Mod p Hecke algebras and dual equivariant cohomology

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(joint work with Cédric Pépin)

Let $F/\mathbb{Q}_p$ be a finite extension field with residue field $\mathbb{F}_q$ and let $G = \mathrm{GL}_n(F)$. Let $I \subset G$ be the standard Iwahori subgroup of G. The classical Deligne-Langlands conjecture for Hecke modules, proved in the middle of the 1980's by Kazhdan-Lusztig [10], is an incarnation of the (tame) local Langlands correspondance for G. It predicts a parametrization of the simple modules of the complex Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{C}} = \mathbb{C}[I \setminus G/I]$ in terms of certain pairs $(s,t) \in \widehat{\mathbf{G}}^2$ where $sts^{-1} = t^q$. Here, $\widehat{\mathbf{G}} = \mathrm{GL}_n(\mathbb{C})$ is the complex Langlands dual group of G. Let $\widehat{\mathcal{B}}$ be the complex flag variety of $\widehat{\mathbf{G}}$ together with its $\widehat{\mathbf{G}}$-action given by translations. One of the steps in the proof of the conjecture is the construction of a suitable $\mathcal{H}_{\mathbb{C}}$-action on the equivariant K-theory $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\mathbb{C}}$ [11]. It identifies the action of the center $Z(\mathcal{H}_{\mathbb{C}})$ with the scalar multiplication by $K^{\widehat{\mathbf{G}}}(\mathrm{pt})_{\mathbb{C}} = R(\widehat{\mathbf{G}})_{\mathbb{C}}$ where $R(\widehat{\mathbf{G}})$ denotes the representation ring of the algebraic group $\widehat{\mathbf{G}}$.

The idea of studying various cohomology theories of the flag variety by means of Hecke operators (nowadays called Demazure operators) goes back to earlier work of Demazure [3, 4]. Of course, much of all this extends to reductive groups more general than GL_n.

In this extended abstract we report on the very first steps towards a mod p version of Kazhdan-Lusztig theory which has the aim to understand better the geometry of the modular Hecke algebras which appear in the mod p local Langlands program. We change slightly notation and let from now on $\widehat{\mathbf{G}} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ be the mod p Langlands dual group of G and let $\widehat{\mathcal{B}}$ be its flag variety. Let $\mathcal{H}^{(1)}$ be the pro-p Iwahori-Hecke algebra of G, with coefficients in $\bar{\mathbb{F}}_p$ [15]. The center of $\mathcal{H}^{(1)}$ contains a subring $Z^{\circ}(\mathcal{H}^{(1)})$ which is isomorphic to the monoid algebra $\bar{\mathbb{F}}_p[\Lambda^+]$ of the dominant cocharacters Λ^+ of G [12]. In turn, the classical Steinberg isomorphism identifies the monoid algebra $\bar{\mathbb{F}}_p[\Lambda^+]$ with $R(\widehat{\mathbf{G}})_{\bar{\mathbb{F}}_p}$. An element $s \in \widehat{\mathbf{G}}$ is called supersingular, if the associated character $Z^{\circ}(\mathcal{H}^{(1)}) \to \bar{\mathbb{F}}_p$ is supersingular, i.e. sends the noninvertible fundamental dominant cocharacters to zero. An irreducible $\mathcal{H}^{(1)}$-module is called *supersingular*, if $Z^{\circ}(\mathcal{H}^{(1)})$ acts via a supersingular character and this notion generalizes to finite length modules [16]. The interest in this notion comes from the mod p Langlands correspondence:

Theorem (Breuil, Colmez, Grosse-Klönne, Vignéras [1, 2, 7, 8, 14]). There is an exact and fully faithful functor from the category of supersingular $\mathcal{H}^{(1)}$-modules to the category of $\mathrm{Gal}(\bar{F}/F)$-representations over $\bar{\mathbb{F}}_p$.

Let $\mathbb{T} = T(\mathbb{F}_q)$ be the finite diagonal torus of G and let $\mathbb{T}^{\vee}$ be its set of characters. Let $W_0 \subset W_{\mathrm{aff}} \subset W$ be the finite, affine and Iwahori Weyl group of G respectively. The algebra $\mathcal{H}^{(1)}$ decomposes into finitely many components

$$\mathcal{H}^{(1)} = \prod_{\gamma \in \mathbb{T}^\vee / W_0} \mathcal{H}^\gamma$$

and, consequently, so does the category of $\mathcal{H}^{(1)}$-modules. If $|\gamma| = 1$ or $|\gamma| = W_0$, the γ-component is called *of Iwahori type* or *regular* respectively. The structure of a general γ-component is a 'mixture' between these two extreme cases.

Let γ be an Iwahori component. Then

$$\mathcal{H}^\gamma \simeq \mathcal{H} := \bar{\mathbb{F}}_p[I \setminus G / I]$$

is isomorphic to the Iwahori-Hecke algebra over $\bar{\mathbb{F}}_p$ (which itself is the component associated to the orbit of the trivial character of $\mathbb{T}$). Fix a set S_0 of simple reflections and a set $S_{\text{aff}} = S_0 \cup \{s_0\}$ of simple affine reflections for W_0 and W_{aff} respectively. The quadratic relations in $\mathcal{H}$ are given by $T_s^2 = -T_s$ for $s \in S_{\text{aff}}$.

Theorem 1. There exists a unique algebra homomorphism

$$\mathscr{A} : \mathcal{H} \longrightarrow \text{End}_{R(\widehat{\mathbf{G}})_{\bar{\mathbb{F}}_p}}(K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\bar{\mathbb{F}}_p})$$

such that

- $\mathscr{A}(T_s) = -D_s \quad \forall s \in S_0$ (Demazure operator)
- $\mathscr{A}|_{Z(\mathcal{H})} : Z(\mathcal{H}) \xrightarrow{\sim} R(\widehat{\mathbf{G}})_{\bar{\mathbb{F}}_p}$ (Ollivier-Steinberg isomorphism).

Remark: Note that $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}) = \mathbb{Z}[\Lambda]$ and $K^{\widehat{\mathbf{G}}}(\text{pt}) = \mathbb{Z}[\Lambda]^{W_0}$ as abstract rings. In this setting the Demazure operator is given by

$$D_s(a) = \frac{a - s(a)}{1 - e^{\alpha^\vee}}$$

for all $a \in \mathbb{Z}[\Lambda]$ where $s = s_\alpha$ with associated simple root α [4].

Given $s \in \widehat{\mathbf{G}}$ we let $\mathcal{H}_s = \mathcal{H} \otimes_{Z(\mathcal{H}),s} \bar{\mathbb{F}}_p$. If s is semisimple, then, by localisation [13], $K^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}})_{\bar{\mathbb{F}}_p} \otimes_{R(\widehat{\mathbf{G}})_{\bar{\mathbb{F}}_p},s} \bar{\mathbb{F}}_p = K(\widehat{\mathcal{B}}^s)_{\bar{\mathbb{F}}_p}$ where $\widehat{\mathcal{B}}^s \subset \widehat{\mathcal{B}}$ denotes the s-fixed points. Then $\mathcal{H}_s$ acts on $K(\widehat{\mathcal{B}}^s)_{\bar{\mathbb{F}}_p}$. The $\bar{\mathbb{F}}_p$-dimension of $K(\widehat{\mathcal{B}}^s)_{\bar{\mathbb{F}}_p}$ is at most $|\widehat{\mathcal{B}}^s| = |W_0|$.

Theorem 2. Let $n \leq 3$. Fix a semisimple element $s \in \widehat{\mathbf{G}}$ which is supersingular. All n-dimensional simple supersingular $\mathcal{H}_s$-modules appear as subquotients of $K(\widehat{\mathcal{B}}^s)_{\bar{\mathbb{F}}_p}$ with multiplicity one.

Let now γ be a regular orbit. Then

$$\mathcal{H}^\gamma \simeq \bar{\mathbb{F}}_p^{|\gamma|} \otimes'_{\bar{\mathbb{F}}_p} \mathcal{H}^{\text{nil}}$$

is a certain smash product between the direct product ring $\bar{\mathbb{F}}_p^{|\gamma|}$ and Kostant-Kumar's *Nil Hecke ring* $\mathcal{H}^{\text{nil}}$ with coefficients in $\bar{\mathbb{F}}_p$ [9]. The quadratic relations in $\mathcal{H}^{\text{nil}}$ are given by $T_s^2 = 0$ for $s \in S_{\text{aff}}$. Moreover, $Z^\circ(\mathcal{H}^\gamma) := Z^\circ(\mathcal{H}^{(1)}) \cap \mathcal{H}^\gamma$ is a proper subring of $Z(\mathcal{H}^\gamma)$.

We let $\widehat{\mathcal{B}}^\gamma$ be the disjoint union of $|\gamma|$ copies of $\widehat{\mathcal{B}}$ and consider its equivariant intersection theory $CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^\gamma)$ [5, 6]. It is a module over $S(\widehat{\mathbf{G}}) = CH^{\widehat{\mathbf{G}}}(\mathrm{pt})$ together with an action perm : $W \to W_0 \curvearrowright \widehat{\mathcal{B}}^\gamma$ which permutes the factors. Note that

$$CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}) = \mathrm{Sym}(\Lambda) \qquad \text{and} \qquad CH^{\widehat{\mathbf{G}}}(\mathrm{pt}) = \mathrm{Sym}(\Lambda)^{W_0}$$

as abstract rings and the Demazure operator is given as

$$D_s(a) = \frac{a - s(a)}{\alpha^\vee}$$

for all $a \in \mathrm{Sym}(\Lambda)$ where $s = s_\alpha$ [3]. On these objects, we finally invert the invariant $\chi_n = \eta_1 \cdots \eta_n$ where the $\eta_i(x) = \mathrm{diag}(1, ..., 1, x, 1, ..., 1)$ (x in position i) are the standard basis elements for Λ.

Theorem 3. There exists a unique algebra homomorphism

$$\mathscr{A}^\gamma : \mathcal{H}^\gamma \longrightarrow \mathrm{End}_{S(\widehat{\mathbf{G}})[\chi_n^{-1}]_{\bar{\mathbb{F}}_p}}(CH^{\widehat{\mathbf{G}}}(\widehat{\mathcal{B}}^\gamma)[\chi_n^{-1}]_{\bar{\mathbb{F}}_p})$$

such that

- $\mathscr{A}^\gamma|_{\mathcal{H}^{\mathrm{nil}}}(T_s) = -D_s \circ \mathrm{perm}(s) \quad \forall s \in S_0$
- $\mathscr{A}^\gamma|_{Z^\circ(\mathcal{H}^\gamma)} : Z^\circ(\mathcal{H}^\gamma) \hookrightarrow S(\widehat{\mathbf{G}})[\chi_n^{-1}]_{\bar{\mathbb{F}}_p}$.

Theorem 4. Let $n \leq 3$. Fix a semisimple element $s \in \widehat{\mathbf{G}}$ which is supersingular. All n-dimensional simple supersingular $\mathcal{H}_s^\gamma$-modules appear as subquotients of $CH(\widehat{\mathcal{B}}^{\gamma,s})[\chi_n^{-1}]_{\bar{\mathbb{F}}_p}$ with multiplicity one.

We believe that these theorems can be generalized to any n and to all γ-components of the pro-p Hecke algebra $\mathcal{H}^{(1)}$. Let $\varphi, v \in \mathrm{Gal}(\bar{F}/F)$ be a Frobenius lift and a monodromy generator respectively. Any tame n-dimensional $\mathrm{Gal}(\bar{F}/F)$-representation over $\bar{\mathbb{F}}_p$ leads to a couple $(s = \rho(\varphi), t = \rho(v)) \in \widehat{\mathbf{G}}^2$ with $sts^{-1} = t^q$. A natural question is then the following: Can one parametrize the supersingular $\mathcal{H}_s^{(1)}$-modules appearing in the cohomology of $\widehat{\mathcal{B}}^s$ via the parameter t? This is work in progress.

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