsilon]$. We have a map

$$c(\rho_1) : G \to \mathrm{Ad}(\rho) : g \mapsto m_g$$

where $(1+\epsilon m_g)\rho(g) := \rho_1(g)$. It is a 1-cocycle inducing an isomorphism of k-vector spaces

$$D_N(k[\epsilon]) \xrightarrow{\cong} H^1(G, \mathrm{Ad}(\rho))$$

(cf. [Maz89], p.399). The *finiteness condition* (in the sense of [loc.cit.], 1.1) is satisfied since G contains an open pro-p subgroup of finite rank (cf. [DdSMS99], Cor. 8.33). We therefore have $d_1 := \dim_k H^1(G, \mathrm{Ad}(\rho)) < \infty$ (cf. [Maz97], Prop. §21.2a). In this situtation our theorem (3.8) is therefore equivalent to the existence statement [Maz89], Prop.1. In particular, the deformation ring $R = R(\rho)$ is a (noetherian) quotient of $o[[x_1,...,x_{d_1}]]$. Let $d_2 := \dim_k H^2(G, \mathrm{Ad}(\rho))$ and $\mathfrak{m} \subset o$ the maximal ideal. We have

$$\text{Krull}\dim(R/\mathfrak{m}R) \geq d_1 - d_2$$

with equality in case $d_2 = 0$ (the *unobstructed case*); in this latter case $R = o[[x_1,...,x_{d_1}]]$ ([loc.cit.], Prop.2).

Remark: It is conjectured that in fact Krull $\dim(R/\mathfrak{m}R) = d_1 - d_2$ (compare [Gou01], p. 287). In [Böc98] this conjecture is proved in many cases. In case G equals a global Galois group it has been pointed out by B. Mazur that this conjecture should be viewed as a generalization of *Leopoldt's conjecture* (cf. [Maz89], 1.10).

3.6. **The case of a character.** Let k be finite of characteristic $p > 0$ and N such that $|I| = 1$. Under a reasonable technical assumption we will compute the universal deformation ring R and the universal deformation. As in the compact case R will not depend on the particular choice of such an N (cf. Prop. 5.1 below).

Recall that any commutative local pseudocompact ring is *henselian* (cf. [Nag62], Thm. V.30.3). Suppose A is such a ring with maximal ideal $\mathfrak{m}$ and finite residue field $k = A/\mathfrak{m}$. We have a short exact sequence

$$1 \longrightarrow 1 + \mathfrak{m} \longrightarrow A^\times \longrightarrow k^\times \longrightarrow 1$$

which is canonically split. Let $s : k^\times \to o^\times$ be the splitting in the case $A = o$.

We assume in the following that the Hausdorff abelianization $\bar{G} := G/\overline{[G,G]}$ of G is topologically finitely generated. Since there are only finitely many continuous group homomorphisms $G \to k[\epsilon]^\times$ the universal deformation ring is noetherian. By [Eme], Prop. 6.4.1 the inclusion of the maximal compact open subgroup $\bar{G}_0$ into $\bar{G}$ induces a (noncanonical) isomorphism $\bar{G}_0 \times \mathbb{Z}^r \cong \bar{G}$ of locally $\mathbb{Q}_p$-analytic groups for some unique integer $r \geq 0$. If Γ denotes the pro-p completion functor (cf. [RZ00], 3.2) we have $\Gamma(\bar{G}) = \mathbb{Z}_p^r \times \Gamma(\bar{G}_0)$. The canonical homomorphism $\bar{G} \to \Gamma(\bar{G})$ is therefore continuous with respect to the quotient topology on $\bar{G}$ and hence, so is the composed homomorphism

$$G \to \bar{G} \to \Gamma(\bar{G}). \tag{12}$$

Let $R := o[[\Gamma(\bar{G})]]$. It is a profinite local noetherian o-algebra in $\hat{C}$ endowed with a continuous homomorphism $\Gamma(\bar{G}) \to R^\times$.

After these preliminaries let $\bar{\chi} : G \to k^\times$ be the continuous homomorphism describing the G-action on N. Composing $\bar{\chi}$ and the map (12) with the splitting s and the homomorphism $\Gamma(\bar{G}) \to R^\times$ respectively we obtain two continuous homomorphisms $\chi_0 : G \to o^\times$ and $\gamma : G \to R^\times$.

Proposition 3.11. *The homomorphism*

$$\chi^{univ} = \chi_0 \cdot \gamma : G \longrightarrow R^\times$$

equals the universal deformation.

Proof: The following argument is a generalization of [Gou01], Prop. 3.13. Fix a compact open subgroup $H \subseteq G$. First note that $H \xrightarrow{\bar{\chi}} k^\times \to o^\times$ is a continuous homomorphism between profinite groups whence a continuous algebra homomorphism $R[[H]] \to R[[o^\times]]$. The homomorphism $o^\times \to R^\times$ induced by the algebra structure of R is continuous whence a continuous algebra homomorphism $R[[o^\times]] \to R$ by the universal property of o[[.]] applied to the profinite ring R. The composite

$$R[[H]] \longrightarrow R[[o^\times]] \longrightarrow R$$

then coincides with χ_0 on the subring $R[H]$. Secondly, the continuous map γ restricted to H has image in $1+\mathfrak{m}_R \subseteq R^\times = k^\times \times (1+\mathfrak{m}_R)$ since $\Gamma(\bar{G})$ is a pro-p group. On the one hand, this yields a continuous algebra homomorphism

$$R[[H]] \longrightarrow R[[1+m_R]] \longrightarrow R$$

that coincides with γ on the subring $R[H]$. We therefore see $\chi^{univ} \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(R)^{\mathrm{fl}}$. On the other hand, it shows that χ^{univ} is a deformation of $\bar{\chi}$.

Now suppose $[\chi]$ is a deformation of $\bar{\chi}$ to a noetherian ring $A \in \hat{C}$. Since $1+m_A$ is an abelian pro-p group the continuous homomorphism $\chi_0^{-1} \cdot \chi : G \to 1+m_A$ factores through $G \to \Gamma(\bar{G})$. Indeed, it evidently factores through $G \to \bar{G}$ inducing two continuous maps into $1+m_A$: on the source $\mathbb{Z}^r$ with the discrete topology and on the source $\bar{G}_0$ with the induced topology as open subgroup of $\bar{G}$. By a straightforward argument these maps remain continuous when giving both sources the topology coming from the families of subgroups of finite p-power index. This yields the claim.

All in all we obtain a continuous map

$$f_\chi : R = o[[\Gamma(\bar{G})]] \longrightarrow o[[1+m_A]] \longrightarrow A$$

that specializes χ^{univ} to χ. It follows that χ^{univ} is the universal deformation. $\square$

We give an example in which our hypothesis on $\bar{G}$ is satisfied.

Proposition 3.12. *Let $\mathbb{G}$ denote a connected reductive group over $\mathbb{Q}_p$ and let G denote the group of its $\mathbb{Q}_p$-rational points. Then $\bar{G}$ is topologically finitely generated.*

Proof: If Z denotes the center of G the natural homomorphism $Z \times [G,G] \to G$ has finite kernel and cokernel. Being a torus Z is topologically finitely generated and, hence, so is $\bar{G}$. $\square$

Example: If $G = \mathrm{GL}_n(\mathbb{Q}_p)$ we have

$$\bar{G} = G/[G,G] = \mathbb{Q}_p^\times = p^{\mathbb{Z}} \times U \times \mu$$

where $U = 1 + p^\kappa \mathbb{Z}_p$ with $\kappa = 2$ if $p = 2$ and 1 otherwise and where μ denotes the subgroup of roots of unity in $\mathbb{Q}_p^\times$. The group U is topologically isomorphic to $\mathbb{Z}_p$ whence

$$\Gamma(\bar{G}) = \mathbb{Z}_p \times \mathbb{Z}_p \times \mu'$$

with μ' the group with κ elements. Thus $R = o[[x_1, x_2]] \otimes_o o[\mu']$.

3.7. **Deformation conditions.** It is a formality that the usual formalism of *deformation conditions* works in our setting. We suppose in the following that the deformation functor D_N is representable.

Given $A \in \hat{C}$ assume that some elements of $D_N(A)$ have been designated to be "of type $\mathcal{P}$" and that this property is preserved under the base change $D_N(\phi)$ associated to morphisms $\phi : A \to B$ in $\hat{C}$. We obtain a subfunctor

$$\mathcal{D}_N \subseteq D_N$$

by putting $\mathcal{D}_N(A) := \{M \in D_N(A) : M \text{ of type } \mathcal{P}\}$ for $A \in \hat{C}$.

Proposition 3.13. *The following conditions are necessary and sufficient for the representability of $\mathcal{D}_N$:*

1. *$N \in D_N(k)$ has property $\mathcal{P}$.*
2. *Given a diagram $A_1 \to A_0 \leftarrow A_2$ in C, any deformation of N to the fiber product $A_1 \times_{A_0} A_2$ whose base changes to A_1 and A_2 are of type $\mathcal{P}$ is of type $\mathcal{P}$.*

(3) *If $A \in \hat{C}$ is an inverse limit of objects A_i in C and the basechange to A_i of a deformation M of N to A is of type $\mathcal{P}$ for each i then M is of type $\mathcal{P}$.*

Proof: Granting the representability of D_N the first two conditions are tantamount to the fact that $\mathcal{D}_N$ is left-exact and the third condition asserts that $\mathcal{D}_N$ preserves arbitrary inverse limits. The claim is therefore a direct consequence of Thm. 3.4. $\square$

Let us give two examples of prominent deformation conditions. Write $Z \subseteq G$ for the center of G. Let $A \in C$. We say a deformation M of N to A *has a central character* if Z acts on M via a group homomorphism $Z \to A^\times$. If $Z = \mathbb{Q}_p^\times$ so that $p \in Z$ we say a deformation M of N to A has *uniformizer acting trivially* if p acts trivially on M. We denote these properties by $\mathcal{P}_i, i = 1, 2$ respectively.

Proposition 3.14. *The deformation functor corresponding to $\mathcal{P}_i$ is representable if and only if N is of type $\mathcal{P}_i$.*

Proof: We treat the case $\mathcal{P}_1$ of a central character. The other case is similar. By definition of the G-action $\mathcal{P}_1$ is preserved under base change and it remains to show that property (1) implies (2)-(3) above. Let $A_3 = A_1 \times_{A_0} A_2, M \in D_N(A_3)$ and χ_i the central character of the base change $M_i \in D_N(A_i)$. By topological freeness and compatibility between base change and direct product we are reduced to $M = A_3$. Recalling that A_3 equals the equalizer in $A_1 \times A_2$ of the maps $A_i \to A_0$ the character $\chi_1 \times \chi_2 : Z \to A_1^\times \times A_2^\times$ is seen to take values in $A_3^\times$. Hence, we obtain (2) and (3) follows by a similar argument. $\square$

4. Proof of the main result

We fix an object $A \in \hat{C}$ and a compact open subgroup $H \subseteq G$. We identify once and for all $N \simeq k^I$ in $\mathfrak{PM}(k)$ by means of a pseudobasis for N. As already observed the cardinality $|I|$ of such a basis is an invariant of N. Let k be finite.

Recall the topological ring

$$M_I(A) := \mathrm{End}_{\mathfrak{PM}(A)}(A^I)$$

endowed with the compact-open topology. According to [Bou89], X.§3.4 Thm. 3 a jointly continuous action

$$A[[H]] \times A^I \longrightarrow A^I$$

is the same as a continuous A-algebra homomorphism $A[[H]] \to M_I(A)$. Having this in mind we follow a strategy of B. Mazur (cf. [Maz89]) to rewrite the functor D_N in a more accessible way. Namely, let E_A denote the set of pseudocompact augmented G-representations on the A-module A^I that lift N. It is evidently functorial in A.

Let $\mathrm{GL}_I(A) := M_I(A)^\times$ be the group of units in $M_I(A)$. By what we have just said we may think of an element of E_A as a commutative diagram

$$\begin{array}{ccc} A[[H]] & \xrightarrow{cont} & M_I(A) \\ \uparrow \subseteq & & \uparrow \\ A[H] & \xrightarrow{\subseteq} & A[G] \end{array} \tag{13}$$

that reduces via $A \to k$ to the corresponding diagram for N. Note here, that the G-action on any $M \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ is necessarily by continuous automorphisms (cf. Lem. 3.1) whence the right vertical arrow. The group $\mathrm{GL}_I(A)$ acts on $M_I(A)$

from the left via conjugation. By acting on the right-upper corner of diagrams (13) this induces an action of the subgroup

$$G_A := \ker\big(\mathrm{GL}_I(A) \to \mathrm{GL}_I(k)\big) = 1 + \prod_I \oplus_I m_A$$

on the set E_A.

Lemma 4.1. *There is a bijection*

$$E_A/G_A \xrightarrow{\cong} D_N(A)$$

natural in A.

Proof: Let $\pi : A \to k$ be the residue homomorphism. The map $A^I \mapsto [A^I, 1 \otimes \pi^I]$ induces an injective map from E_A/G_A to $D_N(A)$. It is surjective by Lem. 3.6. □

Let us now assume that N has only scalar endomorphisms. Let $\bar{\rho}$ denote the corresponding element in E_k. Given $\rho \in E_A$ write

$$C(\rho) \subseteq M_I(A)$$

for the A-algebra equal to the centralizer in $M_I(A)$ of the image of $\rho : A[G] \to M_I(A)$. Recall that a surjection $\phi : A \to B$ in C is called a *small extension* if $\ker\phi$ equals a nonzero principal ideal which is annihilated by the maximal ideal of A. It is well-known that every surjection in C factors into a finite composite of small extensions (e.g. [Gou01], Problem 3.1).

Lemma 4.2. *Let $A \in C$. One has $C(\rho) = A$ for all $\rho \in E_A$.*

Proof: If $A \to B$ is a small extension in C with kernel generated by $t \in A$ and $\rho \in E_A$ then the isomorphism (2) shows that $\ker\big(M_I(A) \to M_I(k)\big)$ is killed by t. Furthermore, $C(\bar{\rho}) = k$ since N has only scalar endomorphisms. Taken these facts together the claim is a straightforward generalization of the arguments given in the proof of [Gou01], Lem. 3.8. □

As a corollary the functor D_N is *continuous* in the usual sense:

Corollary 4.3. *Given $A = \varprojlim_n A_n \in \hat{C}$ the natural map*

$$D_N(A) \xrightarrow{\cong} \varprojlim_n D_N(A_n) \tag{14}$$

is a bijection.

Proof: By Lem. 3.3 we may assume that A_n is an artinian quotient of A so that $A_{n+1} \to A_n$ is surjective for all n. It follows that the maps

$$G_{A_{n+1}} \longrightarrow G_{A_n} \tag{15}$$

are surjective for all n. To check surjectivity of the map (14) let $([M_n, \alpha_n])_n$ be an element of the projective limit. A straightforward argument, using the surjectivity of (15) shows the existence of isomorphisms $\beta_n : M_{n+1} \otimes_{A_{n+1}} A_n \xrightarrow{\cong} M_n$ compatible with the α_n. Passing to the projective limit using $\varprojlim_n A_n[[H]] = A[[H]]$ (and similarly for $A[H], A[G]$) yields a pseudocompact augmented G-representation on $M := \varprojlim_{\beta_n} M_n$. By the lemma below M is topologically free over A and therefore the desired preimage. For the injectivity let M, M' be representatives of two classes in $D_N(A)$ together with isomorphisms $M_n \simeq M'_n$ for all n which are compatible with reductions. Let ρ, ρ' be the corresponding elements in E_A. A straightforward argument using the surjectivity of

$$C(\rho_{n+1}) \longrightarrow C(\rho_n) \tag{16}$$

(Lem. 4.2) shows that we may assume the isomorphisms $M_n \simeq M'_n$ to be compatible with $A_{n+1} \to A_n$. Passage to projective limits yields an isomorphism $M \simeq M'$. □

The following statement was used in the preceding proof.

Lemma 4.4. *Let $A = \varprojlim_n A_n \in \hat{C}$ with $A_n \in C$ an artinian pseudocompact quotient of A for all n. Let $(M_n)_n$ be a projective system where each M_n is a pseudocompact topologically free A_n-module. The transition map $M_{n+1} \to M_n$ is supposed to be continuous and compatible with $A_{n+1} \to A_n$. Then $M := \varprojlim_n M_n$ equipped with the projective limit topology is a pseudocompact topologically free A-module.*

Proof: Via the quotient map $A \to A_n$ we may view each M_n as a pseudocompact A-module. It follows that M is a pseudocompact A-module and, according to the proof of [Gab70], Prop. 0.3.7, that M is topologically free over A if the autofunctor $(.)\,\hat{\otimes}_A M$ on $\mathfrak{PM}(A)$ is exact. So suppose that

$$\mathcal{E} : 0 \longrightarrow N' \longrightarrow M' \xrightarrow{\phi} P' \longrightarrow 0$$

is a short exact sequence in $\mathfrak{PM}(A)$. Let $\mathfrak{m}_n$ be the kernel of $A \to A_n$ and put $M'_n = \overline{\mathfrak{m}_n M'}$. Since $\mathfrak{m}_n$ is closed it follows easily from [Gab70], 0.3.2 that $M' = \varprojlim_n M'/M'_n$. Putting $N'_n = N \cap M'_n$ and $P'_n = \phi(M'_n)$ yields the exact sequence

$$\mathcal{E}_n : 0 \longrightarrow N'/N'_n \longrightarrow M'/M'_n \xrightarrow{\phi} P'/P'_n \longrightarrow 0$$

of artinian A-modules for all n. Since $\cap M'_n = 0$ we have $\cap N'_n = \cap P'_n = 0$ and $\varprojlim_n \mathcal{E}_n = \mathcal{E}$ by exactness of $\varprojlim_n$ on $\mathfrak{PM}(A)$. Since $\hat{\otimes}_A$ commutes with projective limits we obtain isomorphisms of topological A-modules

$$\mathcal{E}\,\hat{\otimes}_A M \xrightarrow{\cong} \varprojlim_n \mathcal{E}_n\,\hat{\otimes}_A M_n \xrightarrow{\cong} \varprojlim_n \mathcal{E}_n\,\hat{\otimes}_{A_n} M_n.$$

Since M_n is topologically free over A_n for all n the functor $(.)\,\hat{\otimes}_A M$ is seen to be exact. □

According to Thm. 3.4 and Cor. 4.3 D_N is representable if its restriction to C is left exact. Since $D_N(k)$ is a singleton this reduces to verify that D_N respects fiber products. Let therefore

$$A_3 = A_1 \times_{A_0} A_2$$

be a fiber product in C. Writing $E_i := E_{A_i}$ and $G_i := G_{A_i}$ and invoking Lem. 4.1 we have to show that the natural map of sets

$$b : E_3/G_3 \to E_1/G_1 \times_{E_0/G_0} E_2/G_2 \tag{17}$$

is a bijection.

Lemma 4.5. *The correspondance $A \mapsto M_I(A)$ is a functor from C to topological rings.*

Proof: Let us show that $M_I(\phi)$ is continuous at zero for a morphism $\phi : A \to B$ in C. For an open B-submodule V of B^I let $W(V) \subseteq M_I(B)$ be the subset consisting of homomorphisms with image contained in V (and similar notation for B replaced by A). Since B^I is compact it suffices to see that $M_I(\phi)^{-1}(W(V))$ is open. If U denotes the preimage in A^I of V under ϕ^I then $W(U) \subseteq M_I(\phi)^{-1}(W(V))$. This shows that $M_I(\phi)$ is continuous and the rest is clear. □

Lemma 4.6. *The natural map*

$$M_I(A_3) \xrightarrow{\cong} M_I(A_1) \times_{M_I(A_0)} M_I(A_2) \tag{18}$$

is an isomorphism of topological rings when the target is equipped with the topology induced by the direct product topology on $M_I(A_1) \times M_I(A_2)$.

Proof: The map in question, say ϕ, is certainly a continuous ring homomorphism. We have a chain of canonical isomorphisms of abstract groups

$$M_I(A_1) \times_{M_I(A_0)} M_I(A_2) \simeq \prod_I (\oplus_I A_1) \times_{(\oplus_I A_0)} (\oplus_I A_2) \simeq \prod_I \oplus_I A_3 \simeq M_I(A_3)$$

whence ϕ is bijective. Since all spaces A_i^I are compact it follows easily from the definition of the compact-open topology that the inverse ϕ^{-1} is continuous. □

Lemma 4.7. *The map b is surjective.*

Proof: We adapt an argument of M. Dickinson (cf. [Gou01],Appendix 1) to our situation. For this it will be convenient to think of an augmented G-representation ρ in $E_A, A \in C$ as taking values $\rho(g)$ in infinite $I \times I$-matrices. We shall therefore write suggestively $c\rho c^{-1} := c.\rho$ for $c \in G_A$. Let

$$([\rho], [\sigma]) \in E_1/G_1 \times_{E_0/G_0} E_2/G_2.$$

Let m_0 be the maximal ideal of A_0. Since A_0 is artinian we have $m_0^n = 0$ for some $n \geq 1$. We first prove by induction on n that $[\rho]$ and $[\sigma]$ have representatives in E_1 and E_2 respectively whose images coincide in E_0.

To do this let $\phi : A_1 \to A_0$ and $\psi : A_2 \to A_0$ be the transition maps in the fiber product. By the induction hypothesis we may assume $n > 1$ and that $\phi_*(\rho) = \psi_*(\sigma)$ mod m_0^{n-1}. Here, we abbreviate $\phi_* = E_A(\phi)$ and similarly for ψ. Pick an element $c \in G_0$ such that $c\phi_*(\rho)c^{-1} = \psi_*(\sigma)$ in E_0. We show that there are elements $g \in G_1, h \in G_2$ such that $\phi_*(g\rho g^{-1}) = \psi_*(h\sigma h^{-1})$. This proves the claim.

Reducing c mod m_0^{n-1} centralizes the image of the reduction $\phi_*(\rho)$ mod m_0^{n-1} and therefore (Lem. 4.2) we may assume that $c = 1 + l$ with $l \in \prod_I \oplus_I m_0^{n-1}$. Since l has entries in the finite dimensional k-vector space m_0^{n-1} we may apply *mutatis mutandis* [Gou01], App. 1,Lem. 9.3 and arrive at

$$l = \lambda 1 + \psi(m_2) - \phi(m_1)$$

with a scalar $\lambda \in m_0^{n-1}$ and $m_i \in \prod_I \oplus_I A_i$. From now on the claim follows formally from the computations given in the proof of [loc.cit.], App. 1, Lemma 9.5.

By what we have just shown we may now suppose that $(\rho, \sigma) \in E_1 \times_{E_0} E_2$. The couple gives therefore rise to a continuous homomorphism

$$A_1[[H]] \times_{A_0[[H]]} A_2[[H]] \longrightarrow M_I(A_1) \times_{M_I(A_0)} M_I(A_2)$$

compatible with the G-action. Composing it with the obvious continuous ring homomorphism

$$A_3[[H]] \longrightarrow A_1[[H]] \times_{A_0[[H]]} A_2[[H]]$$

as well as the inverse of the map (18) yields a preimage in E_3. □

Let again $A_3 = A_1 \times_{A_0} A_2$ in C. For $\rho_i \in E_i$ write

$$G(\rho_i) \subseteq G_i$$

for the stabilizer of ρ_i in G_i. According to Lem. 4.2

$$G(\rho_i) = \big(1 + \prod_I \oplus_I m_{A_i}\big) \cap C(\rho_i) = 1 + m_{A_i}.$$

Lemma 4.8. *The map b is injective.*

Proof: Assume $b([\rho]) = b([\sigma])$ with $[\rho], [\sigma] \in E_3/G_3$. For $\rho \in E_3$ write ρ_i for the image in E_i and similarly for $\sigma \in E_3$. Pick $(g_1, g_2) \in G_1 \times G_2$ with $\rho_i = g_i.\sigma_i$ in E_i. The "top left entry" of the $I \times I$-matrix of $g_i \in G_i = 1 + \prod_I \oplus_I m_{A_i}$ lies in $1 + m_{A_i} \subseteq A_i^\times$. Multiplying by a scalar we may therefore assume this entry is equal to one. Let $\bar{g}_i$ denote the image of g_i in G_0. Since $\rho_0 = \sigma_0$ we have $\bar{g}_2^{-1}\bar{g}_1 \in G(\rho_0) = 1 + m_{A_0}$. Comparing top left entries we see that $\bar{g}_1 = \bar{g}_2$ whence $(g_1, g_2) \in G_1 \times_{G_0} G_2 = G_3$. This element conjugates σ to ρ whence $[\rho] = [\sigma]$. $\square$

This completes the proof of Thm. 3.8.

5. Functoriality

This section briefly illustrates that the usual functorial properties of the universal deformation ring $R = R(G, o, N)$ hold in our setting. Granting the universal property of R this is almost a formality.

5.1. **Morphisms.** Let I be any set and consider the functor $A \mapsto M_I(A)$ from $\hat{C}$ to topological $M_I(o)$-algebras. Suppose $N \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ with a pseudobasis indexed by I. Fix an isomorphism $N \simeq k^I$. Given any morphism of functors

$$\delta : M_I \longrightarrow M_J$$

we may compose the diagram for N

$$\begin{array}{ccc} k[[H]] & \xrightarrow{cont} & M_I(k) \\ \subseteq\uparrow & & \uparrow \\ k[H] & \xrightarrow{\subseteq} & k[G] \end{array}$$

in the obvious way with $\delta(k)$ and obtain an element $N' \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$. Evidently N' has a pseudobasis indexed by J which is why we refer to this procedure as *change of range* for N. We denote by $\mathrm{Mod}_{G,0}^{\mathrm{pro\,aug}}(k)$ the category consisting of the same objects as $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ but with the following morphisms. Given $N, N' \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ there is a morphism $N \to N'$ if and only if N' arises from N via change of range. In this situation a *morphism*

$$(G, o, N) \longrightarrow (G', o', N')$$

consists by definition of

(a) a morphism $\varphi : G' \to G$ of locally $\mathbb{Q}_p$-analytic groups (*change of group*),
(b) a local homomorphism $\iota : o \to o'$ of commutative complete local noetherian rings making o' a pseudocompact o-algebra (*change of base*),
(c) a morphism $N \to N'$ in $\mathrm{Mod}_{G,0}^{\mathrm{pro\,aug}}(k)$ (*change of range*).

The effect of such a morphism $(G, o, N) \to (G', o', N')$ on universal deformation rings is in complete analogy to the compact case (cf. [Maz89], 1.3) which is why we omit the details here.

Remark: To show the limits of this analogy let k be finite and consider the Pontryagin duality $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k) \simeq \mathrm{Mod}_G^{\mathrm{sm}}(k)$. The left hand side inherits a duality coming from the smooth contragredient. Due to the asymmetry of $M_I(.) = \prod_I \oplus_I(.)$ with respect to "rows and columns" N there does not seem to be a naive duality relating the universal deformation rings of N and its dual (comp. [Maz89], 1.3 (a.2)). For similar reasons there is no naive "determinant" morphism (comp. [loc.cit.], 1.3 (a.3)).

5.2. **Tensor product.** We explain the effect of the tensor product on $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ on universal deformation rings. Again, this is in analogy to loc.cit. but since we will deduce the important proposition 5.1 from it we give some details. Let $N, N' \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$ and $N'' := N \hat{\otimes}_k N' \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k)$. Given $A, A' \in \hat{C}$ with maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ respectively the results of sect. 2 show that for any $A, A' \in \hat{C}$ the o-algebra $A'' := A \hat{\otimes}_o A'$ lies again in $\hat{C}$ (and has maximal ideal $\mathfrak{m}'' = \mathfrak{m} \hat{\otimes} A' + A \hat{\otimes} \mathfrak{m}'$). Given $[M, \alpha]$ and $[M', \alpha]$ in $D_N(A)$ and $D_{N'}(A')$ respectively the compatibility of $\hat{\otimes}$ with direct products shows that

$$M'' := M \hat{\otimes}_o M' \in \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A'')^{\mathrm{fl}}$$

and

$$M''/\overline{\mathfrak{m}''M''} \xrightarrow{\cong} M/\overline{\mathfrak{m}M} \hat{\otimes}_o M'/\overline{\mathfrak{m}'M'} \xrightarrow{\cong} N''$$

where the first isomorphism is canonical and the second induced by $\alpha \otimes \alpha'$. Hence, $[M'', \alpha \otimes \alpha'] \in D_{A''}(N'')$ whence a morphism of functors $D_N \times D_{N'} \to D_{N''}$. If all functors involved are representable the usual Yoneda lemma yields a homomorphism

$$h(N, N') : R'' \longrightarrow R \hat{\otimes}_o R'$$

in $\hat{C}$ such that the system $(N, N') \mapsto h(N, N')$ satisfies the usual commutativity and associativity relations. Now let N, N' and N'' be as above but fix $[M, \alpha] \in D_N(o)$. We obtain a homomorphism (*contraction with a lifting*)

$$R'' \stackrel{h(N,N')}{\longrightarrow} R \hat{\otimes}_o R' \stackrel{M}{\longrightarrow} R'$$

satisfying the usual commutativity relations (loc.cit.). It follows formally from these relations that if $|I| = 1$ (so that the G-action on N is given by a character) this homomorphism is always an isomorphism in $\hat{C}$ (the *twisting morphism* by M) and satisfies the evident homomorphic property in the variable M.

As we have observed before there is a canonical splitting $s : k^\times \to o^\times$ in case k is finite. Granting the discussion above the following proposition follows therefore as in loc.cit.

Proposition 5.1. *Let k be finite and suppose D_N is representable. The universal deformation ring $R(G, o, N)$ depends on N only up to twisting by characters.*

6. Applications to p-adic Banach space representations

6.1. **Unitary Deformations.** We keep the above notations but assume that o equals the ring of integers in a finite extension K of $\mathbb{Q}_p$. In particular, k is finite of characteristic $p > 0$. Let ϖ be a uniformizer for o. Recall that a *Banach space representation of G over K* is a K-Banach space V together with a linear G-action such that the map

$$G \times V \longrightarrow V$$

giving the action is continuous (cf. [ST02]). Together with continuous G-equivariant K-linear maps these objects form a category $\mathrm{Ban}_G(K)$. We denote by $\mathrm{Ban}_G(K)^{\leq 1}$ the category consisting of K-Banach spaces $(V, ||.||)$ such that $||V|| \subseteq |K|$ endowed with a continuous G-action such that $||.||$ is G-invariant (i.e. $||gv|| = ||v||$ for all $g \in G, v \in V$). We let morphisms be G-equivariant norm-decreasing K-linear maps. Elements $(V, ||.||) \in \mathrm{Ban}_G(K)^{\leq 1}$ will be called *unitary* representations.

Lemma 6.1. *Suppose V is a K-Banach space representation of G with a G-invariant norm defining the topology. Then there is an equivalent norm $||.||$ on V which is G-invariant and such that $||V|| \subseteq |K|$.*

Proof: If $||.||'$ denotes the G-invariant norm on V we put

$$||v|| := \inf\{r \in K : r \geq ||v||'\}$$

for $v \in V$. Then $||.||$ is equivalent to $||.||'$ and G-invariant. $\square$

Remark: It follows from this lemma that the full subcategory of $\mathrm{Ban}_G(K)$ consisting of elements V that admit a G-invariant norm is equivalent to the isogeny category of $\mathrm{Ban}_G(K)^{\leq 1}$.

After these preliminaries consider an element $(V, ||.||) \in \mathrm{Ban}_G(K)^{\leq 1}$. The unit ball $V^0 := \{v \in V : ||v| \leq 1\}$ evidently inherits a G-action and

$$\bar{V} := V^0/\varpi V^0$$

defines a smooth G-representation over k. We obtain in this way a functor

$$\mathrm{Ban}_G(K)^{\leq 1} \longrightarrow \mathrm{Mod}_G^{\mathrm{sm}}(k).$$

If $\bar{V}$ is admissible smooth $(V, ||.||)$ is called *admissible*.

Theorem 6.2. *Let ρ be a smooth G-representation over k which admits only scalar endomorphisms (e.g admissible and absolutely irreducible). Let $N = \rho^\vee$. There is a canonical and natural bijection between the o-valued points of $R(G, o, N)$ and the set of isomorphism classes of unitary G-Banach space representations V over K such that $\bar{V} \simeq \rho$.*

Proof: Pick a compact open subgroup $H \subseteq G$. Given $V \in \mathrm{Ban}_H(K)^{\leq 1}$ we may equip the o-module $V^d := \mathrm{Hom}_o(V^0, o)$ with the topology of pointwise convergence and the contragredient H-action. The H-equivariant version of the discussion in [ST02], (proof of) Thm. 1.2 shows that $V \mapsto V^d$ establishes an equivalence of categories

$$(.)^d : \mathrm{Ban}_H(K)^{\leq 1} \xrightarrow{\cong} \mathfrak{PM}(o[[H]])^{\mathrm{fl}}.$$

It is evidently compatible with the restriction functors relative to a compact open subgroup $H' \subseteq H$. Taking into account the G-action it therefore restricts to an equivalence

$$(.)^d : \mathrm{Ban}_G(K)^{\leq 1} \xrightarrow{\cong} \mathrm{Mod}_G^{\mathrm{pro\,aug}}(o)^{\mathrm{fl}}$$

on the faithfully embedded subcategory $\mathrm{Ban}_G(K)^{\leq 1}$. We may now form a diagram of functors

$$\begin{array}{ccc} \mathrm{Ban}_G(K)^{\leq 1} & \xrightarrow[(.)^d]{\sim} & \mathrm{Mod}_G^{\mathrm{pro\,aug}}(o)^{\mathrm{fl}} \\ \Big\downarrow{\scriptstyle V \mapsto \bar{V}} & & \Big\downarrow{\scriptstyle \pi^*} \\ \mathrm{Mod}_G^{\mathrm{sm}}(k) & \xrightarrow[(.)^\vee]{\sim} & \mathrm{Mod}_G^{\mathrm{pro\,aug}}(k) \end{array} \tag{19}$$

where the right perpendicular arrow refers to the base change relative to the residue map $\pi : o \to k$ and the lower horizontal arrow equals Pontryagin duality. A straightforward equivariant version of [Pas10], Lem. 5.4 proves the diagram to be commutative. It follows that the functor $(.)^d$ induces a natural and canonical bijection between the set of isomorphism classes in $\mathrm{Ban}_G(K)^{\leq 1}$ of $(V, ||.||)$ with $\bar{V} \simeq \rho$ and $D_N(o)$. $\square$

6.2. **Principal series representations of** $\mathrm{GL}_2(\mathbb{Q}_p)$**.** To illustrate the above methods we let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and compute the fibers of the reduction map in principal series representations. Besides deformation theory our computations strongly rely on results of M. Emerton concerning the functor of *ordinary parts* (cf. [Eme10a], [Eme10b]). To start with let P and $\overline{P}$ be the Borel subgroup of G consisting of upper triangular and lower triangular matrices respectively. Given two smooth characters $\chi_i : \mathbb{Q}_p^\times \longrightarrow A^\times, i = 1, 2$ for $A \in C$ we view $\chi = \chi_1 \otimes \chi_2$ as a smooth character of the diagonal torus T in G in the obvious way. Define

$$\mathrm{Ind}_{\overline{P}}^G(\chi) = \{f : G \to A | f \text{ locally constant}, f(\overline{p}g) = \chi(\overline{p})f(g), p \in \overline{P}, g \in G\}$$

with G-action by right translations. It is a smooth admissible G-representation over A in the sense of [Eme10a], Def. 2.2.5. Finally, let $\epsilon(a) = a|a| \in \mathbb{Z}_p^\times$ for all $a \in \mathbb{Q}_p^\times$ and write $\bar{\epsilon}$ for the induced smooth character $\mathbb{Q}_p^\times \to k^\times$.

If V^0 denotes the unit ball of an element $(V, ||.||)$ in $\mathrm{Ban}_G(K)^{\leq 1}$ we write $V_n := V^0/\varpi^n V^0$ and $o_n := o/\varpi^n o$ for all n.

Lemma 6.3. *The o_n-module V_n is faithful for all n.*

Proof: By the same argument as in case $n = 1$ the diagram (19) remains commutative when we replace k by o_n, ϖ by ϖ^n and restrict to topologically free o_n-modules in the lower right corner. But then $(V_n)^\vee$ is topologically free and therefore V_n is faithful. □

After these preliminaries let us fix smooth characters $\overline{\chi}_i : \mathbb{Q}_p^\times \longrightarrow k^\times, i = 1, 2$ for which $\bar{\chi}_1\bar{\chi}_2^{-1} \neq 1, \bar{\epsilon}$. The G-representation $\mathrm{Ind}_{\overline{P}}^G(\bar{\chi})$ is then admissible and absolutely irreducible (cf. [BL94]). In particular, it admits only scalar endomorphisms (cf. Lem. 3.10). Suppose our chosen Banach space representation V satisfies

$$\bar{V} = V^0/\varpi V^0 \simeq \mathrm{Ind}_{\overline{P}}^G(\bar{\chi}).$$

Lemma 6.4. *Let $n \geq 1$ and suppose there is an isomorphism $V_n \xrightarrow{\cong} \mathrm{Ind}_{\overline{P}}^G(\overline{\chi}^n)$ with some smooth $o_n^\times$-valued character $\overline{\chi}^{(n)} = \chi_1^{(n)} \otimes \chi_2^{(n)}$. Then there is a smooth $o_{n+1}^\times$-valued character $\overline{\chi}^{(n+1)} = \chi_1^{(n+1)} \otimes \chi_2^{(n+1)}$ and a commutative diagram of smooth G-representations*

$$\begin{array}{ccc} V_{n+1} & \xrightarrow[\varphi_{n+1}]{\sim} & \mathrm{Ind}_{\overline{P}}^G(\overline{\chi}^{n+1}) \\ \downarrow{\scriptstyle mod\ \varpi^n} & & \downarrow{\scriptstyle mod\ \varpi^n} \\ V_n & \xrightarrow[\varphi_n]{\sim} & \mathrm{Ind}_{\overline{P}}^G(\overline{\chi}^n). \end{array}$$

Proof: Granting the above lemma this is a straightforward generalization of [Eme10b], Prop. 4.1.5. □

On the other hand, given a continuous character $\chi : T \to o^\times$, we may define

$$^{\mathrm{c}}\mathrm{Ind}_{\overline{P}}^G(\chi) = \{f : G \to K | f \text{ continuous}, f(\overline{p}g) = \chi(\overline{p})f(g), p \in \overline{P}, g \in G\}$$

with G acting by right translations. Equipped with the supremum norm taken over the compact space $\overline{P}\backslash G$ it constitutes an admissible unitary G-Banach space representation over K, the so-called *ordinary continuous principal series*. Its irreducibility properties are well-known (cf. [Eme06], Prop. 5.3.4). The reduction mod ϖ of its unit ball is of the form $\mathrm{Ind}_{\overline{P}}^G(\bar{\chi})$. Recall that $\kappa = 2$ if $p = 2$ and 1 else. Let μ' denote the group with κ elements.

Theorem 6.5. *If $V \in \mathrm{Ban}_G(K)^{\leq 1}$ such that $\bar{V} \simeq \mathrm{Ind}_{\bar{P}}^{G}(\bar{\chi})$ then $V = {}^{\mathrm{c}}\mathrm{Ind}_{\bar{P}}^{G}(\chi)$ with a lift χ of $\overline{\chi}$ to $o^{\times}$. The isomorphism classes of such V are therefore in bijection to the product of four copies of the maximal ideal (ϖ) with two copies of μ'.*

Proof: The above lemma together with [Eme10a], Lem. 4.1.1 implies that the injective map

$$\chi \mapsto {}^{\mathrm{c}}\mathrm{Ind}_{\bar{P}}^{G}(\chi)$$

from unitary lifts of $\bar{\chi}$ to unitary lifts of $\bar{V}$ is surjective. Using the above theorem it remains to compute $D_N(o)$ where $N = \bar{\chi}^{\vee} = \bar{\chi}^{-1}$. But the example after Cor. 3.12 shows that

$$R(T, o, N) = o[[x_1, x_2, x_3, x_4]] \otimes_o o[\mu' \times \mu'].$$

□

Remark: Let $H^{\bullet}\mathrm{Ord}_P$ denote the δ-functor associated to the functor of ordinary parts Ord_P relative to P. The proof of Lem. 6.4 relies on the fact that

$$H^1\mathrm{Ord}_P(\mathrm{Ind}_{\bar{P}}^{G}(\bar{\chi})) \simeq \bar{\chi}_2\bar{\epsilon}^{-1} \otimes \bar{\chi}_1\bar{\epsilon}$$

as smooth T-representations (cf. [Eme10b], Thm. 4.1.3 (1)). A generalization of the above corollary to other groups than $\mathrm{GL}_2(\mathbb{Q}_p)$ is therefore related to a better understanding of (higher) ordinary parts.

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