STABLE FLATNESS OF NONARCHIMEDEAN HYPERENVELOPING ALGEBRAS

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Abstract. Let L be a completely valued nonarchimedean field and $\mathfrak{g}$ a finite dimensional Lie algebra over L. We show that its hyperenveloping algebra $\mathfrak{F}(\mathfrak{g})$ agrees with its Arens-Michael envelope and, furthermore, is a stably flat completion of its universal enveloping algebra. As an application, we prove that the Taylor relative cohomology for the locally convex algebra $\mathfrak{F}(\mathfrak{g})$ is naturally isomorphic to the Lie algebra cohomology of $\mathfrak{g}$.

Keywords: Lie algebras, stable flatness, representation theory.

1. Introduction

Let L be a p-adic local field and G a locally L-analytic group. A distinguished topological algebra appearing in the locally analytic representation theory of G is the hyperenveloping algebra $\mathfrak{F}(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$ of G ([14]). It equals a certain canonical locally convex completion of the ordinary universal enveloping algebra $U(\mathfrak{g})$, similar to the well-known Arens-Michael envelope ([3]). In the theory of complex Lie algebras it is important to know when these completions are stably flat (or, in other words, Taylor absolute localizations, cf. [7]). Vaguely speaking, a continuous ring extension between topological algebras $\theta : A \to B$ is stably flat if the restriction functor θ_* identifies the category of topological B-modules with a full subcategory of topological A-modules in a way that leaves certain homological relations invariant. We recall that such extensions are of central importance in complex non-commutative operator theory, partly because they leave the joint spectrum invariant ([7], [13]).

Turning back to our nonarchimedean setting it is therefore natural to ask for nonarchimedean analogues of these complex results. In this brief note we give a positive answer in full generality: given a finite dimensional Lie algebra $\mathfrak{g}$ over an arbitrary completely valued nonarchimedean field we define its hyperenveloping algebra $\mathfrak{F}(\mathfrak{g})$ and deduce that it always coincides with the Arens-Michael envelope. Our main result is then that the natural map

$$U(\mathfrak{g}) \to \mathfrak{F}(\mathfrak{g})$$

is stably flat. We remark that the first (second) result is in contrast to (in accordance with) the complex situation ([7]).

To give an application of our results recall that cohomology theory for locally analytic G-representations ([5]) follows Taylor's general approach of a homology theory for topological algebras ([12],[13]) where the locally convex algebra of locally analytic distributions on G takes up the role of the base algebra. In view of the close relation between this latter algebra and $\mathfrak{F}(\mathfrak{g})$ ([14]) it is to be expected that locally analytic cohomology is closely related to the cohomology relative to the base algebra $\mathfrak{F}(\mathfrak{g})$. As a consequence of our flatness result we find this latter cohomology to be naturally isomorphic to the usual Lie algebra cohomology of $\mathfrak{g}$.

2. The result

Throughout this note we freely use basic notions of nonarchimedean functional analysis as presented in [9]. Let L be a completely valued nonarchimedean field. We begin by recalling the necessary relative homological algebra following [5],[7],[12]. We emphasize that as in [5],[13] (but in contrast to [7]) our preferred choice of topological tensor product is the completed inductive topological tensor product $\hat{\otimes}_L$. By a *topological algebra* A we mean a complete Hausdorff locally convex L-space together with a separately continuous multiplication. For a topological algebra A we denote by $\mathcal{M}_A$ the category of complete Hausdorff locally convex L-spaces with a separately continuous left A-module structure (to ease notation we denote the right version by the same symbol). Morphisms are continuous module maps and the Hom-functor is denoted by $\mathcal{L}_A(.,.)$. A morphism is called *strong* if it is strict with closed image and if both its kernel and its image admit complements by closed L-subspaces. The category $\mathcal{M}_A$ is endowed with a structure of exact category by declaring a sequence to be *s-exact* if it is exact as a sequence of abstract vector spaces and if all occuring maps are strong. Finally, a module $P \in \mathcal{M}_A$ is called *s-projective* if the functor $\mathcal{L}_A(P,.)$ transforms short s-exact sequences into exact sequences of abstract L-vector spaces. A *projective resolution* of $M \in \mathcal{M}_A$ is an augmented complex $P_\bullet \rightarrow M$ which is s-exact and where each P_n is s-projective. A standard argument shows that $\mathcal{M}_A$ has enough projectives and that any object admits a projective resolution. As usual for a left resp. right A-module N resp. M we denote by $M\hat{\otimes}_A N$ the quotient of $M\hat{\otimes}_L N$ by the closure of the subspace generated by elements of the form $ma \otimes n - m \otimes an, a \in A, m \in M, n \in N$. Given a projective resolution $P_\bullet \rightarrow M$ we define as usual

$$\mathcal{T}or_*^A(M,N) := h_*(P_\bullet\hat{\otimes}_A N)\ ,\ \mathcal{E}xt_A^*(M,N) := h^*(\mathcal{L}_A(P_\bullet, N))$$

for $M, N \in \mathcal{M}_A$. These L-vector spaces do not depend on the choice of $P_\bullet$ and have the usual functorial properties.

Given a topological algebra A so is the opposite algebra A^{op} and we may form the enveloping algebra $A^e := A\hat{\otimes}_L A^{op}$ as a topological algebra. Given a continuous ring homomorphism between topological algebras $\theta : A \rightarrow B$ we may define a functor $B^e\hat{\otimes}_{A^e}(.)$ from the category of A-bimodules $\mathcal{M}_{A^e}$ to the category of B-bimodules $\mathcal{M}_{B^e}$. The map θ is called *stably flat* (or an *absolute localization*, cf. [7],[13]) if the above functor transforms every projective resolution of A^e into a projective resolution of B^e.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over L and let $U(\mathfrak{g})$ be its enveloping algebra. Denote by $M_{\mathfrak{g}}$ the category of all (abstract) left $\mathfrak{g}$-modules. Fix a real number $r > 1$. Let $\mathfrak{x}_1, ..., \mathfrak{x}_d$ be an ordered L-basis of $\mathfrak{g}$ with $d = \dim_L \mathfrak{g}$. Using the associated *Poincaré-Birkhoff-Witt-basis* for $U(\mathfrak{g})$ we define a vector space norm on $U(\mathfrak{g})$ via

$$||\sum_\alpha d_\alpha \mathfrak{X}^\alpha||_{\mathfrak{X},r} = \sup_\alpha |d_\alpha| r^{|\alpha|} \tag{2.1}$$

where $\mathfrak{X}^\alpha := \mathfrak{x}_1^{\alpha_1} \cdots \mathfrak{x}_d^{\alpha_d},\ \alpha \in \mathbb{N}_0^d$. We call the Hausdorff completion of $U(\mathfrak{g})$ with respect to the family of norms $||.||_{\mathfrak{X},r},\ r > 1$ the *hyperenveloping algebra* of $\mathfrak{g}$. Being a Hausdorff completion it comes equipped with a natural map

$$\theta : U(\mathfrak{g}) \to \mathfrak{F}(\mathfrak{g}).$$

Endowing the source with the finest locally convex topology we will see below that θ is a continuous homomorphism between well-defined topological algebras in the above sense. In particular, we have the categories $\mathcal{M}_{U(\mathfrak{g})}$ and $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$ at our disposal. Finally, the Hausdorff completion of $U(\mathfrak{g})$ with respect to *all* submultiplicative seminorms on $U(\mathfrak{g})$ is called the *Arens-Michael envelope* $\hat{U}(\mathfrak{g})$ of $U(\mathfrak{g})$ (cf. [3], chap. V, [7], 6.1).

Theorem 2.1. *The homomorphism θ is stably flat. It induces a natural isomorphism of topological algebras*

$$\hat{U}(\mathfrak{g}) \xrightarrow{\cong} \mathfrak{F}(\mathfrak{g}).$$

As with any stably flat homomorphism we obtain that the restriction functor θ_* identifies $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$ with a full subcategory of $\mathcal{M}_{U(\mathfrak{g})}$ (cf. [13], Prop. 1.2) leaving certain homological relations invariant ([loc.cit.], Prop. 1.4). Since in our setting $U(\mathfrak{g})$ has the finest locally convex topology one may go one step further and pass to abstract Lie algebra cohomology.

Corollary 2.2. *Given $M, N \in \mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$ the restriction functor θ_* induces natural vector space isomorphisms*

$$\mathcal{T}\mathrm{or}_*^{\mathfrak{F}(\mathfrak{g})}(M,N) \cong \mathrm{Tor}_*^{U(\mathfrak{g})}(M,N)\ ,\ \mathcal{E}\mathrm{xt}^*_{\mathfrak{F}(\mathfrak{g})}(M,N) \cong \mathrm{Ext}^*_{U(\mathfrak{g})}(M,N).$$

3. The proof

3.1. **Norms on the hyperenveloping algebra.** We begin with two simple lemmas on the norm $||.||_{\mathfrak{X},r},\ r > 1$. Let $c_{ijk} \in L,\ 1 \le i,j,k \le d$ denote the structure constants of $\mathfrak{g}$ attached to the basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$.

Lemma 3.1. *Suppose $|c_{ijk}| \le 1$ for all $1 \le i,j,k \le d$. Then $||.||_{\mathfrak{X},r}$ is multiplicative.*

Proof. Note first that L is a filtered ring (in the sense of [10], sect. 1) via its absolute value and, by multiplicativity, the associated graded ring is an integral domain. Now put $\mathfrak{X}^\alpha \mathfrak{X}^\beta =: \sum_\gamma c_{\alpha\beta,\gamma}\mathfrak{X}^\gamma$ with $c_{\alpha\beta,\gamma} \in L$. By hypothesis $\mathfrak{X}^\alpha\mathfrak{X}^\beta = \mathfrak{x}_1^{\alpha_1+\beta_1}\cdots\mathfrak{x}_d^{\alpha_d+\beta_d} + \mathfrak{y}$ where $||\mathfrak{y}||_{\mathfrak{X},r} < ||\mathfrak{x}_1^{\alpha_1+\beta_1}\cdots\mathfrak{x}_d^{\alpha_d+\beta_d}||_{\mathfrak{X},r}$. Hence $\sup_\gamma |c_{\alpha\beta,\gamma}|r^{|\gamma|} = ||\mathfrak{X}^\alpha\mathfrak{X}^\beta||_{\mathfrak{X},r} = r^{|\alpha|+|\beta|}$ and therefore $|c_{\alpha\beta,\gamma}| \le r^{|\alpha|+|\beta|-|\gamma|}$ for all α,β,γ. It follows easily from this that $||.||_{\mathfrak{X},r}$ is submultiplicative (cf. [10], Prop. 4.2). Putting

$$F^sU(\mathfrak{g}) := \{\lambda \in U(\mathfrak{g}), ||\lambda||_{\mathfrak{X},r} \le p^{-s}\},$$

$s \in \mathbb{R}$ turns $U(\mathfrak{g})$ into a filtered ring (cf. [loc.cit.], end of sect. 2). Using the hypothesis again we obtain for $i < j$ that

$$||\mathfrak{x}_i\mathfrak{x}_j - \mathfrak{x}_j\mathfrak{x}_i||_{\mathfrak{X},r} \le r < r^2 = ||\mathfrak{x}_i\mathfrak{x}_j||_{\mathfrak{X},r}.$$

The associated graded ring is therefore a polynomial ring over $gr^{\cdot} L$ in the principal symbols $\sigma(\mathfrak{x}_j)$. Thus, it is an integral domain and therefore the norm on $U(\mathfrak{g})$ must be multiplicative. □

Lemma 3.2. *The locally convex topology on $\mathfrak{F}(\mathfrak{g})$ induced by the family $||.||_{\mathfrak{X},r},\ r > 1$ is independent of the choice of $\mathfrak{x}_1, ..., \mathfrak{x}_d$.*

Proof. Since a homothety in $L^\times$ obviously induces an equivalent locally convex topology we are reduced to show: given two bases $\mathfrak{X}$ and $\mathfrak{Y}$ of $\mathfrak{g}$ with integral structure constants there is a real constant C such that for any $\lambda \in U(\mathfrak{g})$ and any real $r > 1$ we have $||\lambda||_{\mathfrak{Y},r} \leq ||\lambda||_{\mathfrak{X},Cr}$. For all $i = 1, ..., d$ let $\mathfrak{x}_i =: \sum_{j=1,...,d} a_{ij}\mathfrak{y}_j,\ a_{ij} \in L$ and let $C := \max_{ij}(|a_{ij}|)$. If $\lambda = \sum_\alpha d_\alpha \mathfrak{X}^\alpha$ then

$$||\lambda||_{\mathfrak{Y},r} \leq \sup_\alpha |d_\alpha| (Cr)^{|\alpha|} = ||\lambda||_{\mathfrak{X},Cr}$$

using that $||.||_{\mathfrak{Y},r}$ is multiplicative by Lem. 3.1. □

Recall that a *Hopf* $\hat{\otimes}$*-algebra* is a Hopf algebra object in the braided monoidal category of topological algebras. Without recalling more details on these definitions (cf. [7], sect. 2) we state the

Lemma 3.3. *$\mathfrak{F}(\mathfrak{g})$ is a Hopf $\hat{\otimes}$-algebra with invertible antipode.*

Proof. The locally convex vector space $\mathfrak{F}(\mathfrak{g})$ is visibly a Fréchet space ([9], §8) whence the inductive and projective tensor product topologies on $\mathfrak{F}(\mathfrak{g}) \otimes_L \mathfrak{F}(\mathfrak{g})$ coincide ([loc.cit.], Prop. 17.6). Fixing a basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}$ the topology on the topological algebra $\mathfrak{F}(\mathfrak{g})\hat{\otimes}_L\mathfrak{F}(\mathfrak{g})$ may therefore be described by the tensor product norms $||.||_{\mathfrak{X},r} \otimes ||.||_{\mathfrak{X},r}, r > 1$. A direct computation now shows that the usual Hopf algebra structure on $U(\mathfrak{g})$ extends to the completion $\mathfrak{F}(\mathfrak{g})$. □

Remark: Suppose L is a p-adic local field, G is a locally L-analytic group and $C_1^{an}(G, L)$ denotes the stalk at $1 \in G$ of germs of L-valued locally analytic functions on G. Then $\mathfrak{F}(\mathfrak{g}) = C_1^{an}(G, L)'_b$ as locally convex vector spaces ([4], Prop. 1.2.8) in accordance with the complex situation (cf. [8],[7]). Analogous to this situation (cf. [7], sect. 8) the space $C_1^{an}(G, L)$ inherits a Hopf $\hat{\otimes}$-algebra structure by functoriality in G. The structure on $\mathfrak{F}(\mathfrak{g})$ may then be obtained by passing to strong duals.

Corollary 3.4. *Suppose that the natural continuous map*

$$\mathfrak{F}(\mathfrak{g})\hat{\otimes}_{U(\mathfrak{g})}L \longrightarrow L \tag{3.1}$$

is a topological isomorphism. If there exists a projective resolution $P_\bullet \to L$ in $\mathcal{M}_{U(\mathfrak{g})}$ such that the base extension via θ

$$\mathfrak{F}(\mathfrak{g})\hat{\otimes}_{U(\mathfrak{g})}P_\bullet \longrightarrow \mathfrak{F}(\mathfrak{g})\hat{\otimes}_{U(\mathfrak{g})}L \tag{3.2}$$

is a s-exact complex in $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$, then θ is stably flat.

Proof. This follows from the above lemma together with [7], Prop. 3.7. Note that the proof of the latter proposition directly carries over to our setting since it is based on formal properties of Hopf $\hat{\otimes}$-algebras with invertible antipodes in certain braided monoidal categories. □

Remark: A.Y. Pirkovski calls homomorphisms between complex topological algebras satisfying (*mutatis mutandis*) the conditions (3.1) and (3.2) *weak localizations.* The content of [loc.cit.], Prop. 3.7 is then that a homomorphism between Hopf $\hat{\otimes}$-algebras with invertible antipodes is a localization if and only if it is a weak localization.

By Lem. 3.1 the map θ induces inclusions

$$U(\mathfrak{g}) \subseteq \hat{U}(\mathfrak{g}) \subseteq \mathfrak{F}(\mathfrak{g}). \tag{3.3}$$

On the other hand, suppose $||.||$ is a submultiplicative semi-norm on $U(\mathfrak{g})$. Choose a basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}$ with integral structure constants and let $r > 1$ such that $r \geq \max_j ||\mathfrak{x}_j||$. Given $\lambda \in U(\mathfrak{g})$ write $\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathfrak{X}^\alpha,\ d_\alpha \in L$ and compute

$$||\lambda|| \leq \sup_\alpha |d_\alpha|\ ||\mathfrak{X}^\alpha|| \leq \sup_\alpha |d_\alpha| r^\alpha = ||\lambda||_{\mathfrak{x},r}.$$

Hence, the second inclusion in (3.3) is surjective and the second part of the theorem is proved.

3.2. **Contracting homotopies.** We fix a real $r > 1$, a L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}$ with integral structure constants and let $||.||_r := ||.||_{\mathfrak{x},r}$. Recall the homological standard complex $U_\bullet := U(\mathfrak{g}) \otimes_L \bigwedge^\bullet \mathfrak{g}$ with differential $\partial = \psi + \phi$ where

$$\psi(\lambda \otimes \mathfrak{x}_1 \wedge ... \wedge \mathfrak{x}_q) = \sum_{s<t} (-1)^{s+t} \lambda \otimes [\mathfrak{x}_s, \mathfrak{x}_t] \wedge \mathfrak{x}_1 \wedge ... \wedge \widehat{\mathfrak{x}_s} \wedge ... \wedge \widehat{\mathfrak{x}_t} \wedge ... \wedge \mathfrak{x}_q,$$

$$\phi(\lambda \otimes \mathfrak{x}_1 \wedge ... \wedge \mathfrak{x}_q) = \sum_s (-1)^{s+1} \lambda \mathfrak{x}_s \otimes \mathfrak{x}_1 \wedge ... \wedge \widehat{\mathfrak{x}_s} \wedge ... \wedge \mathfrak{x}_q$$

(cf. [2]). Let I_q be the collection of indices $1 \leq i_1 < ... < i_q \leq d$ and let $\lambda_q = \sum_{I \in I_q} u_I \otimes x_I \in U_q$ with $u_I \in U(\mathfrak{g}), x_I = \mathfrak{x}_{i_1} \wedge ... \wedge \mathfrak{x}_{i_q} \in \bigwedge^q \mathfrak{g}$ be an element. If $\sum_q \lambda_q \in U_\bullet$ we let

$$||\sum_q \lambda_q||_r := \sup_q r^q \sup_{I \in I_q} ||u_I||_r. \tag{3.4}$$

By Lem. 3.1 $U_\bullet$ becomes in this way a faithfully normed left $U(\mathfrak{g})$-module (in the sense of [1], Def. 2.1.1/1).

Lemma 3.5. *The differential ∂ is norm-decreasing on $(U_\bullet, ||.||_r)$.*

Proof. Let c_{ijk} denote the structure constants of $\mathfrak{x}_1, ..., \mathfrak{x}_d$. Since $|c_{ijk}| \leq 1$ and since $||.||_r$ is multiplicative on $U(\mathfrak{g})$ (Lem. 3.1) we obtain

$$\begin{aligned} ||\partial(\lambda_q)||_r &\leq \sup_{I \in I_q} || \sum_{s<t} (-1)^{s+t} u_I \otimes (\sum_k c_{stk} \mathfrak{x}_k) \wedge \mathfrak{x}_1 \wedge ... \wedge \widehat{\mathfrak{x}_s} \wedge ... \wedge \widehat{\mathfrak{x}_t} \wedge ... \wedge \mathfrak{x}_q \\ &\quad + \sum_s (-1)^{s+1} u_I \mathfrak{x}_s \otimes \mathfrak{x}_1 \wedge ... \wedge \widehat{\mathfrak{x}_s} \wedge ... \wedge \mathfrak{x}_q)||_r \\ &\leq \sup_{I \in I_q} \max (\sup_{s<t} r^{q-1} ||u_I||_r, \sup_s r^q ||u_I||_r) \\ &\leq \sup_{I \in I_q} r^q ||u_I||_r = ||\lambda_q||_r. \end{aligned}$$

□

In the following we endow $U_\bullet$ with the locally convex topology induced by the family of norms $||.||_r, r > 1$. The last result then implies that ∂ is continuous.

Recall that a *contracting homotopy* on an augmented homological complex of L-vector spaces $X_\bullet \xrightarrow{\epsilon} L$ is an L-linear map $\eta : L \to X_0$ together with a family of L-linear maps $s_q : X_q \to X_{q+1},\ q \geq 0$ satisfying

$$\epsilon \circ \eta(x) = x, \quad (\partial_{q+1} \circ s_q + s_{q-1} \circ \partial_q)(y) = y,$$

$$s_0 \circ \eta(x) = 0, \quad \partial_1 \circ s_0(z) = z - (\eta \circ \epsilon)(z)$$

for $x \in L,\ y \in X_q,\ z \in X_0,\ q \geq 1$ (e.g. [6], V.1.1.4). Now suppose that $X_\bullet \to L$ is an augmented complex in $\mathcal{M}_A$ for some topological L-algebra A. If it admits a contracting homotopy such that the maps η and $s_q,\ q\geq 0$ are continuous then it is s-exact ([12], remark after Def. 1.5).

Recall that the augmented complex $U_\bullet \xrightarrow{\epsilon} L$ has a distinguished contracting homotopy s ([loc.cit.], V.1.3.6.2). To review its construction let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. To ease notation we denote its natural augmentation by ϵ as well. Let

$$S_\bullet := S(\mathfrak{g}) \otimes_L \dot{\bigwedge} \mathfrak{g} \xrightarrow{\epsilon} L$$

be the augmented *Koszul complex* ([loc.cit.], V.1.3.3) attached to the vector space $\mathfrak{g}$ with differential ϕ (*mutatis mutandis*). Note that the choice of $\mathfrak{x}_1,...,\mathfrak{x}_d$ induces an isomorphism

$$f : U_\bullet \xrightarrow{\cong} S_\bullet \tag{3.5}$$

as L-vector spaces compatible with ϵ. The augmented complex $S_\bullet \to L$ comes equipped with the following contracting homotopy $\bar{s}$ depending on the basis $\mathfrak{x}_1,...,\mathfrak{x}_d$ (cf. [loc.cit.], (1.3.3.4)). In case $d=1$ it is given by the structure map $\eta : L \to S(\mathfrak{g})$ together with $\bar{s}_0 := S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes \bigwedge^1 \mathfrak{g}$ defined via $\bar{s}_0(\mathfrak{x}_1^n) = \mathfrak{x}_1^{n-1} \otimes \mathfrak{x}_1$ for all $n\in\mathbb{N}$ and $\bar{s}_0(1)=0$. In general, the definition is extended to the tensor product

$$f^1_\bullet : S^1_\bullet \otimes_L \cdots \otimes_L S^d_\bullet \xrightarrow{\cong} S_\bullet$$

by general principles (cf. [loc.cit.], V.1.3.2.). Here, $S^j_\bullet$ equals the Koszul complex of the vector space $L\mathfrak{x}_j$ and $f^1_\bullet$ comes from functoriality of $S_\bullet$ applied to $\mathfrak{g} = \oplus_j L\mathfrak{x}_j$. One obtains from $\bar{s}$ the desired homotopy s on $U_\bullet$ as follows: pulling $\bar{s}$ back to $U_\bullet$ via f gives an L-linear map σ on $U_\bullet$. It gives rise to maps

$$\sigma^{(n)} : U_\bullet \longrightarrow U_\bullet,\ \sigma_q^{(n)} : U_q \longrightarrow U_{q+1}$$

(cf. [loc.cit.], Lem. V.1.3.5) having the property: for fixed $x\in U_q$ the sequence $(\sigma_q^{(n)}(x))_{n\in\mathbb{N}} \subseteq U_{q+1}$ becomes eventually stationary ([loc.cit.], remark after formula V.1.3.6.2). Then

$$s_q(x) := \lim_n \sigma_q^{(n)}(x)$$

defines the desired contracting homotopy s on $U_\bullet \to L$.

Given $||.||_r, r>1$ on $U_\bullet$ we may use the basis $\mathfrak{x}_1,...,\mathfrak{x}_d$ to define in complete analogy to (2.1) and (3.4) norms on $S(\mathfrak{g})$ and $S_\bullet$ which we also denote by $||.||_r$.

Lemma 3.6. *The homotopy $\bar{s}$ is norm-decreasing on $(S_\bullet, ||.||_r)$.*

Proof. By induction on $d=\dim_L \mathfrak{g}$ we may endow the left-hand side of the isomorphism $f^1_\bullet$ with the following norm:

$$||\lambda||_r := \sup_{s+t=q} \inf_{(\lambda_s),(\mu_t)} ||\lambda_s||_r\, ||\mu_t||_r$$

where $\lambda \in (S^i_\bullet \otimes_L S^j_\bullet)_q = \oplus_{s+t=q} S^i_s \otimes_L S^j_t$ is of the form $\lambda = \sum_{s+t=q}(\sum \lambda_s \otimes \mu_t)$ and the infimum is taken over all possible representations $\sum \lambda_s \otimes \mu_t$ of the (s,t)-component of λ. We claim that $f^1_\bullet$ is isometric. Again by induction we are reduced to prove the claim for f^2_q where

$$f^2_\bullet : S^{<d}_\bullet \otimes_L S^d_\bullet \xrightarrow{\cong} S_\bullet$$

and $S_\bullet^{<d}$ equals the Koszul complex of $\oplus_{j<d}L\mathfrak{x}_j$. Fix $q \geq 0$. By definition of $||.||_r$ the decomposition

$$(S_\bullet^{<d} \otimes S_\bullet^d)_q = \oplus_{s+t=q} S_s^{<d} \otimes S_t^d$$

is orthogonal. By definition of f_q^2 and since the elements $\{1 \otimes x_{I_q}\}_{I_q}$ are orthogonal in S_q, f_q^2 preserves this orthogonality in S_q. It therefore suffices to fix $s+t=q$ and prove $||f_q^2(\lambda)||_r = ||\lambda||_r$ for $\lambda \in S_s^{<d} \otimes_L S_t^d$. In both cases ($s=q$ and $s=q-1$) this is a straightforward computation whence $f_\bullet^1$ is indeed isometric. Next we prove that $\bar{s}$ is norm-decreasing on the left-hand side of the isomorphism $f_\bullet^1$. For $d=1$ this follows since η and $\bar{s}_0$ are certainly norm-decreasing. By induction we may suppose that this is true on the complex $S_\bullet^{<d}$ and consider the tensor product $S_\bullet^{<d} \otimes_L S_\bullet^d$. Let $\lambda \in (S_\bullet^{<d} \otimes_L S_\bullet^d)_q$. Suppose $q=0$ and hence $\lambda \in L$. It is then clear that $||s(\lambda)||_r = ||\eta(\lambda) \otimes 1||_r = ||\lambda||_r$ where the first identity follows from formula [loc.cit.], V.1.3.2.2. So assume $q>0$. Write $\lambda = \sum_{s+t=q}(\sum \lambda_s \otimes \mu_t)$. Then

$$\bar{s}(\lambda) = \sum_{s+t=q, s>0} \sum \bar{s}(\lambda_s) \otimes \mu_t + \sum_{s+t=q, s=0} \sum \bar{s}(\lambda_s) \otimes \mu_t + \eta\epsilon(\lambda_s) \otimes \bar{s}(\mu_t).$$

according to the formulas [loc.cit.], V.1.3.2.2/1.3.2.3. Using the induction hypothesis on the right-hand side one obtains $||\bar{s}(\lambda)||_r \leq ||\lambda||_r$ as desired. □

Lemma 3.7. *The homotopy s is continuous with respect to the locally convex topology on $U_\bullet$.*

Proof. Fix the norm $||.||_r$, $r>1$ on $U_\bullet$ and $S_\bullet$. By construction, the map f appearing in (3.5) becomes isometric. Hence, Lem. 3.6 shows the L-linear map $\sigma = f^{-1} \circ \bar{s} \circ f$ on $U_\bullet$ to be norm-decreasing. The augmentation $\epsilon : U_0 \to L$ and the differential ∂ are also norm-decreasing (the latter by Lem. 3.5). Invoking the maps $\sigma^{(n)}$ from above we deduce from $\sigma^{(0)} = \sigma$ and the formula

$$\sigma^{(n)} - \sigma^{(n-1)} = \sigma(1 - \epsilon - \partial\epsilon - \epsilon\partial)^n$$

([loc.cit.], V.1.3.5.4) by induction that all $\sigma^{(n)}$ are norm-decreasing. Now the contracting homotopy s of $U_\bullet$ is defined as the pointwise limit $s_q(x) := \lim_n \sigma_q^{(n)}(x)$, $x \in U_q$. Since the sequence $\sigma_q^{(n)}(x)$ for $n \to \infty$ becomes eventually stationary s is seen to be norm-decreasing on $(U_\bullet, ||.||_r)$. Since $r>1$ was arbitrary the proof is complete. □

3.3. **Stable flatness.** We prove the remaining part of the theorem and the corollary of section 2, respectively.

Proof. By Cor. 3.4 it suffices to show that the natural continuous map

$$\mathfrak{F}(\mathfrak{g}) \hat{\otimes}_{U(\mathfrak{g})} L \longrightarrow L \tag{3.6}$$

is a topological isomorphism and secondly, that there exists a projective resolution $P_\bullet \to L$ in $\mathcal{M}_{U(\mathfrak{g})}$ such that the base extension via θ

$$\mathfrak{F}(\mathfrak{g}) \hat{\otimes}_{U(\mathfrak{g})} P_\bullet \longrightarrow \mathfrak{F}(\mathfrak{g}) \hat{\otimes}_{U(\mathfrak{g})} L \tag{3.7}$$

is a s-exact complex in $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$.

The augmentation ideal in $\mathfrak{F}(\mathfrak{g})$ clearly generates the augmentation ideal of $U(\mathfrak{g})$. Hence, the map (3.6) is a bijection between finite dimensional L-vector spaces and therefore topological. To prove the second condition we let $P_\bullet := U_\bullet$ together with the augmentation ϵ. The complex $U_\bullet$ consists of s-projective modules (which are even *s-free* in the sense of [7], sect. 1) and, admitting the (continuous) contracting homotopy s, the augmented complex $P_\bullet \to L$ is s-exact. Using associativity of $\hat{\otimes}$ the base extension (3.7) may be identified with $\mathfrak{F}(\mathfrak{g}) \otimes_L \dot{\bigwedge} \mathfrak{g} \to L$ and thus, equals the Hausdorff completion of the topologized complex $U_\bullet \to L$. By continuity (Lem. 3.7) the map s extends to this completion yielding a continuous contracting homotopy on (3.7). □

Proof. Let $M, N \in \mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$. Any projective resolution $P_\bullet \to M$ in $\mathrm{Mod}(\mathfrak{g})$ by abstract free $U(\mathfrak{g})$-modules is a projective resolution in $\mathcal{M}_{U(\mathfrak{g})}$ when endowed with the finest locally convex topology. One may check that [7], Prop. 3.3 remains valid in our setting whence the natural map $\mathfrak{F}(\mathfrak{g})\hat{\otimes}_{U(\mathfrak{g})} M \to M$ is an isomorphism in $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$. Hence, by stable flatness $\mathfrak{F}(\mathfrak{g})\hat{\otimes}_{U(\mathfrak{g})} P_\bullet \to M$ is a projective resolution of M in $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$. The claims follow now from the isomorphisms of complexes

$$\mathcal{L}_{\mathfrak{F}(\mathfrak{g})}(\mathfrak{F}(\mathfrak{g})\hat{\otimes}_{U(\mathfrak{g})} P_\bullet, N) \simeq \mathcal{L}_{U(\mathfrak{g})}(P_\bullet, N) \simeq \mathrm{Hom}_{\mathfrak{g}}(P_\bullet, N)$$

and

$$N\hat{\otimes}_{\mathfrak{F}(\mathfrak{g})}(\mathfrak{F}(\mathfrak{g})\hat{\otimes}_{U(\mathfrak{g})} P_\bullet) \simeq N\hat{\otimes}_{U(\mathfrak{g})} P_\bullet \simeq N \otimes_{U(\mathfrak{g})} P_\bullet$$

where the last isomorphisms in both rows follow from the fact that,in each degree, $P_\bullet$ carries the finest locally convex topology. □

Remark: Following [12], Def. 2.3 one may also introduce relative Hochschild cohomology in our nonarchimedean setting. Another proof of the corollary (at least for the Tor-groups) may then be deduced using analogues of [13], Prop. 1.4 and [7], Prop. 3.4.

References

1. S. Bosch, U. Güntzer, R. Remmert, *Non-Archimedean Analysis. Berlin Heidelberg New York: Springer 1984*
2. C. Chevalley, S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras. Trans. AMS 63 (1948)*
3. A. Ya. Helemski, *Banach and Poylnormed algebras: General Theory, Representations, Homology. Nauka, Moscow 1989 (Russian). Engl. Transl.: Oxford University Press 1993*
4. J. Kohlhaase, *Invariant distributions on p-adic analytic groups. Duke Math. J. 137 (2007)*
5. J. Kohlhaase, *The cohomology of locally analytic representations. Münster: Preprint (2007)*
6. M. Lazard, *Groupes analytiques p-adiques. Publ. Math. Inst. Hautes Étud. Sci. 26 (1965)*
7. A.Y. Pirkovski, *Stably flat completions of universal enveloping algebras. Diss. Math. 441 (2006)*
8. P.K. Raševskii *Associative superenvelope of a Lie algebra and its regular representation and ideals. (Russian) Trudy MMO 15 (1966)*
9. P. Schneider, *Nonarchimedean Functional Analysis. Berlin Heidelberg New York: Springer 2002*
10. P. Schneider, J. Teitelbaum, *Algebras of p-adic distributions and admissible representations. Inv. Math. 153 (2003)*
11. P. Schneider, J. Teitelbaum, *Locally analytic distributions and p-adic representation theory, with applications to GL_2. J. Am. Math. Soc. 15 (2002)*
12. J.L. Taylor, *Homology and cohomology for topological algebras. Adv. Math. 9 (1972)*
13. J.L. Taylor, *A general framework for a multi-operator functional calculus. Adv. Math. 9 (1972)*
14. J. Teitelbaum, *Algèbres de Hecke et représentations de Banach et localement analytiques des groupes de Lie p-adiques. Course notes. Available at: http://evenements.institut.math.jussieu.fr/ecole_ete_2008/files/Teitelbaum−Main.Notes.pdf*

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