VERMA MODULES OVER p-ADIC ARENS-MICHAEL ENVELOPES OF REDUCTIVE LIE ALGEBRAS

TOBIAS SCHMIDT

ABSTRACT. Let K be a locally compact p-adic field, $\mathfrak{g}$ a split reductive Lie algebra over K and $U(\mathfrak{g})$ its universal enveloping algebra. We investigate the category $\mathcal{C}_\mathfrak{g}$ of coadmissible modules over the p-adic Arens-Michael envelope $\hat{U}(\mathfrak{g})$ of $U(\mathfrak{g})$. Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra. The main result gives a canonical equivalence between the classical parabolic BGG category of $\mathfrak{g}$ relative to $\mathfrak{p}$ and a certain explicitly given highest weight subcategory of $\mathcal{C}_\mathfrak{g}$. This completely clarifies the "Verma module theory" over $\hat{U}(\mathfrak{g})$.

CONTENTS

1. INTRODUCTION

Let p be a prime number and let K be a locally compact p-adic field. Let G be a d-dimensional p-adic Lie group defined over K with a split reductive Lie algebra $\mathfrak{g}$. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$.

The Arens-Michael envelope $\hat{U}(\mathfrak{g})$ of $U(\mathfrak{g})$ equals the completion of $U(\mathfrak{g})$ with respect to all submultiplicative seminorms. Being an interesting p-adic power series envelope of $U(\mathfrak{g})$ in its own right it is also an important technical tool in the study of locally analytic G-representations (e.g. [30] for a short introduction). As a ring it is best understood as Fréchet-Stein algebra (in the sense of Schneider-Teitelbaum, cf. [28]), i.e. a noncommutative version of the ring of holomorphic functions on rigid analytic affine d-space. From this angle the coherent module sheaves on affine d-space are generalized to the abelian category $\mathcal{C}_{\mathfrak{g}}$ of *coadmissible* (left) $\hat{U}(\mathfrak{g})$-modules. Due to the close relation of $\hat{U}(\mathfrak{g})$ to the locally analytic distribution algebra of G the latter category is a first approximation to the category of admissible locally analytic G-representations. Besides the short note [26] there are practically no results about the specific structure of $\mathcal{C}_{\mathfrak{g}}$ so far.

In this note we single out certain full subcategories of $\mathcal{C}_{\mathfrak{g}}$ and establish canonical equivalences to certain well-known highest weight categories over $\mathfrak{g}$. In particular, this completely clarifies the "Verma module theory" over $\hat{U}(\mathfrak{g})$. To be more precise, let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra. On the one hand, we then have the well-known parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ in the sense of Bernstein-Gelfand-Gelfand and Rocha-Caridi ([5], [24]). This is a certain full abelian subcategory of finitely generated $\mathfrak{g}$-modules with appropriate finiteness conditions for the action of the Levi subalgebra and the nilpotent radical of $\mathfrak{p}$ respectively. It is known to be artinian and noetherian and allows a block decomposition with respect to the central action. Any block is equivalent to a category of finitely generated modules over a quasi-hereditary finite dimensional K-algebra. The structure of the latter algebras was made explicit by work of W. Soergel ([29]). Prominent objects in $\mathcal{O}^{\mathfrak{p}}$ are the generalized Verma modules in the sense of J. Lepowsky ([19]).

In defining a genuine p-adic counterpart of $\mathcal{O}^{\mathfrak{p}}$ over $\hat{U}(\mathfrak{g})$ we build upon a certain weight theory for topological Fréchet modules over commutative Fréchet algebras ([13]). Applying it to the Arens-Michael envelope $\hat{U}(\mathfrak{h})$ of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ enables us to explicitly define certain highest weight categories $\hat{\mathcal{O}}^{\mathfrak{p}}$ within $\mathcal{C}_{\mathfrak{g}}$ by imposing appropriate compactness conditions on the weights related to $\mathfrak{p}$. The main objects turn out to be certain Verma type modules whose properties closely parallel the classical case. In particular, they admit unique irreducible quotients parametrized by the linear dual of $\mathfrak{h}$ and any irreducible object in $\hat{\mathcal{O}}^{\mathfrak{p}}$ occurs like this. To go further, the existence of the p-adic Harish-Chandra homomorphism ([18]) leads to a decomposition of $\hat{\mathcal{O}}^{\mathfrak{p}}$ with respect to central characters χ of $\hat{U}(\mathfrak{g})$. The blocks $\hat{\mathcal{O}}^{\mathfrak{p}}_{\chi}$ are noetherian and artinian. This makes possible to prove the following main result.

As with any Arens-Michael envelope there is a natural map $U(\mathfrak{g}) \to \hat{U}(\mathfrak{g})$. We show that base change along this map induces an equivalence of categories $\mathcal{O}^{\mathfrak{p}} \xrightarrow{\cong} \hat{\mathcal{O}}^{\mathfrak{p}}$ which preserves the central blocks, the Verma modules and their irreducible quotients. A quasi-inverse can be given explicitly. The proof of our main result builds on results of [26] and well-known properties of the categories $\mathcal{O}^{\mathfrak{p}}$.

We assemble some information on quasi-hereditary algebras and BGG-reciprocity in an appendix.

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2. Diagonalisable modules

We begin by reviewing a notion of semisimplicity for topological Fréchet-modules (developed by T. Féaux deLacroix, cf. [13]). The exposition is adapted to our purposes. For all notions of nonarchimedean functional analysis we refer to P. Schneider's monograph [27].

Let K be a locally compact p-adic field and $\mathcal{H}$ a commutative K-algebra. Let $\mathcal{H}^*$ denote the set of K-*valued weights* of $\mathcal{H}$, i.e. the set of K-algebra homomorphisms $\mathcal{H} \to K$. A subset $Y \subseteq \mathcal{H}^*$ is called *relatively compact* if there are finitely many elements $h_1, ..., h_l$ in $\mathcal{H}$ such that the map

$$Y \longrightarrow K^l, \ \lambda \mapsto (\lambda(h_1), ..., \lambda(h_l))$$

is injective with relatively compact image. Let $\mathcal{M}(\mathcal{H})$ denote the category whose objects are K-Fréchet spaces M endowed with an action of $\mathcal{H}$ by continuous K-linear endomorphisms. Morphisms are continuous K-linear maps compatible with $\mathcal{H}$-actions.

Let $\lambda \in \mathcal{H}^*$. Following [13] a nonzero $m \in M$ is called a λ-*weight vector* if $h.m = \lambda(h).m$ for all $h \in \mathcal{H}$. In this case λ is called a *weight of M*. The closure M_λ in M of the K-vector space generated by all λ-weight vectors is called the λ-*weight space* of M. The module M is called $\mathcal{H}$-*diagonalisable* if there is a set of weights $\Pi(M) \subseteq \mathcal{H}^*$ with the property: to every $m \in M$ there exists a family $\{m_\lambda \in M_\lambda\}_{\lambda \in \Pi(M)}$ converging cofinite against zero in M and satisfying

$$m = \sum_{\lambda \in \Pi(M)} m_\lambda.$$

Given an $\mathcal{H}$-diagonalisable module M we may form the abstract $\mathcal{H}$-module

$$M^{ss} = \oplus_{\lambda \in \Pi(M)} M_\lambda$$

(depending on the choice of $\Pi(M)$).

Proposition 2.0.1. *Let M be $\mathcal{H}$-diagonalisable with a relatively compact set of weights $\Pi(M)$. The following hold:*

(i) *Given $m = \sum_{\lambda \in \Pi(M)} m_\lambda$ in M the weight components m_λ are uniquely determined by m. If M is contained in a closed $\mathcal{H}$-invariant subspace of M then so are all m_λ.*

(ii) *M has no other weights besides the set $\Pi(M)$.*

(iii) *Suppose additionally that $\dim_K M_\lambda < \infty$ for all $\lambda \in \Pi(M)$. The map*

$$(*) \quad N \mapsto N \cap M^{ss}$$

induces an inclusion preserving bijection between the $\mathcal{H}$-invariant closed subspaces of M and the abstract $\mathcal{H}$-invariant subspaces of M^{ss}. The inverse is given by passing to the closure in M.

(iv) *If in the situation of (iii) M admits additionally an action of a K-algebra $\mathcal{H} \subseteq \mathcal{A}$ that stabilizes M^{ss} then the bijection $(*)$ descends to $\mathcal{A}$-invariant objects.*

Proof. This follows from Satz 1.3.19 and Kor. 1.3.22 of [13]. Note that K-Fréchet spaces are Hausdorff, complete and barrelled. $\square$

We let $\mathcal{D}(\mathcal{H})$ denote the full subcategory of $\mathcal{M}(\mathcal{H})$ whose objects are $\mathcal{H}$-diagonalisable modules M over a relatively compact set of weights $\Pi(M) \subseteq \mathcal{H}^*$ with finite dimensional weight spaces M_λ. By the proposition, given $M \in \mathcal{D}(\mathcal{H})$ the definition of M^{ss} depends solely on M and coincides with the *socle* of the abstract $\mathcal{H}$-module M. Let Vec_K be the category of abstract K-vector spaces. The following proposition is easily checked.

Proposition 2.0.2. *The forgetful functor to Vec_K endowes $\mathcal{D}(\mathcal{H})$ with the structure of exact category. The latter is stable under passage to closed $\mathcal{H}$-invariant subspaces and to the corresponding quotients. The functor on $\mathcal{D}(\mathcal{H})$*

$$M \mapsto M^{ss}$$

into the category of abstract $\mathcal{H}$-modules is faithful and exact.

3. Highest weight categories

Let p be a prime number. Throughout this section K denotes a locally compact p-adic field. Let $|.|$ be a nonarchimedean valuation on K with $|p| = p^{-1}$ inducing its topology.

3.1. **Fréchet-Stein algebras.** In [28] P. Schneider and J. Teitelbaum introduce the notion of Fréchet-Stein algebra and show that locally analytic distribution algebras of compact p-adic Lie groups are of such type (see remark below). Since Arens-Michael envelopes of Lie algebras over K are another example of this type (see below) we briefly review the definition.[1] A K-Fréchet algebra A is called (two-sided) *Fréchet-Stein* if there is a sequence $q_1 \leq q_2 \leq ...$ of algebra norms on A defining its Fréchet topology and such that for all $m \in \mathbb{N}$ the completion A_m of A with respect to q_m is a left and right noetherian K-Banach algebra and a flat left and right A_{m+1}-module via the natural map $A_{m+1} \to A_m$. Any such algebra A gives rise to a certain full subcategory $\mathcal{C}_A$ of all (left) A-modules, the *coadmissible modules*. As Fréchet-Stein algebras are typically non-noetherian $\mathcal{C}_A$ serves as a well-behaved replacement for the category of all finitely generated (left) A-modules. Instead of giving all details of the construction (cf. [28], §3) we summarize some basic properties of $\mathcal{C}_A$ in the following proposition.

Proposition 3.1.1. *Let A be a Fréchet-Stein algebra.*
- (i) *The direct sum of two coadmissible A-modules is coadmissible.*
- (ii) *the (co)kernel and (co)image of an arbitrary A-linear map between coadmissible A-modules is coadmissible.*
- (iii) *the sum of two coadmissible submodules of a coadmissible A-module is coadmissible.*
- (iv) *any finitely generated submodule of a coadmissible A-module is coadmissible.*
- (v) *any finitely presented A-module is coadmissible.*
- (vi) *$\mathcal{C}_A$ is an abelian category.*
- (vii) *any coadmissible A-module M is equipped with a canonical Fréchet topology making it a topological A-module. Any A-linear map between two coadmissible A-modules is continuous and strict with closed image with respect to canonical topologies.*

Proof. [28], Cor. 3.4/3.5 and Lem. 3.6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Let A be a Fréchet-Stein algebra. We will make much use of the following basic property of the canonical topology.

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[1]Our definition is adapted to our purposes and slightly more restrictive than in [28].

Lemma 3.1.2. *For any coadmissible A-module M and any abstract A-submodule $N \subseteq M$ the following are equivalent:*

    (i) *N is coadmissible.*
    (ii) *M/N is coadmissible.*
    (iii) *N is closed in the canonical topology of M.*

Proof. [28], Lem. 3.6. □

Remark 3.1.3. Let G denote a locally K-analytic group. With respect to the convolution product the strong dual $D(G)$ of the K-vector space of locally analytic functions on G is a topological algebra, the so-called *locally analytic distribution algebra* of G. The main result of [loc.cit.] proves that, in case G is compact, $D(G)$ is a Fréchet-Stein algebra. This enables the authors to develop a general theory of *admissible* locally analytic G-representations generalizing the classical notion ([7]) of an *admissible* smooth G-representation.

The theory is modelled according to the example $G = \mathbb{Z}_p$, the additive group of p-adic integers. In this case, the Fourier isomorphism of Y. Amice ([1]) identifies $D(G)$ with the ring of holomorphic functions on the rigid analytic open unit disc. The latter is a quasi-Stein space in the sense of R. Kiehl ([17]).

3.2. **Arens-Michael envelopes.** An *Arens-Michael K-algebra* is a locally convex K-algebra topologically isomorphic to a projective limit of K-Banach algebras. For the theory of such algebras (over the complex numbers) we refer to the book by A.Y. Helemskii ([14], chap. V). Given a locally convex K-algebra A its *Arens-Michael envelope* $\hat{A}$ equals the Hausdorff completion of A with respect to the family of all continuous submultiplicative seminorms on A. It is universal with respect to continuous K-algebra homomorphisms of locally convex K-algebras into Arens-Michael algebras. It comes equipped with a continuous algebra homomorphism

$$A \longrightarrow \hat{A}$$

with dense image. This construction gives a functor

$$A \mapsto \hat{A}$$

from locally convex K-algebras to Arens-Michael algebras which is compatible with projective tensor products and passage to quotients by twosided ideals (cf. [22], 6.1 for the complex case; the proofs generalize).

Let $\mathfrak{g}$ be a Lie algebra over K of dimension d and let $U(\mathfrak{g})$ be its universal enveloping algebra endowed with the finest locally convex topology. Let $\hat{U}(\mathfrak{g})$ be its Arens-Michael envelope. It will be convenient to realize $\hat{U}(\mathfrak{g})$ in the following explicit way. Fix a K-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}$. Using the associated PBW-basis for $U(\mathfrak{g})$ we may define for each $r > 0$ a vector space norm on $U(\mathfrak{g})$ via

$$(3.2.0) \qquad\qquad ||\sum_\alpha d_\alpha \mathfrak{X}^\alpha||_{\mathfrak{x},r} = \sup_\alpha |d_\alpha| r^{|\alpha|}$$

where $\mathfrak{X}^\alpha := \mathfrak{x}_1^{\alpha_1} \cdots \mathfrak{x}_d^{\alpha_d}$, $\alpha \in \mathbb{N}_0^d$ and $|\alpha| := \alpha_1 + \cdots + \alpha_d$.

Proposition 3.2.1. *The Hausdorff completion of $U(\mathfrak{g})$ with respect to the family of norms $||.||_{\mathfrak{x},r}$, $r > 1$ is an Arens-Michael algebra. The canonical homomorphism from $\hat{U}(\mathfrak{g})$ into it is a topological algebra isomorphism.*

Proof. It is easy to see that each norm $||.||_{\mathfrak{x},r}$, $r > 1$ is submultiplicative and that this family is cofinal in the directed set of all submultiplicative seminorms on $U(\mathfrak{g})$ (cf. [26]) $\qquad\square$

Remark 3.2.2. In analogy to the complex hyperenveloping algebra introduced by P.K. Rasevskii (cf. [23]) the completion of $U(\mathfrak{g})$ with respect to the norms $||.||_{\mathfrak{x},r}$, $r > 1$ is sometimes called the *p-adic hyperenveloping algebra* of $\mathfrak{g}$ ([26],[30]).

The above discussion shows that we have a functor

$$\mathfrak{g} \mapsto \hat{U}(\mathfrak{g})$$

from finite dimensional Lie algebras over K to Arens-Michael K-algebras satisfying the obvious compatibilities with respect to products/projective tensor products and passage to quotients. It is immediate that everything we said above may be applied *mutatis mutandis* to the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$.

Proposition 3.2.3. *The algebras $\hat{U}(\mathfrak{g})$ and $\hat{S}(\mathfrak{g})$ are Fréchet-Stein algebras which are integral domains. The K-linear isomorphism $U(\mathfrak{g}) \simeq S(\mathfrak{g})$ induced by the choice of basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ extends to a topological isomorphism $\hat{U}(\mathfrak{g}) \simeq \hat{S}(\mathfrak{g})$.*

Proof. [25], Thm. 2.3 and [26], Thm. 2.1. The corresponding noetherian Banach algebras arise as the completions with respect to single norms $||.||_{\mathfrak{x},r}$. $\qquad\square$

For future reference we denote the completion of $U(\mathfrak{g})$ with respect to the norm $||.||_{\mathfrak{x},r}$ by $U_r(\mathfrak{g})$. It is a noetherian Banach algebra and the natural map $\hat{U}(\mathfrak{g}) \to U_r(\mathfrak{g})$ is flat ([28], Remark 3.2).

Let V and W be two locally convex K-spaces. We denote the completed projective tensor product of V and W over K by $V \hat{\otimes}_K W$.

Lemma 3.2.4. *Suppose $\mathfrak{g}_1, ...\mathfrak{g}_n$ are Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathfrak{g}$ as K-vector spaces. There exists a unique isomorphism f of topological bimodules*

$$\hat{U}(\mathfrak{g}_1) \hat{\otimes}_K \cdots \hat{\otimes}_K \hat{U}(\mathfrak{g}_n) \xrightarrow{\cong} \hat{U}(\mathfrak{g})$$

such that $f(u_1 \hat{\otimes} \cdots \hat{\otimes} u_n) = u_1 \cdots u_n$ for $u_i \in \hat{U}(\mathfrak{g}_i)$. Similarly for S instead of U.

Proof. We have the usual PBW-isomorphism of bimodules

$$U(\mathfrak{g}_1) \otimes_K \cdots \otimes_K U(\mathfrak{g}_n) \xrightarrow{\cong} U(\mathfrak{g})$$

([9], Prop. 2.2.10) and similarly for S. In the case of S the latter is even an isomorphism of K-algebras and compatibility of Arens-Michael envelopes with projective tensor products yields the claim. The second claim of Prop. 3.2.3 applied to all algebras $\mathfrak{g}_1, ..., \mathfrak{g}_n$ and $\mathfrak{g}$ then yields the claim for U. $\qquad\square$

We conclude this paragraph with some remarks in case $\mathfrak{g}$ is abelian. By the universal property of the Arens-Michael envelope any weight $\hat{U}(\mathfrak{g}) \to K$ (cf. sect. 2) is automatically continuous. The map

$$\hat{U}(\mathfrak{g})^* \xrightarrow{\cong} \mathfrak{g}^*, \lambda \mapsto [\mathfrak{x} \mapsto \lambda(\mathfrak{x})]$$

therefore identifies the set $\hat{U}(\mathfrak{g})^*$ canonically with the K-linear dual $\mathfrak{g}^*$ of $\mathfrak{g}$. This identification is compatible with the isomorphism of locally convex K-algebras (cf. prop. 3.2.3)

$$(3.2.4) \qquad\qquad \mathfrak{X} : \hat{U}(\mathfrak{g}) \xrightarrow{\cong} \mathcal{O}(\mathbb{A}_K^{d,an})$$

mapping a chosen Lie algebra basis $\mathfrak{X} := \{\mathfrak{x}_1, ... \mathfrak{x}_d\}$ to a system of coordinates on $\mathbb{A}_K^{d,an}$. Here, $\mathbb{A}_K^{d,an}$ denotes the rigid analytic affine d-space over K ([6], 9.3.4) and $\mathcal{O}(\mathbb{A}_K^{d,an})$ equals its ring of rigid analytic functions viewed as a locally convex algebra in the usual way.

3.3. Reductive Lie algebras.
From now on $\mathfrak{g}$ is a split reductive Lie algebra over K. We refer to [9] for the basic structure of such algebras. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{b}$ be a Borel subalgebra containing $\mathfrak{h}$, Φ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and Φ^+ and Δ the set of positive and simple roots, respectively. Let W denote the Weyl group of Φ. Denote by $\mathfrak{n}$ and $\mathfrak{n}^-$ the nilpotent radicals of $\mathfrak{b}$ and $\mathfrak{b}^-$ respectively. We have $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and $\mathfrak{h} \simeq \mathfrak{b}/\mathfrak{n}$ canonically. Let $\mathfrak{h}^*$ denote the K-linear dual and put $l := \dim_K \mathfrak{h}$. For each root $\alpha \in \Phi$ let $\mathfrak{g}_\alpha$ be the one dimensional root space in $\mathfrak{g}$. Finally, we let $\Lambda_r \subseteq \Lambda$ be the root lattice and the integral weight lattice respectively. Λ contains the subsemigroup of dominant integral weights Λ^+.

In the following we will fix a Chevalley basis $\{\mathfrak{x}_\beta, \beta \in \Phi, h_\alpha, \alpha \in \Delta\}$ of the derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. We fix once and for all a K-basis for the center $\mathfrak{c}$ of $\mathfrak{g}$ which, together with $\{h_\alpha\}_{\alpha \in \Delta}$ gives rise to a K-basis of $\mathfrak{h}$. Throughout this work we will work with this fixed choice of K-basis of $\mathfrak{h}$ and call it $\mathfrak{H}$.

3.4. Generalized Verma modules.
Generalized Verma modules (GVM) for parabolic subalgebras of reductive Lie algebras were first introduced by J. Lepowsky ([19]). For an extensive treatment of such modules we refer to V. Mazorchuk's monograph ([20]).
Let $\mathfrak{p}_I$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and let $I \subseteq \Delta$ be the associated subset of simple roots. Let $\Phi_I \subseteq \Phi$ be the corresponding root system with positive roots Φ_I^+, negative roots Φ_I^- and Weyl group $W_I \subseteq W$. Let

$$\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$$

be a Levi decomposition of $\mathfrak{p}_I$ with Levi subalgebra $\mathfrak{l}_I$ and nilpotent radical $\mathfrak{u}_I \subseteq \mathfrak{b}$. Since $\mathfrak{l}_I$ is reductive there is a further decomposition

$$\mathfrak{l}_I = \mathfrak{g}_I \oplus \mathfrak{z}_I$$

where $\mathfrak{g}_I$ and $\mathfrak{z}_I$ denote the derived algebra and the center of $\mathfrak{l}_I$ respectively.

Setting $\mathfrak{h}_I := \oplus_{\alpha \in I} K h_\alpha$ defines a Cartan subalgebra of the semisimple algebra $\mathfrak{g}_I$ such that

$$\mathfrak{h} = \mathfrak{h}_I \oplus \mathfrak{z}_I.$$

This induces a decomposition $\mathfrak{h}^* = \mathfrak{h}_I^* \oplus \mathfrak{z}_I^*$. For each $\lambda \in \mathfrak{h}^*$ denote by λ_I its projection to $\mathfrak{h}_I^*$. Finally, let $\alpha^\vee$ denote the dual root to a given $\alpha \in \Phi$ and define

$$\Lambda_I^+ := \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in I\}.$$

Let $\lambda \in \Lambda_I^+$. Denote by $L_I(\lambda)$ the irreducible finite dimensional $\mathfrak{g}_I$-module with highest weight λ_I viewed as $\mathfrak{l}_I$-module (by letting $\mathfrak{z}_I$ act through the projection of λ to $\mathfrak{z}_I^*$). The generalized

Verma module (GVM) of weight λ relative to $\mathfrak{p}_I$ is the left $U(\mathfrak{g})$-module

$$M_I(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} L_I(\lambda)$$

where $L_I(\lambda)$ is inflated to a $\mathfrak{p}_I$-module via the projection $\mathfrak{p}_I \to \mathfrak{p}_I/\mathfrak{u}_I = \mathfrak{l}_I$.

In case $I = \emptyset$ let $M(\lambda), N(\lambda), L(\lambda)$ be the classical Verma module of weight λ, its unique maximal submodule and its unique irreducible quotient respectively. The module $M_I(\lambda)$ is generated by a maximal vector $m^+ \in M_I(\lambda)$ (i.e. $\mathfrak{n}.m^+ = 0$) and, thus, admits a surjection $M(\lambda) \to M_I(\lambda)$. Hence $L(\lambda)$ equals the unique irreducible quotient of $M_I(\lambda)$.

3.5. **The parabolic BGG category.** We keep the notation of the preceding subsections and fix a parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_I$ of $\mathfrak{g}$. Let $\mathrm{Mod}(U(\mathfrak{g}))$ denote the category of all left $U(\mathfrak{g})$-modules. Let us recall the parabolic BGG category [2] in the sense of A. Rocha-Caridi ([24]). It equals the full subcategory of $\mathrm{Mod}(U(\mathfrak{g}))$ consisting of modules M such that

  (i) M is finitely generated as $U(\mathfrak{g})$-module;
  (ii) viewed as a $U(\mathfrak{l}_I)$-module, M is the direct sum of finite dimensional simple modules;
  (iii) M is locally $\mathfrak{u}_I$-finite.

Here, (iii) means that $U(\mathfrak{u}_I)m$ is finite dimensional for each $m \in M$.

It is known that $\mathcal{O}^{\mathfrak{p}}$ is a K-linear, abelian, artinian and noetherian category which is closed under submodules and quotients ([15]). There are two extreme cases: the case $I = \emptyset$ recovers the classical category $\mathcal{O}$ in the sense of Bernstein-Gelfand-Gelfand ([4]), while $I = \Delta$ yields the (semisimple) category of finite dimensional $U(\mathfrak{g})$-modules. Obviously, $\mathfrak{p} \subset \mathfrak{p}'$ implies $\mathcal{O}^{\mathfrak{p}'} \subset \mathcal{O}^{\mathfrak{p}}$. We summarize a few more properties of $\mathcal{O}^{\mathfrak{p}}$ in the following theorem. Let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$ and χ a central character $Z(\mathfrak{g}) \to K$.

Theorem 3.5.1.   (i) *The modules $M_I(\lambda), \lambda \in \Lambda_I^+$ belong to $\mathcal{O}^{\mathfrak{p}}$;*
  (ii) *the modules $L_I(\lambda), \lambda \in \Lambda_I^+$ exhaust the set of irreducible objects in $\mathcal{O}^{\mathfrak{p}}$;*
  (iii) *$\mathcal{O}^{\mathfrak{p}} = \oplus_\chi \mathcal{O}_\chi^{\mathfrak{p}}$ where $\mathcal{O}_\chi^{\mathfrak{p}}$ consists of modules M_χ such that $(\ker \chi)^{n(m)}.m = 0$ for some $n(m) \geq 1$ and all $m \in M_\chi$;*
  (iv) *$\mathcal{O}^{\mathfrak{p}}$ has enough projective objects;*
  (v) *there is a bijection between irreducible objects and indecomposable projective objects in $\mathcal{O}^{\mathfrak{p}}$;*
  (vi) *each $\mathcal{O}_\chi^{\mathfrak{p}}$ is (noncanonically) equivalent to a category of finitely generated right modules over a finite-dimensional K-algebra $A_\chi^{\mathfrak{p}}$.*

Proof. The proofs mentioned in [20], §5 all generalize immediately to our setting of a split reductive K-algebra $\mathfrak{g}$. Note that the K-rationality of the block decomposition in (iii) follows from the (generalized) Harish-Chandra map [loc.cit.], §4.3 (compare also the argument in the proof of Prop. 4.2.1 below) like this. Let $M \in \mathcal{O}^{\mathfrak{p}}$. Viewed as a $\mathfrak{l}_I$-module M decomposes into the direct sum over isotypic components M_λ where λ equals an isomorphism class of finite dimensional simple $\mathfrak{l}_I$-modules. Since $\mathfrak{g}$ is split the highest weight of any such finite-dimensional simple $\mathfrak{l}_I$-module is an element, denoted λ again, of the K-linear dual of $\mathfrak{h}$. If we decompose λ into a sum of two linear forms on the Cartan subalgebra of $\mathfrak{g}_I$ and the center $\mathfrak{z}_I$ of $\mathfrak{l}_I$ respectively

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[2]Traditionally, the categories $\mathcal{O}^{\mathfrak{p}}$ are defined for semi-simple (complex) Lie algebras. Following [21] we extend their definition here to general reductive Lie algebras over K.

we obtain a character of the tensor product of the algebras $Z(\mathfrak{g}_I)$ and $S(\mathfrak{z}_I)$. Here, $S(\cdot)$ refers as usual to the symmetric algebra. Let K' be a finite field extension of K and let m be an element of $K' \otimes_K M_\lambda$ on which $Z(\mathfrak{g})$ acts through a K'-valued character χ. By the very construction of the generalized Harish-Chandra map

$$\varphi_I : Z(\mathfrak{g}) \longrightarrow Z(\mathfrak{g}_I) \otimes_K S(\mathfrak{z}_I)$$

we then have $\chi = \lambda \circ \varphi_I$ and, thus, χ takes values in K. $\qquad\square$

The category $\mathcal{O}_\chi^{\mathfrak{p}}$ satisfies an analogue of the classical BGG reciprocity principle ([5]) or, equivalently, the algebra $A_\chi^{\mathfrak{p}}$ appearing in (v) is a so-called *BGG algebra*. For more information in this direction and on the related class of *quasi-hereditary algebras* we refer to the appendix.

Let $\Gamma_I \subseteq \mathfrak{h}^*$ be the positive cone over the set $\Phi^+ \setminus \Phi_I^+$, i.e.

$$\Gamma_I := \mathbb{Z}_{\geq 0}(\Phi^+ \setminus \Phi_I^+).$$

Put $\Gamma = \Gamma_\emptyset$. For $\lambda, \mu \in \mathfrak{h}^*$ we define a partial order on $\mathfrak{h}^*$ as usual via

$$\lambda \geq \mu$$

if $\lambda - \mu \in \Gamma$. It will be convenient to have the following weight characterization of $\mathcal{O}^{\mathfrak{p}}$ as a subcategory of $\mathcal{O}$.

Lemma 3.5.2. *Let $M \in \mathcal{O}$. Then $M \in \mathcal{O}^{\mathfrak{p}}$ if and only if the $\mathfrak{h}$-weights of M lie in a finite union of cosets of the form $\lambda - \Gamma_I$.*

Proof. Put $\mathfrak{n}_I^- := \oplus_{\alpha \in \Phi_I^-} \mathfrak{g}_\alpha$ so that $\mathfrak{n}^- = \mathfrak{n}_I^- \oplus \mathfrak{u}_I^-$. According to [15], Prop. 9.3 we are reduced to show that M satisfies the above condition on weights if and only if it is locally $\mathfrak{n}_I^-$-finite.
In any case, $M \in \mathcal{O}$ is finitely generated by $\mathfrak{h}$-weight vectors $m_1, ..., m_s$. Since any $U(\mathfrak{g})m_i \subseteq M$ is $\mathfrak{h}$-semisimple it suffices to treat the case $s = 1$. From the decompositions

$$\mathfrak{g} = \mathfrak{u}_I^- \oplus \mathfrak{p}_I = \mathfrak{u}_I^- \oplus \mathfrak{l}_I \oplus \mathfrak{u}_I = \mathfrak{u}_I^- \oplus \mathfrak{g}_I \oplus \mathfrak{z}_I \oplus \mathfrak{u}_I$$

and

$$\mathfrak{g}_I = \mathfrak{n}_I^- \oplus \mathfrak{h}_I \oplus \mathfrak{n}_I$$

with $\mathfrak{h} = \mathfrak{h}_I \oplus \mathfrak{z}_I$ and $\mathfrak{h}_I$ the Cartan subalgebra of the semisimple algebra $\mathfrak{g}_I$ we obtain

$$\mathfrak{g} = \mathfrak{u}_I^- \oplus \mathfrak{h} \oplus \mathfrak{n}_I^- \oplus \mathfrak{n}$$

and therefore

$$U(\mathfrak{g}) = U(\mathfrak{u}_I^-) \otimes_K U(\mathfrak{h}) \otimes_K U(\mathfrak{n}_I^-) \otimes_K U(\mathfrak{n})$$

as bimodules. Now if $M \in \mathcal{O}$ is additionally locally $\mathfrak{n}_I^-$-finite then multiplying m with $U(\mathfrak{n}_I^-) \otimes_K U(\mathfrak{n})$ produces a finite-dimensional $\mathfrak{h}$-stable subspace generated by finitely many $\mathfrak{h}$-weight vectors of weights λ. Multiplying these with $U(\mathfrak{u}_I^-)$ produces only weights of the form $\lambda - \Gamma_I$.
Conversely, let M satisfy the assumption on weights. If λ denotes the weight of m, multiplying m by elements in $(\mathfrak{n}_I^-)^n, n > 0$ produces weights of the form $\lambda - \beta$ with $\beta \in \mathbb{Z}_{\geq 0}\Phi_I^+$. By assumption only finitely many of such weights can occur whence $(\mathfrak{n}_I^-)^n.m = 0$ for some $n > 0$. Hence $\dim_K U(\mathfrak{n}_I^-)m < \infty$ and M is locally $\mathfrak{n}_I^-$-finite.

$\qquad\square$

3.6. **A p-adic counterpart.** We keep the notation developed so far. Recall that $\hat{U}(\mathfrak{g})$ is Fréchet-Stein and denote the category of coadmissible $\hat{U}(\mathfrak{g})$-modules by $\mathcal{C}_{\mathfrak{g}}$ (and similarly for appropriate subalgebras of $\mathfrak{g}$). By functoriality we have a continuous homomorphism $\hat{U}(\mathfrak{h}) \to \hat{U}(\mathfrak{g})$ extending the inclusion $\mathfrak{h} \subset \mathfrak{g}$. We apply the notions of section 2 to the commutative K-algebra $\hat{U}(\mathfrak{h})$ and the set of elements in $\mathfrak{H}$. Restriction of scalars via $\hat{U}(\mathfrak{h}) \to \hat{U}(\mathfrak{g})$ induces a faithful and exact functor

$$\mathcal{C}_{\mathfrak{g}} \longrightarrow \mathcal{M}(\hat{U}(\mathfrak{h})).$$

Lemma 3.6.1. *Let $M \in \mathcal{M}(\hat{U}(\mathfrak{h}))$ be $\hat{U}(\mathfrak{h})$-diagonalisable with a set of weights $\Pi(M) \subseteq \mathfrak{h}^*$ contained in finitely many cosets of the form $\lambda - \Gamma_I$. Then $\Pi(M)$ is relatively compact.*

Proof. Invoking the basis elements $\mathfrak{H} = \{h_1, ..., h_l\}$ the map

$$\iota : \Pi(M) \to K^l, \lambda \mapsto (\lambda(h_1), ..., \lambda(h_l))$$

is injective. It suffices to see that $\iota(\Gamma_I)$ is relatively compact. Since each root in Φ is trivial on $\mathfrak{c}$ this image lies in the closed subspace $K^{|\Delta|} \subseteq K^l$. Since each $h_\alpha, \alpha \in \Delta$ is part of a Chevalley basis we have $h_\alpha(\beta) \in \mathbb{Z}$ for all $\beta \in \Phi$. It follows that the closure of $\iota(\Gamma_I)$ lies in the closure of $\mathbb{Z}^{|\Delta|}$, i.e. in $\mathbb{Z}_p^{|\Delta|}$. $\square$

This lemma enables us to single out the following subcategory of $\mathcal{C}_{\mathfrak{g}}$. Let $\mathfrak{p} = \mathfrak{p}_I$.

Definition 3.6.2. The category $\hat{\mathcal{O}}^{\mathfrak{p}}$ for $\hat{U}(\mathfrak{g})$ equals the full subcategory of $\mathcal{C}_{\mathfrak{g}}$ consisting of coadmissible modules M satisfying:

(1) M is $\hat{U}(\mathfrak{h})$-diagonalisable with $\Pi(M)$ contained in the union of finitely many cosets of the form $\lambda - \Gamma_I, \lambda \in \mathfrak{h}^*$.
(2) All weight spaces $M_\lambda, \lambda \in \Pi(M)$ are finite dimensional over K.

We let $\hat{\mathcal{O}} := \hat{\mathcal{O}}^{\mathfrak{b}}$ in case $I = \emptyset$. Obviously $\mathfrak{p} \subset \mathfrak{p}'$ implies $\hat{\mathcal{O}}^{\mathfrak{p}'} \subset \hat{\mathcal{O}}^{\mathfrak{p}}$. Before we exhibit a class of interesting objects in $\hat{\mathcal{O}}^{\mathfrak{p}}$ we list some basic formal properties.

Proposition 3.6.3.   (i) *The direct sum in $\mathcal{C}_{\mathfrak{g}}$ of two objects of $\hat{\mathcal{O}}^{\mathfrak{p}}$ is in $\hat{\mathcal{O}}^{\mathfrak{p}}$*

(ii) *the (co)kernel and (co)image of an arbitrary $\hat{U}(\mathfrak{g})$-linear map between objects in $\hat{\mathcal{O}}^{\mathfrak{p}}$ is in $\hat{\mathcal{O}}^{\mathfrak{p}}$*
(iii) *the sum of two coadmissible submodules of an object in $\hat{\mathcal{O}}^{\mathfrak{p}}$ is in $\hat{\mathcal{O}}^{\mathfrak{p}}$*
(iv) *any finitely generated submodule of an object in $\hat{\mathcal{O}}^{\mathfrak{p}}$ is in $\hat{\mathcal{O}}^{\mathfrak{p}}$*
(v) *$\hat{\mathcal{O}}^{\mathfrak{p}}$ is an abelian category.*

Proof. This follows from Prop. 3.1.1 and Lem. 2.0.2. $\square$

Lemma 3.6.4. *For any object M in $\hat{\mathcal{O}}^{\mathfrak{p}}$ and any abstract $\hat{U}(\mathfrak{g})$-submodule $N \subseteq M$ the following are equivalent:*

(i) *$N \in \hat{\mathcal{O}}^{\mathfrak{p}}$*
(ii) *$M/N \in \hat{\mathcal{O}}^{\mathfrak{p}}$*
(iii) *N is closed in the canonical topology of M.*

Proof. Lem. 3.1.2. $\square$

Recall the exact category $\mathcal{D}(\hat{U}(\mathfrak{h}))$ of section 2.

Lemma 3.6.5. *Let $M \in \hat{\mathcal{O}}^{\mathfrak{p}}$. The map $N \mapsto N \cap M^{ss}$ defines an inclusion preserving bijection between subobjects of $M \in \mathcal{D}(\hat{U}(\mathfrak{h}))$ and abstract $U(\mathfrak{h})$-submodules of M^{ss}. It descends to a bijection between subobjects of $M \in \hat{\mathcal{O}}^{\mathfrak{p}}$ and abstract $U(\mathfrak{g})$-submodules of M^{ss}.*

Proof. M is $U(\mathfrak{h})$-diagonalisable with set of weights $\Pi(M)$ and finite dimensional weight spaces. The first statement follows thus from propositions 2.0.1 and 2.0.2. For the second statement observe that the K-subalgebra $\mathcal{A}$ of $\hat{U}(\mathfrak{g})$ generated by $\hat{U}(\mathfrak{h})$ and $U(\mathfrak{g})$ stabilizes M^{ss} (e.g. [9], Prop. 7.1.2). Again by Prop. 2.0.1 the bijection descends to closed $\mathcal{A}$-invariant subobjects of $M \in \mathcal{D}(\hat{U}(\mathfrak{h}))$ and abstract $U(\mathfrak{g})$-submodules of M^{ss}. The $\mathcal{A}$-action on such a subobject $N \subseteq M$ uniquely extends to $\hat{U}(\mathfrak{g})$ making N a subobject of $M \in \hat{\mathcal{O}}^{\mathfrak{p}}$ according Lem. 3.6.4. $\square$

Example 3.6.6. Let M be a finite dimensional $\mathfrak{g}$-module. Since the endomorphism algebra $\mathrm{End}_K(M)$ has a natural and unique K-Banach topology the $U(\mathfrak{g})$-action uniquely extends to $\hat{U}(\mathfrak{g})$ yielding $M \in \mathcal{C}_{\mathfrak{g}}$. By standard highest weight theory for reductive Lie algebras ([9]) M is $\hat{U}(\mathfrak{h})$-diagonalisable with a finite set of weights contained in Λ^+. Hence $M \in \hat{\mathcal{O}}^{\mathfrak{p}}$ and we have an exact and fully faithful embedding from the finite dimensional $\mathfrak{g}$-modules into $\hat{\mathcal{O}}^{\mathfrak{p}}$.

3.7. *p-adic Verma modules.* We exhibit Verma type modules in $\hat{\mathcal{O}}^{\mathfrak{p}}$. Let $\lambda \in \Lambda_I^+$. Consider the finite dimensional irreducible $\mathfrak{l}_I$-module $L_I(\lambda)$. As explained in the example above the $\mathfrak{l}_I$-action extends to $\hat{U}(\mathfrak{l}_I)$. Invoking the map $\hat{U}(\mathfrak{p}_I) \to \hat{U}(\mathfrak{l}_I)$ we may form the left $\hat{U}(\mathfrak{g})$-module

$$\hat{M}_I(\lambda) := \hat{U}(\mathfrak{g}) \otimes_{\hat{U}(\mathfrak{p}_I)} L_I(\lambda).$$

Proposition 3.7.1. *The module $\hat{M}_I(\lambda)$ lies in $\hat{\mathcal{O}}^{\mathfrak{p}}$ and we have $\hat{M}_I(\lambda)^{ss} = M_I(\lambda)$. There is a canonical inclusion preserving bijection between subobjects of $\hat{M}_I(\lambda)$ and abstract $U(\mathfrak{g})$-submodules of $M_I(\lambda)$. In particular, the topological $\hat{U}(\mathfrak{g})$-module $\hat{M}_I(\lambda)$ is topologically irreducible if and only if the abstract $U(\mathfrak{g})$-module $M_I(\lambda)$ is irreducible.*

Proof. We first show that $\hat{M}_I(\lambda)$ is coadmissible. The left module $M = \hat{U}(\mathfrak{g}) \otimes_K L_I(\lambda)$ is coadmissible being isomorphic to a direct sum over finitely many copies of $\hat{U}(\mathfrak{g})$. Its submodule N generated by the elements $x \otimes 1 - 1 \otimes x$ where x runs through a K-basis of $\mathfrak{p}_I$ is coadmissible. Being closed it contains all elements of the form $y \otimes 1 - 1 \otimes y$ with $y \in \hat{U}(\mathfrak{p}_I)$ whence M/N coincides with $\hat{M}_I(\lambda)$. Hence $\hat{M}_I(\lambda)$ is coadmissible and its canonical topology arises as a quotient topology from M. In particular, the natural map

$$\hat{M}_I(\lambda) \xrightarrow{\cong} \hat{U}(\mathfrak{g}) \hat{\otimes}_{\hat{U}(\mathfrak{p}_I)} L_I(\lambda)$$

into the completed projective tensor product of the locally convex $U(\mathfrak{p}_I)$-modules $\hat{U}(\mathfrak{g})$ and $L_I(\lambda)$ is a topological isomorphism. Let now $\mathfrak{u}_I^- = \oplus_{\Phi^- \setminus \Phi_I^-} \mathfrak{g}_\alpha$ so that $\mathfrak{u}_I^- \oplus \mathfrak{p}_I = \mathfrak{g}$. Applying Lem. 3.2.4 to the latter decomposition and recalling that the completed projective tensor product is associative we obtain that

$$\hat{M}_I(\lambda) = \hat{U}(\mathfrak{u}_I^-) \otimes_K L_I(\lambda)$$

as left $\hat{U}(\mathfrak{u}_I^-)$-modules. Contemplating the $\hat{U}(\mathfrak{h})$-action on this representation we see that $\hat{M}_I(\lambda) \in \hat{\mathcal{O}}^{\mathfrak{p}}$ and that $\hat{M}_I(\lambda)^{ss} = U(\mathfrak{u}_I^-) \otimes_K L_I(\lambda) = M_I(\lambda)$. The final two statements follow now from Lem. 3.6.5. $\square$

We recall at this point that the irreducibility properties of generalized Verma modules are dependent-at least in the case of regular weights- on antidominance properties of the inducing character. To be more precise, let $\rho := 1/2 \sum_{\alpha \in \Phi^+} \alpha$ and consider the following condition on a weight $\lambda \in \Lambda_I^+$:

$$(*) \quad \langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{Z}_{>0} \text{ for all } \beta \in \Phi^+ \setminus \Phi_I.$$

In case $I = \emptyset$ we recover the usual definition of antidominance ([9]). Put $w \cdot \lambda = w(\lambda + \rho) - \rho, w \in W$ for the usual *dot-action* of W on $\mathfrak{h}^*$. If the stabilizer with respect to this action of λ is trivial we call λ *regular*.

Theorem 3.7.2. *Let $\lambda \in \Lambda_I^+$.*

(i) *If λ satisfies (*), then $M_I(\lambda)$ is irreducible.*
(ii) *If λ is regular and $M_I(\lambda)$ is irreducible, then λ satisfies (*).*

The preceding theorem is due to Wallach-Conze-Berline-Duflo-Jantzen for which we refer to [15], Thm. 9.12. In case $I = \emptyset$ it holds true without the regularity condition in (ii) and is due to Bernstein-Gelfand- Gelfand for which we refer to [9], Thm. 7.6.24. We also mention that there is a much deeper irreduciblity criterion for generalized Verma modules which avoids any regularity conditions and is due to H.C. Jantzen. We refer to [15], 9.13 for a concise account.

3.8. **Highest weight modules.** In the following it will be convenient-at least in case $I = \emptyset$ -to introduce in our setting the notion of a highest weight module.
So let $\mathfrak{p} = \mathfrak{b}$ be the Borel subalgebra. Let M be for a moment an arbitrary coadmissible $\hat{U}(\mathfrak{g})$-module. As usual, a *maximal vector* of weight $\lambda \in \mathfrak{h}^*$ in M is a nonzero element $m \in M_\lambda$ such that $\mathfrak{n}.m = 0$. We call a coadmissible module M a *highest weight module with highest weight λ* if it is a cyclic $\hat{U}(\mathfrak{g})$-module on a maximal vector in M_λ.

Remark 3.8.1. It follows directly from the definition of $\hat{\mathcal{O}}$ that any $M \in \hat{\mathcal{O}}$ has a maximal vector. In particular, any irreducible object in $\hat{\mathcal{O}}$ is a highest weight module according to property (iv) of Prop. 3.6.3.

Lemma 3.8.2. *The coadmissible module $\hat{M}(\lambda)$ is a highest weight module of weight λ.*

Proof. This follows as a special case from (the proof of) Prop. 3.7.1.

$\square$

Proposition 3.8.3. *Let $M \in \mathcal{C}_\mathfrak{g}$ be a highest weight module on a maximal vector $m \in M$ of weight $\lambda \in \mathfrak{h}^*$. We have the following:*

(a) *M is $\hat{U}(\mathfrak{h})$-diagonalisable with a compact set of weights $\Pi(M)$ satisfying $\mu \leq \lambda$ for $\mu \in \Pi(M)$.*
(b) *One has $dim_K M_\mu < \infty$ and $dim_K M_\lambda = 1$ for all $\mu \in \Pi(M)$. In particular, $M \in \hat{\mathcal{O}}$ and M is a finite length object in $\hat{\mathcal{O}}$.*
(c) *Each nonzero quotient of M by a coadmissible submodule is again a highest weight module.*
(d) *Each coadmissible submodule of M generated by a maximal vector $m \in M$ of weight $\mu < \lambda$ is proper. In particular, if M is a simple object then all its maximal vectors lie in $K.m$ and hence $End_{\hat{U}(\mathfrak{g})}(M) = K$.*

(e) M has a unique maximal subobject and a unique simple quotient object and, hence, is indecomposable in $\mathcal{C}_{\mathfrak{g}}$.

(f) Let M, N be two highest weight modules of weights λ and μ respectively. We have $dim_K Hom_{\hat{U}(\mathfrak{g})}(M, N) < \infty$. If $\lambda \neq \mu$ then M and N are nonisomorphic. If M and N are simple objects and $\lambda = \mu$ then $M \simeq N$.

Proof. Since M is a quotient object of $\hat{M}(\lambda)$ in $\mathcal{C}_{\mathfrak{g}}$ we obtain an $U(\mathfrak{g})$-linear surjection

$$M(\lambda) = \hat{M}(\lambda)^{ss} \longrightarrow M^{ss}$$

by right exactness of $(.)^{ss}$. In particular, M^{ss} is a highest weight module of weight λ in $\mathcal{O}$. All properties follow then from classical results on highest weight modules in $\mathcal{O}$ (e.g. [15], Thm. 1.2). $\square$

Let $\hat{L}(\lambda)$ denote the unique simple quotient of $\hat{M}(\lambda)$ so that

$$(3.8.3) \qquad\qquad\qquad \hat{L}(\lambda)^{ss} \simeq L(\lambda).$$

Corollary 3.8.4. *The map $\lambda \mapsto [\hat{L}(\lambda)]$ is a bijection from $\mathfrak{h}^*$ onto the set of isomorphism classes of irreducible objects of $\hat{\mathcal{O}}$.*

Proof. This follows as in the classical case of category $\mathcal{O}$ using 3.8.3. $\square$

4. Block decomposition and the main result

4.1. p-adic Harish-Chandra homomorphism.
We begin by recalling some standard results on the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ ([9]). Recall that the usual adjoint action of $\mathfrak{g}$ on itself extends to an action of $\mathfrak{g}$ by derivations on $U(\mathfrak{g})$ and $S(\mathfrak{g})$. Let $U(\mathfrak{g})^{\mathfrak{g}}$ and $S(\mathfrak{g})^{\mathfrak{g}}$ denote the K-algebras of invariants.

Let $\gamma^{\sharp}$ be the algebra automorphism of $S(\mathfrak{h})$ sending a polynomial function f on $\mathfrak{h}^*$ to the function $\lambda \mapsto f(\lambda - \rho)$. Let $U(\mathfrak{g})_0$ be the commutant of $\mathfrak{h}$ in $U(\mathfrak{g})$. Then

$$I := U(\mathfrak{g})\mathfrak{n}^+ \cap U(\mathfrak{g})_0 = \mathfrak{n}^- U(\mathfrak{g}) \cap U(\mathfrak{g})_0$$

is a two-sided ideal in $U(\mathfrak{g})_0$ such that $U(\mathfrak{g})_0 = U(\mathfrak{h}) \oplus I$. The corresponding algebra surjection $\varphi : U(\mathfrak{g})_0 \to U(\mathfrak{h})$ is called the *Harish-Chandra homomorphism* relative to $\mathfrak{b}$. The map

$$\psi := \gamma^{\sharp} \circ \varphi|_{Z(\mathfrak{g})} : Z(\mathfrak{g}) \xrightarrow{\cong} S(\mathfrak{h})^W$$

is an algebra isomorphism independent of the choice of $\mathfrak{b}$.

Remark 4.1.1. There is an important extension of this construction to the case of a general parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. This *generalized Harish-Chandra homomorphism* is due to Drozd-Ovsienko-Futorny ([11]) and is a central tool in the study of generalized Verma modules. Since we will not make use of it here (but see proof of thm. 3.5.1 above) we refer to [20], 4.3 for a detailed description.

It will be convenient to extend the above isomorphism ψ to Arens-Michael envelopes. That this is possible follows from work of J. Kohlhaase on the center of p-adic distribution algebras ([18]). We summarize the relevant results.

Proposition 4.1.2. *The $\mathfrak{g}$-action on $S(\mathfrak{g})$ and $U(\mathfrak{g})$ extends to Arens-Michael envelopes and the same holds for the Weyl action on $S(\mathfrak{h})$. The algebra of invariants $\hat{U}(\mathfrak{g})^{\mathfrak{g}}$ coincides with the center $\hat{Z}(\mathfrak{g})$ of $\hat{U}(\mathfrak{g})$ and equals the closure of $Z(\mathfrak{g})$. The homomorphism ψ extends to a topological isomorphism of K-Fréchet algebras*

$$\hat{\psi} : \hat{Z}(\mathfrak{g}) \xrightarrow{\cong} \hat{S}(\mathfrak{h})^W.$$

Proof. All this is contained in [18], sect. 2.1. For example the last statement follows from Prop. 2.1.5 and (proof of) Thm. 2.1.6. $\qquad\square$

Remark 4.1.3. Let $X = \mathbb{A}_K^{l,an}$. The basis $\mathfrak{H} = \{h_1, .., h_l\}$ induces an isomorphism $\mathfrak{H} : \hat{S}(\mathfrak{h}) \xrightarrow{\cong} \mathcal{O}(X)$ (cf. (3.2.4)). It follows that W acts on X by rigid analytic automorphisms. The rigid-analytic quotient X/W exists by finiteness of W according to general principles ([loc.cit.]. The projection

$$X \longrightarrow X/W$$

is a finite morphism ([6], 9.4.4) and as such has finite fibres ([6] Cor. 9.6.3/6). Alltogether $\mathfrak{H}$ induces a topological isomorphism

$$\overline{\mathfrak{H}} : \hat{S}(\mathfrak{h})^W \xrightarrow{\cong} \mathcal{O}(X/W).$$

Finally, this situation is the analytification of an algebraic action on algebraic affine space via the finite group W. The usual description of $S(\mathfrak{h})^W$ as l-dimensional polynomial ring over K ([9]) therefore extends to completions yielding an isomorphim of K-Fréchet algebras

$$\mathcal{O}(X/W) \xrightarrow{\cong} \mathcal{O}(\mathbb{A}_K^{l,an})$$

onto the algebra of holomorphic functions on affine l-space. The above proposition gives thus a very explicit description of the center of $\hat{U}(\mathfrak{g})$. For more details we refer to [18].

4.2. **Central characters.** Recall that the usual *dot action* of W on $\mathfrak{h}^*$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $\lambda \in \mathfrak{h}^*, w \in W$. Since translating the origin of $X = \mathbb{A}_K^{l,an}$ to $-\rho$ is a rigid isomorphism, say γ, the action extends to a dot-action of W on X giving $\bar{\gamma} : X/W \xrightarrow{\cong} X/(W, \cdot)$. Invoking Prop. 4.1.2 the composite $(\bar{\gamma}^{\sharp})^{-1} \circ \overline{\mathfrak{H}} \circ \hat{\psi}$ is a canonical topological isomorphism

$$(4.2.0) \qquad \hat{Z}(\mathfrak{g}) \xrightarrow{\cong} \hat{S}(\mathfrak{h})^W \xrightarrow{\cong} \mathcal{O}(X/W) \xrightarrow{\cong} \mathcal{O}(X/(W, \cdot))$$

of K-Fréchet algebras.

Now let $\lambda \in \mathfrak{h}^*$ and choose an irreducible highest weight module $M \in \hat{\mathcal{O}}$ with maximal vector $m \in M_\lambda$. By Prop. 3.6.5 we have $\text{End}_{\hat{U}(\mathfrak{g})}(M) = K$ whence a continuous character $\chi_\lambda : \hat{Z}(\mathfrak{g}) \longrightarrow K$. Since $\hat{\psi}$ extends the $\gamma^{\sharp}$-twisted Harish-Chandra homomorphism and since $Z(\mathfrak{g}) \subseteq \hat{Z}(\mathfrak{g})$ is dense the resulting map $\lambda \mapsto \chi_\lambda$ is induced by the rigid analytic quotient morphism

$$\pi : X \longrightarrow X/(W, \cdot).$$

In particular, any continuous character $\chi : \hat{Z}(\mathfrak{g}) \to K$ arises, up to a finite extension of K, as some χ_λ.

Moreover, a highest weight module of weight λ has finite length (Prop. 3.6.5) and visibly all Jordan-Hölder factors of such a module have highest weights contained in the fibre of π in $\chi = \pi(\lambda)$.

We turn back to the general case of a parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_I$ of $\mathfrak{g}$. We propose the following straightforward variant of the classical decomposition of $\mathcal{O}^\mathfrak{p}$ in terms of central characters ([15],[5]). Let $M \in \hat{\mathcal{O}}^\mathfrak{p}$ and let $\chi : \hat{Z}(\mathfrak{g}) \to K$ be a central character. Then $\hat{Z}(\mathfrak{g})$ acts on the weight spaces M_λ (λ being a $\hat{U}(\mathfrak{h})$-weight) and we may form the subspace

$$M_\lambda^\chi := \{m \in M_\lambda : (\ker \chi)^n.m = 0 \text{ for some } n = n(m) \geq 1\}.$$

Since $\oplus_\lambda M_\lambda^\chi$ is a $U(\mathfrak{g})$-submodule of M^{ss} its closure M^χ in M is a subobject of M (Lem. 3.6.5). We define the following full subcategory of $\hat{\mathcal{O}}^\mathfrak{p}$:

$$\hat{\mathcal{O}}_\chi^\mathfrak{p} := \{M \in \hat{\mathcal{O}}^\mathfrak{p} : M^\chi = M\}.$$

In case $I = \emptyset$ we write $\hat{\mathcal{O}}_\chi := \hat{\mathcal{O}}_\chi^\mathfrak{b}$.

Proposition 4.2.1. *The category $\hat{\mathcal{O}}_\chi^\mathfrak{p}$ is abelian. The functor*

$$\hat{\mathcal{O}}^\mathfrak{p} \to \hat{\mathcal{O}}_\chi^\mathfrak{p}, \;\; M \mapsto M^\chi$$

is exact and induces an exact and faithful embedding of $\hat{\mathcal{O}}^\mathfrak{p}$ into the direct product $\prod_\chi \hat{\mathcal{O}}_\chi^\mathfrak{p}$ (where χ runs through the K-valued central characters).

Proof. Since the inclusion $\hat{\mathcal{O}}^\mathfrak{p} \subseteq \hat{\mathcal{O}}$ is defined solely in terms of weights (Def. 3.6.2) we are easily reduced to the case $I = \emptyset$. Giving $\hat{\mathcal{O}}_\chi$ the exact structure coming from $\hat{\mathcal{O}}$ let us show that $M \mapsto M^\chi$ is an exact functor. Given a morphism $M \to N$ in $\hat{\mathcal{O}}$ we certainly have maps $M_\lambda \to N_\lambda$ and $M_\lambda^\chi \to N_\lambda^\chi$ for every χ. Taking the sum over all λ and passing to closures with respect to the induced subspace topologies we see that $M \mapsto M^\chi$ is indeed functorial. Using strictness of maps in $\hat{\mathcal{O}}$ with respect to canonical topologies (Prop. 3.1.1 (vii)) the same argument yields its exactness ([6], Cor. 1.1.9/6). It is now clear that the subcategory $\hat{\mathcal{O}}_\chi$ is closed under passage to kernels and cokernels and, thus, abelian.

Now choose topological generators $z_1, ..., z_l$ of $\hat{Z}(\mathfrak{g})$ according to remark 4.1.3. Then M_λ^χ equals the simultaneous generalized eigenspace of the finitely many commuting operators $z_1, ..., z_l$ on the finite dimensional space M_λ corresponding to the ordered set of eigenvalues $\chi(z_1), ..., \chi(z_l)$. In particular, there exists a finite field extension $K \subseteq K'$ of K such that

$$K' \otimes_K M_\lambda = \oplus_{\chi'} (K' \otimes_K M_\lambda)^{\chi'}$$

where the sum runs over all K'-valued central characters χ' and $(K' \otimes_K M_\lambda)^{\chi'}$ is defined in the obvious way. We claim that

$$(K' \otimes_K M_\lambda)^{\chi'} \neq 0 \Rightarrow \chi'(\hat{Z}(\mathfrak{g})) \subseteq K.$$

Indeed, let $m \in K' \otimes_K M_\lambda$ and let $n \geq 1$ be minimal such that $(\ker \chi')^n.m = 0$. On a nonzero $m' \in (\ker \chi')^{n-1}.m$ the center $\hat{Z}(\mathfrak{g})$ operates via χ' and hence $\pi(\lambda) = \chi'$. In particular, χ' is a K-valued point of $X/(W, \cdot)$ which proves the claim.

We therefore have $M_\lambda = \oplus_\chi M_\lambda^\chi$ with χ running through the K-valued characters of $\hat{Z}(\mathfrak{g})$. Together with the obvious equality $M^{ss} \cap M^\chi = \oplus_\lambda M_\lambda^\chi$ this implies $M^{ss} = \oplus_\chi (M^\chi \cap M^{ss})$. It follows from this and properties of $(\cdot)^{ss}$ that the sum $\sum_\chi M^\chi$ is dense and direct in M. In particular, the functor $\hat{\mathcal{O}} \to \prod_\chi \hat{\mathcal{O}}_\chi, M \mapsto (M^\chi)_\chi$ is faithful. $\square$

Proposition 4.2.2. *The categories $\hat{\mathcal{O}}_\chi^\mathfrak{p}$ are artinian and noetherian.*

Proof. With the p-adic Harish Chandra map at hand we may imitate the classical argument ([5],[9]) as follows. Since $\hat{\mathcal{O}}^{\mathfrak{p}}_{\chi} \subseteq \hat{\mathcal{O}}_{\chi}$ is a full subcategory we may suppose $I = \emptyset$. Let $M \in \hat{\mathcal{O}}_{\chi}$ be given and put $V := \sum_{\mu \in \pi^{-1}(\chi)} M_{\mu}$. Since π has finite fibers we have $\dim_K V < \infty$. Suppose $N' \subsetneq N \subseteq M$ are two subobjects. Let $m \in N/N'$ be a maximal vector of some weight μ. Since the subobject $\hat{U}(\mathfrak{g}).m \subseteq N/N'$ is a highest weight module $\hat{Z}(\mathfrak{g})$ operates on m via χ_{μ}. Hence $\chi_{\mu} = \chi$ and $\mu \in \pi^{-1}(\chi)$. By definition $m \in N \cap V$ whence $\dim_K N \cap V > \dim_K N' \cap V$. This shows M to be artinian and noetherian. $\qquad\square$

4.3. **The main result.** In this section we prove the following main result. As with any Arens-Michael envelope (compare 3.2) we have a natural map

$$U(\mathfrak{g}) \to \hat{U}(\mathfrak{g}).$$

Theorem 4.3.1. *The functor $M \mapsto \hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$ induces an equivalence of categories*

$$\mathcal{O}^{\mathfrak{p}} \xrightarrow{\cong} \hat{\mathcal{O}}^{\mathfrak{p}}.$$

A quasi-inverse is given by $(\cdot)^{ss}$. The equivalence identifies $\mathcal{O}^{\mathfrak{p}}_{\chi} \simeq \hat{\mathcal{O}}^{\mathfrak{p}}_{\chi}$ for any K-valued central character χ and hence $\hat{\mathcal{O}}^{\mathfrak{p}} = \prod_{\chi} \hat{\mathcal{O}}^{\mathfrak{p}}_{\chi}$.

According to well-known properties of $\mathcal{O}^{\mathfrak{p}}$ (cf. [15], Thm. 9.8,[20], 5.2) we obtain

Corollary 4.3.2. *The category $\hat{\mathcal{O}}^{\mathfrak{p}}$ has enough injectives and projectives and a duality. Each block $\hat{\mathcal{O}}^{\mathfrak{p}}_{\chi}$ is a highest weight category and (noncanonically) equivalent to a category of finitely generated right modules over a BGG algebra.*

To begin the proof of the theorem let us first assume $I = \emptyset$. Recall (example 3.6.6) the fully faithful embedding from the finite dimensional $\mathfrak{g}$-modules into $\hat{\mathcal{O}}$. Any finitely generated (left) $U(\mathfrak{g})$-module is finitely presented and therefore $M \mapsto \hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$ constitutes a functor F from such modules into $\mathcal{C}_{\mathfrak{g}}$ (Prop. 3.1.1 (v)). Our further investigation relies on the following fact.

Theorem 4.3.3. *The extension $U(\mathfrak{g}) \to \hat{U}(\mathfrak{g})$ is flat.*

Proof. This is a direct consequence of the main result of [26]. Alternatively, one may pass to a finite extension of K and use flatness of adic completion at central ideals of noetherian rings. This yields the flatness of the map $U(\mathfrak{g}) \to U_r(\mathfrak{g})$ for $r \in p^{\mathbb{Q}}$ and then [28], (proof of) Thm. 4.11 gives the claim. $\qquad\square$

In particular, F is exact. It is almost obvious that $F(M(\lambda)) = \hat{M}(\lambda)$ and hence any highest weight module of $\mathcal{O}$ is mapped to a highest weight module in $\hat{\mathcal{O}}$. Since any module M in $\mathcal{O}$ has a finite filtration with graded quotients being highest weight modules ([15], Cor. 1.2) there is a surjection $\oplus_i M_i \xrightarrow{\Sigma} M$ where the source is a finite direct sum of highest weight modules. Since F commutes with direct sums we see $F(M) \in \hat{\mathcal{O}}$. We have thus established an exact functor

$$F : \mathcal{O} \longrightarrow \hat{\mathcal{O}}$$

extending the aforementioned embedding of the finite dimensional modules into $\hat{\mathcal{O}}$.

Proposition 4.3.4. *The functor F is fully faithful. A left quasi-inverse is given by $(\cdot)^{ss}$.*

Proof. If $M \in \mathcal{O}$ and $m \in M_\lambda$ the map $m \mapsto 1 \otimes m$ induces a $U(\mathfrak{h})$-linear homomorphism from M_λ into the λ-weight space of $F(M)^{ss}$. It extends to a $U(\mathfrak{g})$-linear homomorphism $M \to F(M)^{ss}$ natural in M. If M is a Verma module it is bijective according to Prop. 3.7.1. If $M \in \mathcal{O}$ is a highest weight module we consider an exact sequence

$$0 \longrightarrow N \longrightarrow M(\lambda) \longrightarrow M \longrightarrow 0$$

for suitable $\lambda \in \mathfrak{h}^*$. Writing N as a subquotient of the left regular module $U(\mathfrak{g})$ and recalling (proof of Prop. 3.7.1) that $\hat{M}(\lambda)$ equals the completion of $U(\mathfrak{g})/U(\mathfrak{g})J = M(\lambda)$ with respect to the (separated) quotient topology one sees that the natural injection $F(N) \to \hat{M}(\lambda)$ has image equal to the closure of N in $\hat{M}(\lambda)$. Hence, $N \simeq F(N)^{ss}$ by Prop. 2.0.1(iii) and therefore $M \simeq F(M)^{ss}$. Now let $M \in \mathcal{O}$ be arbitrary. By devissage we may assume that M is an extension of highest weight modules. But then $M \simeq F(M)^{ss}$ by the Five lemma. $\square$

Next we will fix a number $r > 1$ in $p^\mathbb{Q}$ and consider the noetherian Banach algebra $U_r(\mathfrak{g})$ (section 3.2). Working with our standard basis (section 3.3) of $\mathfrak{g}$ we see that the inclusion $U(\mathfrak{h}) \subseteq U(\mathfrak{g})$ extends to an isometry $U_r(\mathfrak{h}) \subseteq U_r(\mathfrak{g})$. Similarly we obtain isometries $U_r(\mathfrak{n}), U_r(\mathfrak{n}^-) \subseteq U_r(\mathfrak{g})$. We denote by

$$U_r(\mathfrak{n}^-) \hat{\otimes}_K U_r(\mathfrak{h}) \hat{\otimes}_K U_r(\mathfrak{n})$$

the completion of the tensor product of these subalgebras with respect to the usual tensor product norm (which coincides with the completed projective tensor product, [27], Lem. 17.2).

Lemma 4.3.5. *There PBW-decomposition of Lem. 3.2.4 extends to an isometry of Banach $(U_r(\mathfrak{n}^-), U_r(\mathfrak{n}))$-bimodules*

$$U_r(\mathfrak{n}^-) \hat{\otimes}_K U_r(\mathfrak{h}) \hat{\otimes}_K U_r(\mathfrak{n}) \xrightarrow{\cong} U_r(\mathfrak{g}).$$

Proof. The algebra structure being irrelevant here we may replace U by S. Since $r \in p^\mathbb{Q}$ passing to a finite extension of K reduces us, by faithfully flat descent, to the case $r \in |K^\times|$. We may therefore assume $r = 1$ in which case the result is well-known ([6], Cor. 6.1.1/8). $\square$

Given a coadmissible module M we let

$$M_r := U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M.$$

By the general Fréchet-Stein formalism M_r is a finitely generated Banach $U_r(\mathfrak{g})$-module and the natural map $M \to M_r$ has dense image ([28], §3). The following lemma is due to Benjamin Schraen and I thank him for allowing me to reproduce it here.

Lemma 4.3.6. *We have $\hat{L}(\lambda)_r \neq 0$ for any weight $\lambda \in \mathfrak{h}^*$.*

Proof. Consider the kernel Q of the natural map $\hat{M}(\lambda) \to \hat{L}(\lambda)$. Since $U(\mathfrak{g}) \to U_r(\mathfrak{g})$ is flat the kernel of $\hat{M}(\lambda)_r \to \hat{L}(\lambda)_r$ equals Q_r. Applying the above lemma we see that

$$\hat{M}(\lambda)_r \simeq U_r(\mathfrak{n}^-) \otimes_K K_\lambda$$

whence $\hat{M}(\lambda)_r$ is $U_r(\mathfrak{h})$-diagonalisable with

$$\hat{M}(\lambda)^{ss} = (\hat{M}(\lambda)_r)^{ss}$$

via the inclusion $\hat{M}(\lambda) \subseteq \hat{M}(\lambda)_r$. By Prop. 2.0.2 the modules Q_r and $\hat{L}(\lambda)_r$ are $U_r(\mathfrak{h})$-diagonalisable and it suffices to see that

$$(Q_r)^{ss} \subsetneqq (\hat{M}(\lambda)_r)^{ss}.$$

Now $(Q_r)^{ss} \subseteq Q_r$ is dense. Similarly, the composite map $Q^{ss} \subseteq Q \subseteq Q_r$ has dense image. According to Prop. 2.0.1 we therefore obtain $Q^{ss} = (Q_r)^{ss}$ as abstract $U_r(\mathfrak{h})$-submodules of Q_r. Since $\hat{L}(\lambda) \neq 0$ we arrive at

$$(Q_r)^{ss} = Q^{ss} \subsetneqq \hat{M}(\lambda)^{ss} = (\hat{M}(\lambda)_r)^{ss}.$$

$\square$

Lemma 4.3.7. *The category $\hat{\mathcal{O}}$ is artinian and noetherian.*

Proof. Let $M \in \hat{\mathcal{O}}$. Recall that $\oplus_\chi M^\chi$ is dense in M according to Prop. 4.2.1. Letting $r > 1$ in $p^{\mathbb{Q}}$ we compute

$$M_r = U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M \supseteq U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} (\oplus_\chi M^\chi) = \oplus_\chi (M^\chi)_r.$$

Any nonzero M^χ has a composition series (Prop. 4.2.2) whence $\hat{L}(\lambda)_r \subseteq (M^\chi)_r$ for some weight $\lambda \in \mathfrak{h}^*$ and then $(M^\chi)_r \neq 0$ by the preceding lemma. Since M_r is finitely generated and $U_r(\mathfrak{g})$ is noetherian this means that $M^\chi = 0$ for all but finitely many χ. But then $\oplus_\chi M^\chi$ is closed in M according to Prop. 3.1.1. $\square$

Lemma 4.3.8. *Given $M \in \hat{\mathcal{O}}$ the abstract $U(\mathfrak{g})$-module M^{ss} lies in $\mathcal{O}$. The correspondence $M \mapsto M^{ss}$ is a quasi-inverse to F.*

Proof. By the preceding result we may assume that M is an extension of two simple objects. According to the result (3.8.3) we see that M^{ss} is a finitely generated $U(\mathfrak{g})$-module on which $\mathfrak{h}$ acts semisimple. By our assumption on the weights $\Pi(M)$ the algebra $\mathfrak{n}$ acts locally finite. This means $M^{ss} \in \mathcal{O}$. To prove the second statement it suffices, according to Prop. 4.3.4, to show that $(\cdot)^{ss}$ is right quasi-inverse to F. Let $M \in \hat{\mathcal{O}}$. If we apply $\hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} (\cdot)$ to the inclusion $M^{ss} \subseteq M$ and compose with the map $u \otimes m \mapsto um$ we obtain a morphism $F(M^{ss}) \to M$ in $\hat{\mathcal{O}}$. If K and Q denote its kernel and cokernel respectively we have $K^{ss} = Q^{ss} = 0$ by Prop. 4.3.4 whence $K = Q = 0$ by Prop. 2.0.1. $\square$

This ends the proof of the theorem in case $I = \emptyset$. Now consider the case of a general parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_I$. Since the functors F and $(\cdot)^{ss}$ preserve $\mathfrak{h}$-weight spaces Lem. 3.5.2 shows that our established equivalence $\hat{\mathcal{O}} \simeq \mathcal{O}$ identifies the full subcategories $\mathcal{O}^{\mathfrak{p}} \subseteq \mathcal{O}$ and $\hat{\mathcal{O}}^{\mathfrak{p}} \subseteq \hat{\mathcal{O}}$. It is obvious that this identification respects the central blocks. This finishes the proof of the theorem.

Appendix A. Quasi-hereditary algebras and highest weight categories

We recall some basic facts on quasi-hereditary algebras and highest weight categories. The following formulation is adapted to our purposes. For more details we refer to [10] and [12].

Let K be a field, A a finite dimensional K-algebra, $\mathrm{Mod}_{\mathrm{fg}}(A)$ the category of finitely generated right A-modules and $K_0(A)$ the Grothendieck group of $\mathrm{Mod}_{\mathrm{fg}}(A)$. Let

$$(\Lambda, \leq)$$

be a fixed partially ordered finite set indexing a full set of representatives $(L_\lambda)_{\lambda \in \Lambda}$ for the isomorphism classes of simple right A-modules. The multiplicity of L_λ in a Jordan-Hölder series of a module M will be denoted by $[M : L_\lambda]$. Given $\lambda \in \Lambda$ let P_λ and I_λ be a projective cover and injective hull of L_λ in $\mathrm{Mod}_{\mathrm{fg}}(A)$ respectively.

A collection of *standard modules* for A (relative to the partially ordered set Λ) is a set Δ of modules $\Delta_\lambda \in \mathrm{Mod}_{\mathrm{fg}}(A)$ with the properties $[\Delta_\lambda : L_\lambda] = 1$ with $\mathrm{Top}(\Delta_\lambda) \simeq L_\lambda$ and $[\Delta_\lambda : L_\mu] = 0$ if $\mu \not\leq \lambda$. Given such a set Δ let $\mathcal{F}(\Delta)$ be the full subcategory of $\mathrm{Mod}_{\mathrm{fg}}(A)$ consisting of modules M admitting a finite filtration with graded quotients isomorphic to members of Δ. Given $M \in \mathcal{F}(\Delta)$ the element $[M]$ of $K_0(A)$ can be written as

$$[M] = \sum_{\lambda \in \Lambda} n_\lambda [\Delta_\lambda] = \sum_{\lambda \in \Lambda} n_\lambda \sum_{\mu \in \Lambda} [\Delta_\lambda : L_\mu][L_\mu]$$

with suitable $n_\lambda \in \mathbb{N}$. Choose a numbering $\lambda_1, ..., \lambda_s$ of the elements in Λ such that $\lambda_i < \lambda_j$ implies $i > j$. The matrix $([\Delta_\lambda : L_\mu])_{\lambda,\mu}$ is then unipotent upper triangular and since the elements $[L_\mu]$ form a $\mathbb{Z}$-basis of $K_0(A)$, the coefficients n_λ are uniquely determined. The filtration multiplicities $(M : \Delta_\lambda)$ are therefore independent of the choice of filtration. Finally, the standard module Δ_λ is called *schurian* if $\mathrm{End}_A(\Delta_\lambda)$ is a division ring.

Recall that in this situation A is called (right) *quasi-hereditary* if all standard modules are schurian and we have $P_\mu \in \mathcal{F}(\Delta)$ such that $(P_\mu : \Delta_\mu) = 1$ and $(P_\mu : \Delta_\lambda) = 0$ if $\mu \not\leq \lambda$ for all $\lambda, \mu \in \Lambda$ (cf. [10], §1).

Remark A.0.9. Let A be quasi-hereditary with set of standard modules Δ. If $\leq_1$ is a total ordering on Λ that contains $\leq$ then, trivially, $(A, \leq_1)$ is quasi-hereditary with the same set of standard modules. In dealing with quasi-hereditary algebras we may therefore always assume that $\Lambda = \{1, ..., n\}$, some n, equipped with its natural ordering. In other words, the issue of a *non-adapted* Λ (in the sense of [loc.cit.]) does not arise here.

Remark A.0.10. Let A be quasi-hereditary. Without recalling a precise definition we remark that $\mathrm{Mod}_{\mathrm{fg}}(A)$ is a highest weight category in the sense of Cline-Parshall-Scott (cf. [8], Lem. 3.4).

If A is a quasi-hereditary algebra it is easy to see that each I_λ has a unique largest submodule ∇_λ with $[\nabla_\lambda : L_\mu] = 0$ for $\mu \not\leq \lambda$. The modules ∇_λ are sometimes called the *costandard modules* associated to A (cf. [10], [12]).

Proposition A.0.11. *Let A be quasi-hereditary. Then A has (right) global dimension bounded by $2|\Lambda|$.*

Proof. This follows from [10], Lem. 2.2. □

Remark A.0.12. If the (right) global dimension of a finite dimensional K-algebra A is ≤ 1 then A is (right) hereditary, i.e. all right ideals are projective. ([2], Cor. 5.2). For the extensive and well-understood theory of hereditary algebras we refer to [loc.cit.], chap. VIII.

Proposition A.0.13. *Let A be a quasi-hereditary algebra. Then*

$$(P_\mu : \Delta_\lambda) \cdot d_\lambda = [\nabla_\lambda : L_\mu] \cdot d_\mu$$

where $d_\lambda := \dim_K \mathrm{End}_A(\Delta_\lambda)$ for all $\lambda, \mu \in \Lambda$.

Proof. [10], Lem. 2.5. and [12], Thm. 3   □

A quasi-hereditary algebra is called a *BGG-algebra* if there exists a contravariant involutive autofunctor D on $\mathrm{Mod}_{\mathrm{fg}}(A)$ such that $D(L_\lambda) \simeq L_\lambda$ for all $\lambda \in \Lambda$ ([16]). Such an algebra satisfies the so-called *strong BGG reciprocity*:

Proposition A.0.14. *Let A be a BGG algebra. Then*

$$(P_\mu : \Delta_\lambda) \cdot d_\lambda = [\Delta_\lambda : L_\mu] \cdot d_\mu$$

where $d_\lambda := \dim_K \mathrm{End}_A(\Delta_\lambda)$ for all $\lambda, \mu \in \Lambda$.

Proof. It is easy to see that $D(\nabla_\lambda) \simeq \Delta_\lambda$ for all λ ([12], Lem. 4). The claim follows thus from the above proposition using that D preserves Jordan-Hölder multiplicities.   □

Example A.0.15. Let K be a p-adic local field, $\mathfrak{g}$ a split reductive Lie algebra over K, $\mathfrak{b}$ a Borel subalgebra and $\mathfrak{p} \subseteq \mathfrak{g}$ a parabolic subalgebra containing $\mathfrak{b}$. Denote by $\mathcal{O}^{\mathfrak{p}}$ the parabolic *BGG category* of $\mathfrak{g}$ relative to $\mathfrak{p}$ (cf. 3.5). Let χ be a K-valued character of $Z(\mathfrak{g})$ and $\mathcal{O}^{\mathfrak{p}}_\chi$ the corresponding central block of $\mathcal{O}^{\mathfrak{p}}$. Then $\mathcal{O}^{\mathfrak{p}}_\chi$ is (noncanonically) equivalent to the category of finitely generated (right) modules over a BGG-algebra $A^{\mathfrak{p}}_\chi$. A set of schurian standard modules is given by the GVM's $M_I(\lambda)$ contained in $\mathcal{O}^{\mathfrak{p}}_\chi$ (where $\mathfrak{p} = \mathfrak{p}_I$). The algebra $A^{\mathfrak{p}}_\chi$ arises (a direct consequence of the theorem of Gabriel- Mitchell ([3], Thm. II.1.3 and subsequent exercise)) as the endomorphism algebra of a suitable projective generator of the artinian and noetherian category $\mathcal{O}^{\mathfrak{p}}_\chi$. By work of W. Soergel its structure-at least in the case $\mathfrak{p} = \mathfrak{b}$- can be explicitly determined ([29]).

In case of a complex semisimple algebra and a Borel subalgebra this is the urexample in the theory of quasi-hereditary algebras (cf. [5] and [8], Example 3.3 (c)).

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Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany

E-mail address: toschmid@math.uni-muenster.de