PICARD GROUPS IN p-ADIC FOURIER THEORY

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Abstract. Let $L \neq \mathbb{Q}_p$ be a proper finite field extension of $\mathbb{Q}_p$ and $o \subset L$ its ring of integers viewed as an abelian locally L-analytic group. Let $\hat{o}$ be the rigid L-analytic group parametrizing the locally analytic characters of o constructed by Schneider-Teitelbaum. Let K/L be a finite extension field. We show that the base change $\hat{o}_K$ has a Picard group $Pic(\hat{o}_K)$ which is profinite and that the unit section in $\hat{o}_K$ provides a divisor class of infinite order. In particular, the abelian group $Pic(\hat{o}_K)$ is not finitely generated and is not a torsion group. On the way we show that $\hat{o}_K$ is a nontrivial étale covering of the affine line over K realized via the logarithm map of a Lubin-Tate formal group. We finally prove that rank and determinant mappings induce an isomorphism between $K_0(\hat{o}_K)$ and $\mathbb{Z} \oplus Pic(\hat{o}_K)$.

1. Introduction

Let L be a complete nonarchimedean field and $\mathbf{B}$ a rigid L-analytic open polydisc. A twisted form of $\mathbf{B}$ is a rigid analytic space X over L together with a complete nonarchimedean extension $L \subseteq L'$ and an isomorphism $X_{L'} \xrightarrow{\cong} \mathbf{B}_{L'}$. The study of the forms of $\mathbf{B}$ with respect to a given extension $L \subseteq L'$ is an important yet difficult problem which is still in its infancy. As a first result A. Ducros recently has shown [14] that, if $L \subseteq L'$ is finite and tamely ramified, any form X is already trivial, in the sense that X is itself isomorphic to an open polydisc. He also showed that there are finite wildly ramified extensions that support plenty of nontrivial forms, even in dimension one.

A coarser but more accessible problem concerns the description of basic invariants of such forms such as the Picard group or the Grothendieck group. The aim of this note is to make a first step in this direction and study the Picard group of an interesting form that comes from p-adic representation theory. To give more details, let $L \neq \mathbb{Q}_p$ be from now on a proper finite extension of $\mathbb{Q}_p$ and let $\mathbf{B}$ be the open unit disc of dimension one. In [33] Schneider-Teitelbaum generalize the classical p-adic Fourier theory of Y. Amice [1] for the group $\mathbb{Z}_p$ to the additive group of integers o in L. The key step is the construction of a certain nontrivial form $\hat{o}$ of $\mathbf{B}$ ($\hat{o}$ is even a group object) with respect to the transcendental extension $L \subset \mathbb{C}_p$ having the surprising property of admitting no trivializiation over any discretely valued complete subfield of $\mathbb{C}_p$. In particular, this forces its Picard group to be nontrivial. In [38] J. Teitelbaum suggested to study this Picard group, partly for representation-theoretic reasons which we will describe below.

Slightly more general, we will study the series of Picard groups $Pic(\hat{o}_K)$ where $\hat{o}_K$ denotes the base change of $\hat{o}$ to finite extensions K of L. Although we are not able to determine the structure of $Pic(\hat{o}_K)$ explicitly, our main result shows that it is of enormous size. More precisely, we prove that $Pic(\hat{o}_K)$ is a profinite group and

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that the unit section of $\hat{o}_K$ supports a divisor class of infinite order. In particular, the abelian group $Pic(\hat{o}_K)$ is not finitely generated and is not a torsion group. On the way we show that $\hat{o}_K$ is a nontrivial étale covering of the affine line over K (in the sense of [11]). We finally show that rank and determinant mappings induce an isomorphism $K_0(\hat{o}_K) \xrightarrow{\cong} \mathbb{Z} \oplus Pic(\hat{o}_K)$.

As already indicated the variety $\hat{o}_K$ has a representation-theoretic interpretation. Indeed the additive group o is among the first examples of a compact abelian locally L-analytic group. The generalized Amice-Fourier isomorphism of Schneider-Teitelbaum [33] induces an isomorphism of the ring of holomorphic functions on $\hat{o}_K$ with the K-valued locally analytic distribution algebra of o. Since $\hat{o}_K$ is a quasi-Stein space in the sense of R. Kiehl [21] the group $Pic(\hat{o}_K)$ controls the ideal structure of this distribution algebra and therefore the locally analytic representation theory of o. For example, the rational points of $\hat{o}_K$ are in bijective correspondence with the locally analytic characters $o \to K^\times$.

In the following we briefly outline the article. As with most nontrivial Picard groups there is by no means a straightforward way to compute the structure of $Pic(\hat{o}_K)$. Our strategy is to first determine the local Picard groups corresponding to a suitable open affinoid covering of the quasi-Stein space $\hat{o}_K$ and then 'glue' these informations. We begin by recalling some results on the divisor theory for Dedekind domains. The point is that the affinoid algebra of a twisted form of a nonarchimedean closed disc with respect to a finite extension is a Dedekind domain. In sect. 3 we use descent theory to prove a finiteness result for the Picard group of a certain class of such forms. In sect. 4 we turn to the variety $\hat{o}_K$. We prove that it admits an admissible affinoid covering $\hat{o}_K = \cup_n \hat{o}_{K,n}$ where each $\hat{o}_{K,n}$ is a twisted form of a closed disc with respect to a finite Galois extension (depending on n). The Galois cocycle giving the descent datum comes out of a Lubin-Tate group $\mathbb{G}$ for o and the logarithm $\log_{\mathbb{G}}$ identifies $\hat{o}_{K,n}$ with a finite étale covering of a closed disc whose degree equals q^{en}. Here, e and q equal the ramification index of $L/\mathbb{Q}_p$ and the cardinality of the residue field of L respectively. Passing to the limit in n proves $\hat{o}_K$ to be an étale covering of the affine line over K. The results of sect. 3 may be applied to $\hat{o}_{K,n}$ and yield the finiteness of $Pic(\hat{o}_{K,n})$. Using a spectral sequence argument combined with a vanishing result of L. Gruson [18] we find $Pic(\hat{o}_K) = \varprojlim_n Pic(\hat{o}_{K,n})$. Building on ideas of A. de Jong [10] we show that the zero section of the group $\hat{o}_K$ supports a divisor class of infinite order in $Pic(\hat{o}_K)$. Finally, since the ring of global sections $\mathcal{O}(\hat{o}_K)$ is a Prüferian domain, Serre's theorem from algebraic K-theory [2] implies the result on $K_0(\hat{o}_K)$.

As explained above the points of $\hat{o}$ parametrize the locally analytic characters of o. Generalizing the construction of $\hat{o}$ M. Emerton has introduced such a character variety for any abelian locally L-analytic group which is topologically finitely generated [16]. We conclude this work by briefly explaining how the problem of determining the Picard group of general character varieties can essentially be reduced to the case of (copies of) $\hat{o}$.

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2. Preliminaries on Dedekind domains

We recall some divisor theory for Dedekind domains, cf. [9], VII.§2, thereby fixing some notation. Recall that an integral domain is a *Dedekind domain* if it is noetherian, integrally closed and every nonzero prime ideal is maximal.

Let B be a Dedekind domain with field of fractions F. The free abelian group $D(B)$ on the set $Sp(B)$ of maximal ideals of B is called the *divisor group* of B. Given $P \in Sp(B)$ the localization B_P of B at P is a discrete valuation ring. Let v_P be the associated valuation. We have the well-defined group homomorphism

$$div_B : F^{\times} \longrightarrow D(B)\ ,\ x \mapsto \sum_{P \in Sp(B)} v_P(x)P$$

whose cokernel

$$Cl(B) := D(B)/\mathrm{im}\ div_B$$

is called the *divisor class group*.

On the other hand, let $Pic(B)$ denote the *Picard group* of B, i.e. the group of isomorphism classes of locally free B-modules of rank 1 (with the tensor product of B-modules as group law). There is a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & B^{\times} & \xrightarrow{\subseteq} & F^{\times} & \longrightarrow & Cart(B) & \longrightarrow & Pic(B) & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow{=} & & \downarrow{\iota} & & \downarrow{\bar\iota} & & \\ 1 & \longrightarrow & B^{\times} & \xrightarrow{\subseteq} & F^{\times} & \xrightarrow{div_B} & D(B) & \longrightarrow & Cl(B) & \longrightarrow & 1. \end{array} \tag{1}$$

Here, $Cart(B)$ refers to the group of invertible fractional ideals of B and the map $F^{\times} \to Cart(B)$ is given by $x \mapsto xB$. The map $\iota : Cart(B) \to D(B)$ is given by $I \mapsto \sum_{P \in Sp(B)} v_P(I)P$ where $v_P(I)$ equals the order of the extended fractional ideal in the discretely valued field $Quot(B_P)$. Both ι and $\bar\iota$ are bijections, e.g. [41], Cor. I.3.8.1.

Now suppose that $A \subseteq B$ is a subring which is a Dedekind domain itself such that $A \to B$ is integral or flat. Given $P' \in Sp(A)$, $P \in Sp(B)$ with $P \cap A = P'$ let $e(P/P') \in \mathbb{N}$ denote the ramification index of P over P'. Then $j(P') := \sum e(P/P')P$ induces a well-defined group homomorphism

$$j : D(A) \to D(B)$$

where the sum runs through all $P \in Sp(B), P \cap A = P'$. It factors into a group homomorphism

$$\bar\jmath : Cl(A) \to Cl(B),$$

cf. [9], VII.§1.10 Prop. 14.

Now suppose additionally that $A \subseteq B$ is a finite Galois extension with group G and such that $Cl(B) = 1$. The group G acts on $Sp(B)$ and on $D(B)$. The map j induces an isomorphism $D(A) \xrightarrow{\cong} D(B)^G$. Furthermore, $Quot(A) = F^G$. Taking G-invariants in the lower horizontal row of (1) and using Hilbert 90 yields the exact sequence

$$1 \longrightarrow A^{\times} \longrightarrow Quot(A)^{\times} \longrightarrow D(A) \xrightarrow{\delta} H^1(G, B^{\times}) \longrightarrow 1.$$

We obtain a canonical isomorphism

$$\bar\delta : Cl(A) \xrightarrow{\cong} H^1(G, B^{\times}).$$

3. Class groups of twisted affinoid discs

Let L be a complete non-archimedean field, i.e. a field that is complete with respect to a specified nontrivial non-archimedean absolute value. We assume that the reader is familiar with the classical theory of affinoid spaces over such a field [6].

For any L-affinoid algebra B we denote by $\mathring{B}$ the subring of power-bounded elements, by $\check{B}$ the $\mathring{B}$-ideal of topologically nilpotent elements and by $\tilde{B} := \mathring{B}/\check{B}$ the reduction of B. Passing to the reduction is a covariant functor from L-affinoid algebras to algebras over the residue field of L. If $|.|$ denotes the spectral seminorm on B we have $\mathring{B} = \{b \in B : |b| \leq 1\}$ and $\check{B} = \{b \in B : |b| < 1\}$. Moreover, if the ring B is reduced, the spectral seminorm is a norm and defines the Banach topology of B.

After these preliminaries let K/L be a finite Galois extension and let B be the one dimensional Tate algebra over K, i.e.

$$B = \{\sum_{n \geq 0} a_n z^n, a_n \in K, |a_n| \to 0 \text{ for } n \to \infty\}.$$

Here, we denote the unique extension of the absolute value on L to K also by $|.|$. Suppose A is a L-affinoid algebra equipped with an isomorphism

$$A \otimes_L K \xrightarrow{\cong} B$$

of K-algebras. In other words, A is a *twisted form* of B with respect to the extension K/L, cf. [22], II.§8. Let $G := Gal(K/L)$. The Galois group G acts on $A \otimes_L K$ by $\sigma(a \otimes x) = a \otimes \sigma(x)$ and, via transport of structure, we obtain in this way a semilinear Galois action on the K-algebra B.

Remark: The forms of the affinoid algebra B with respect to the Galois extension K/L are classified by the nonabelian Galois cohomology $H^1(G, \mathrm{Aut}_K(B))$ according to [29]. Here, $\mathrm{Aut}_K(B)$ refers to the automorphism group of the K-algebra B. Any automorphism of B is completely determined by its value on the variable z and induces, by functoriality, an automorphism of the algebra $\tilde{B} = \tilde{K}[z]$. Consequently, the map $f \mapsto f(z)$ induces a group isomorphism between $\mathrm{Aut}_K(B)$ and the group of formal power series

$$a_0 + a_1 z + a_2 z^2 + ...$$

subject to the conditions $|a_0| \leq 1, |a_1| = 1, |a_i| < 1$ for all $i > 1$, cf. [6], Corollary 5.1.4/10. By the enormous size of this group the classification of forms of B seems to be a difficult task. In [29] it is shown that any form of B with respect to a finite tamely ramified extension is trivial, in the sense that it is itself isomorphic to a closed disc. Moreover, it is shown that there are plenty of wildly ramified forms. Apart from these results the author does not know of any results in this direction in the literature.

Lemma 3.1. *The ring A is a Dedekind domain.*

Proof. The ring B is a principal ideal domain. Since the extension $A \subseteq B$ is integral the usual Going Up theorem shows A to be of dimension 1, e.g. [26], Ex. 9.2. Finally, let $a \in Quot(A)$ be integral over A . Since B is integrally closed we have $a \in B$ and thus, by Galois invariance, $a \in A$. □

We therefore have the canonical isomorphism $Cl(A) \xrightarrow{\cong} H^1(G, B^\times)$ at our disposal.

Lemma 3.2. *Suppose that the semilinear action of G maps the principal ideal (z) of B to itself. Then $\mathring{B}^\times \subseteq B^\times$ induces a short split exact sequence*

$$1 \longrightarrow |K^\times|/|L^\times| \longrightarrow H^1(G, \mathring{B}^\times) \longrightarrow H^1(G, B^\times) \longrightarrow 1.$$

Proof. Let $U := \{b_0 + b_1 z + ... \in B^\times : b_0 = 1\}$. Then $B^\times = UK^\times$ and $\mathring{B}^\times = U\mathring{K}^\times$ and the Galois action on B respects these direct products by assumption. Since cohomology commutes with direct sums we obtain, again by Hilbert 90, a short exact sequence

$$1 \longrightarrow H^1(G, \mathring{K}^\times) \longrightarrow H^1(G, \mathring{B}^\times) \longrightarrow H^1(G, B^\times) \longrightarrow 1.$$

The map $UK^\times \to U\mathring{K}^\times, (u,x) \mapsto (u,1)$ induces a splitting of this sequence. Finally, the long exact cohomology sequence induced by the short exact sequence

$$1 \longrightarrow \mathring{K}^\times \longrightarrow K^\times \xrightarrow{|.|} |K^\times| \longrightarrow 1$$

induces, again by Hilbert 90, the isomorphism $|K^\times|/|L^\times| \xrightarrow{\cong} H^1(G, \mathring{K}^\times)$. □

For any real number $0 < m < 1$ we let

$$\mathring{B}^{(m)} := \{b \in \mathring{B} : |b - 1| \leq m\}.$$

It is a Galois stable subgroup of $\mathring{B}^\times$.

Lemma 3.3. *There is a real number $m = m(K)$ with $0 < m < 1$ such that the image of the map*

$$H^1(G, \mathring{B}^{(m)}) \longrightarrow H^1(G, \mathring{B}^\times)$$

induced by the inclusion $\mathring{B}^{(m)} \subseteq \mathring{B}^\times$ is the trivial group $\{1\}$.

Proof. This is a mild generalization of a lemma in [32]. Denote by Tr the trace map of the field extension K/L. Since K/L is separable there is an element $c \in K$ with $Tr(c) = 1$. Then any $0 < m < 1$ such that $m\,|c| < 1$ will do. Indeed, let ψ be a cocycle representing an element of $H^1(G, \mathring{B}^{(m)})$ and consider the element

$$\phi := \sum_{\sigma \in G} \psi(\sigma)\sigma(c) \in B.$$

We have

$$|\phi - 1| = |\sum_{\sigma \in G} (\psi(\sigma) - 1)\sigma(c)| \leq m\,|c| < 1$$

and therefore $\phi \in \mathring{B}^\times$. Given $\tau \in G$ we compute

$$\tau(\phi) = \sum_{\sigma \in G} \psi(\sigma)^\tau (\tau\sigma)(c) = \sum_{\sigma \in G} \psi(\tau)^{-1}\psi(\tau\sigma)(\tau\sigma)(c) = \psi(\tau)^{-1}\phi$$

and, thus, $\psi(\tau) = \phi^{1-\tau}$. Hence, the image of the class of ψ in $H^1(G, \mathring{B}^\times)$ coincides with the class of the coboundary $\tau \mapsto \phi^{1-\tau}$. □

In the situation of the lemma the canonical homomorphism

$$H^1(G, \mathring{B}^\times) \longrightarrow H^1(G, \mathring{B}^\times/\mathring{B}^{(m)}) \qquad (2)$$

is therefore *injective*.

We fix an algebraic closure $K \subseteq \bar{K}$ of the field K and we extend the absolute value from K to $\bar{K}$. Let $r \in |\bar{K}^\times|$ and consider the generalized Tate algebra B_r of all affinoid functions on the closed disc of radius r around zero. It is given by all formal series

$$a_0 + a_1 z + a_2 z^2 + ...$$

subject to $a_n \in K, |a_n| r^n \to 0$ for $n \to \infty$. Suppose $r' \in |\bar{K}^\times|$ is a second radius with $r < r'$. It is well-known that, in case the field K is locally compact, the canonical inclusion

$$f : B_{r'} \xrightarrow{\subset} B_r$$

is a compact continuous linear map between K-Banach spaces, e.g. [30], Example §16. By loc.cit., Remark 16.3 this implies that the image of any bounded $\mathring{K}$-module in $B_{r'}$ has compact closure. We also remark that, since the inclusion $Sp(B_r) \to Sp(B_{r'})$ identifies the source with an affinoid subdomain in the target, the ring extension underlying f is flat, cf. [6], Cor. 7.3.2/6.

We know place ourselves in the following situation. Suppose $r_1 < r_2$ are two numbers in $|K^\times|$, the value group of K. The corresponding affinoid algebra on the closed disc with radius r_i is denoted by B_i. We *suppose* that there is a K-semilinear G-action on B_i stabilizing the principal ideal (z) such that the inclusion map $f : B_2 \to B_1$ is equivariant. We suppose furthermore, that the inclusion of the ring of invariants $A_i := (B_i)^G$ (an affinoid L-algebra) into B_i is a finite Galois extension with group G and that the induced morphism between affinoids $Sp(A_1) \to Sp(A_2)$ is an inclusion making the source an affinoid subdomain in the target. In this situation we prove the

Proposition 3.4. *If the field K is locally compact, the class group $Cl(A_1)$ is a finite group.*

Proof. First, since r_i is in the value group of the ground field K it is almost obvious that the results proved above for the ordinary Tate algebra B hold verbatim for the algebras B_i.

To start with the ring extension $A_2 \subseteq A_1$ is flat and so we have the homomomorphism $j : D(A_2) \to D(A_1)$ from the previous section. On the other hand, the map $Sp(A_1) \to Sp(A_2)$ induces a group homomorphism $\varphi : D(A_1) \to D(A_2)$. According to [6], Prop. 7.2.2/1 (iii) it is a splitting of j, i.e. $j \circ \varphi = id|_{D(A_1)}$. We have similar maps j associated to the flat extensions $A_i \subseteq B_i$ and f. By multiplicativity of the ramification index they assemble to the commutative diagram

$$\begin{array}{ccc} D(A_2) & \longrightarrow & D(B_2) \\ \downarrow & & \downarrow \\ D(A_1) & \longrightarrow & D(B_1). \end{array}$$

It follows that the maps $\delta_i : D(A_i) \to H^1(G, B_i^\times)$ sit in the commutative diagram

$$\begin{array}{ccc} D(A_2) & \xrightarrow{\delta_2} & H^1(G, B_2^\times) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle H^1(f)} \\ D(A_1) & \xrightarrow{\delta_1} & H^1(G, B_1^\times) \end{array}$$

and, consequently, we have the identity

$$H^1(f) \circ \delta_2 \circ \varphi = \delta_1 \circ j \circ \varphi = \delta_1. \qquad (3)$$

After this preliminary discussion we consider for $0 < m < 1$ the following diagram $(+)$ of abelian groups

$$\begin{array}{ccc}
D(A_1) & \xrightarrow{\quad\delta_1\quad} & H^1(G, B_1^\times) \\
\Big\downarrow{\scriptstyle \delta_2\circ\varphi} & & \Big\downarrow{\scriptstyle =} \\
H^1(G, B_2^\times) & \xrightarrow{\quad H^1(f)\quad} & H^1(G, B_1^\times) \\
\Big\downarrow{\scriptstyle s_2} & & \Big\downarrow{\scriptstyle s_1} \\
H^1(G, \mathring{B}_2^\times) & \longrightarrow & H^1(G, \mathring{B}_1^\times) \\
\Big\downarrow & & \Big\downarrow \\
H^1(G, \mathring{B}_2^\times/(\mathring{B}_2^\times \cap \mathring{B}_1^{(m)})) & \longrightarrow & H^1(G, \mathring{B}_1^\times/\mathring{B}_1^{(m)}).
\end{array}$$

Here, the maps s_i are the canonical sections from Lemma 3.2 and the remaining maps are the obvious ones. We prove in a first step that all squares in this diagram are commutative. The identity (3) means that the upper square commutes. Consider the square in the middle. Since $r_i \in |K^\times|$ we have direct product decompositions $B_i^\times = U_i K^\times$ and $\mathring{B}_i^\times = U_i \mathring{K}^\times$ as in the proof of Lemma 3.2 where $U_i := \{b_0 + b_1 z + ... \in B_i^\times : b_0 = 1\}$. These decompositions are respected by f. The middle square is therefore induced by the commutative diagram of Galois modules

$$\begin{array}{ccc}
B_2^\times & \xrightarrow{\ f\ } & B_1^\times \\
\Big\downarrow & & \Big\downarrow \\
\mathring{B}_2^\times & \longrightarrow & \mathring{B}_1^\times
\end{array}$$

where the vertical maps are given by $(u, x) \mapsto (u, 1)$. It is therefore commutative. Finally, the commutativity of the lowest square is clear.

We now prove the assertion of the proposition. In (+) the upper horizontal arrow δ_1 factores into an isomorphism $\bar{\delta}_1 : Cl(A_1) \xrightarrow{\cong} H^1(G, B_1^\times)$. By (2) we may adjust the number $m = m(K)$ so that the lowest vertical arrow on the right hand side becomes injective. Since the right-hand vertical maps are now all injective it suffices to show that the image of the lower horizontal map

$$H^1(G, \mathring{B}_2^\times/(\mathring{B}_2^\times \cap \mathring{B}_1^{(m)})) \longrightarrow H^1(G, \mathring{B}_1^\times/\mathring{B}_1^{(m)})$$

is finite. We claim that already the set $\mathring{B}_2^\times/(\mathring{B}_2^\times \cap \mathring{B}_1^{(m)})$ is finite which yields the claim according to [36], Cor. VIII.§2.2. It is here where we use the local compactness of the field K. Indeed, consider the composite homomorphism

$$h : \mathring{B}_2^\times \xrightarrow{h_1} \mathring{B}_1^\times \xrightarrow{h_2} \mathring{B}_1^\times/\mathring{B}_1^{(m)}$$

with kernel $\mathring{B}_2^\times \cap \mathring{B}_1^{(m)}$. Here, h_1 is induced from the compact map f and h_2 is the canonical projection. Since B_i is reduced the spectral norm induces its Banach topology and so $\mathring{B}_i$ is open and closed in B_i. We equip $\mathring{B}_i^\times$ with the induced topology from $\mathring{B}_i$. The maps h_i are then continuous. We have $\overline{h_1(\mathring{B}_2^\times)} \subseteq \overline{f(\mathring{B}_2)}$ and the right hand side is a compact subset of $\mathring{B}_1$. Hence so is the left hand side. But the units $\mathring{B}_1^\times$ are a closed subset of the complete adic ring $\mathring{B}_1$ and therefore $\overline{h_1(\mathring{B}_2^\times)} \subseteq \mathring{B}_1^\times$. The image $h_2(\overline{h_1(\mathring{B}_2^\times)})$ is compact and contains the image of h. But $\mathring{B}_i^{(m)}$ is open in $\mathring{B}_i^\times$ and therefore the target of h is a discrete space. Hence the image of h must be finite. This proves the assertion. □

4. p-adic Fourier theory

For the basic theory of locally analytic groups over p-adic fields we refer to P. Schneider's monograph [35].

Let $|.|$ be the absolute value on $\mathbb{C}_p$ normalized via $|p| = p^{-1}$. Let

$$\mathbb{Q}_p \subseteq L \subseteq \mathbb{C}_p$$

be a finite extension field. Let $e = e(L/\mathbb{Q}_p)$ be the ramification index, k be the residue field of L and $q = \#k$ its cardinality. Let $o \subseteq L$ be the integers in L and let $e_1, ..., e_{[L:\mathbb{Q}_p]}$ be a $\mathbb{Z}_p$-basis of o. We always view o as an abelian locally L-analytic group of dimension one.

We denote by $\mathbf{B}^s$ the rigid L-analytic open unit disc around zero of dimension $s \geq 1$ and by $\mathbf{B}^s(r)$ a closed subdisc of a real radius r. If $s = 1$ we usually omit it from the notation. Let $z_1, ..., z_{[L:\mathbb{Q}_p]}$ be a set of parameters on the disc $\mathbf{B}^{[L:\mathbb{Q}_p]}$.

Let $\log(1+Z) = Z - Z^2/2 + Z^3/3 - ...$ be the usual logarithm series. The central object of our investigations will be the closed analytic subvariety

$$\hat{o} \subseteq \mathbf{B}^{[L:\mathbb{Q}_p]}$$

defined by the equations

$$e_i \log(1 + z_j) - e_j \log(1 + z_i) = 0$$

for $i, j = 1, ..., [L : \mathbb{Q}_p]$. It is a connected smooth one dimensional rigid L-analytic variety, cf. [33], [38]. As explained in the introduction it is the central object of the p-adic Fourier theory developed in the article [33]. A particular feature of $\hat{o}$ is that for any intermediate complete field $L \subseteq K \subseteq \mathbb{C}_p$ the K-valued points $z \in \hat{o}(K)$ are in natural bijection with the set of K-valued locally analytic characters κ_z of o. This makes $\hat{o}$ a group object.

As explained in the introduction we propose to study the Picard group and the Grothendieck group of

$$\hat{o}_K := \hat{o} \,\hat{\otimes}_L K$$

for any intermediate field $L \subseteq K \subseteq \mathbb{C}_p$ which is a finite extension of L. We remark straightaway that [33], Lem. 3.10 implies that the ideal sheaf corresponding to the zero section in the group $\hat{o}_K$ is an invertible sheaf whose class in $Pic(\hat{o}_K)$ is nontrivial if $L \neq \mathbb{Q}_p$. Since $\hat{o}_K = \mathbf{B}_K$ in case $L = \mathbb{Q}_p$ and $Pic(\mathbf{B}_K) = 1$, cf. [24], Thm. 7.2, this shows

$$Pic(\hat{o}_K) \neq 1 \iff L \neq \mathbb{Q}_p.$$

Our approach to the Picard group of $\hat{o}_K$ rests upon the fact that $\hat{o}$ is a twisted form of $\mathbf{B}$ with respect to the extension $L \subseteq \mathbb{C}_p$ and that the Galois cocycle giving the descent datum comes out of a Lubin-Tate group for o. We give more details in the next subsection.

4.1. **Lubin-Tate groups.**

For a quick introduction to Lubin-Tate theory we suggest [23]. Recall that a *Lubin-Tate formal group* for a fixed prime element $\pi \in o$ is a certain one dimensional commutative formal group $\mathbb{G}$ over o of p-height $[L : \mathbb{Q}_p]$. It comes equipped with a unital ring homomorpism $[.] : o \to \mathrm{End}(\mathbb{G})$. We will assume that $\mathbb{G}$ has the property $[\pi] = \pi X + X^q \in o[[X]]$. This is no essential restriction (loc.cit., Thm. 1.1/3.1).

Viewing $\mathbb{G}$ as a connected p-divisible group let $\mathbb{G}'$ be its Cartier dual and $T(\mathbb{G}')$ be the corresponding Tate module [37]. The latter is a free rank one o-module carrying an action of the absolute Galois group

$$G_L := G(\bar{L}/L)$$

of L which is given by a continuous character

$$\tau' : G_L \longrightarrow o^\times .$$

Denote by $\mathbb{G}_m$ the formal multiplicative group over o. There is a canonical Galois equivariant isomorphism of o-modules

$$T(\mathbb{G}') \cong \mathrm{Hom}_{o_{\mathbb{C}_p}}(\mathbb{G}_{o_{\mathbb{C}_p}}, \mathbb{G}_{m,o_{\mathbb{C}_p}}) \tag{4}$$

where on the right-hand side, G_L acts coefficientwise on formal power series over $o_{\mathbb{C}_p}$ and the o-module structure comes by functoriality from the formal o-module $\mathbb{G}$. Choose once and for all an o-module generator t' for $T(\mathbb{G}')$ and denote by

$$F_{t'}(Z) = \omega Z + ... \in o_{\mathbb{C}_p}[[Z]]Z$$

the corresponding homomorphism of formal groups. According to [10], Lem. 7.3.4, we may identify $\mathbb{G}$ with the rigid L-analytic open unit disc $\mathbf{B}$ around zero and the latter becomes an o-module object in this way. The bijection between $\mathbb{C}_p$-valued points z of $\mathbf{B}$ and $\mathbb{C}_p$-valued locally analytic characters κ_z mentioned in the introduction to this section is then given by

$$\kappa_z(g) = 1 + F_{t'}([g].z)$$

for $g \in o$. This bijection comes in fact from an underlying rigid $\mathbb{C}_p$-analytic isomorphism

$$\kappa : \mathbf{B}_{\mathbb{C}_p} \xrightarrow{\cong} \hat{o}_{\mathbb{C}_p} \tag{5}$$

and the corresponding Galois cocycle is given by

$$\sigma \mapsto [\tau'(\sigma)^{-1}] \tag{6}$$

for $\sigma \in G_L$. Here, $[\tau'(\sigma)^{-1}]$ is viewed as an element of the automorphism group of the algebra $\mathcal{O}(\mathbf{B}_{\mathbb{C}_p})$ in the obvious way. We point out the trivial but useful identity

$$[\tau'(\sigma)^{-1}] = \exp_{\mathbb{G}}(\tau'(\sigma)^{-1}\log_{\mathbb{G}}(Z)) \tag{7}$$

for any $\sigma \in G_L$ where $\exp_{\mathbb{G}}$ and $\log_{\mathbb{G}}$ are the formal exponential and logarithm series of $\mathbb{G}$ respectively. Note also that the linear coefficient ω of the power series $F_{t'}$ is a period (in the sense of p-adic Hodge theory) for the character τ' of the absolute Galois group G_L. Indeed, (4) implies $\sigma.F_{t'} = \tau'(\sigma)F_{t'}$ and therefore

$$\omega^\sigma = \tau'(\sigma)\omega \tag{8}$$

for all $\sigma \in G_L$.

Let

$$L \subseteq L_\infty \subseteq \bar{L}$$

be the algebraic field extension of L obtained by adjoining the p^n-torsion points of the p-divisible group $\mathbb{G}'$ to L for all $n \geq 1$. By the main result of [33], Appendix, the period $\omega \in \mathbb{C}_p$ lies in the closure of L_∞. By construction the isomorphism κ therefore descends from $\mathbb{C}_p$ to this closure. If $L \neq \mathbb{Q}_p$ then, according to [33], Lemma 3.9, the twisted form $\hat{o}$ of $\mathbf{B}$ cannot be trivialized over a discretely valued complete subfield of this closure. We also remark that (4) implies that L_∞ coincides with the field extension of L obtained by adjoining all torsion points of $\mathbb{G}$ and all p-power roots of unity to L. Consequently, it contains wild ramification. It is interesting in this situation to recall from our introduction that A. Ducros has recently shown in [14] that any form of the open unit disc $\mathbf{B}$ with respect to a tamely ramified finite field extension is trivial in the sense that it is itself isomorphic to an open disc. He also showed that there are plenty of nontrivial wildly ramified forms.

After this review we begin our investigations by showing in a first step that, locally, $\hat{o}_K$ is a twisted form of a rigid analytic group on a closed disc which admits trivializations over *finite* extensions of K inside KL_∞. To do this, define for $n \geq 0$ the increasing sequence of radii

$$r_n := r^{1/q^{en}}$$

where $r := p^{-q/e(q-1)}$ and consider

$$\hat{o}_n := \hat{o} \cap \mathbf{B}^{[L:\mathbb{Q}_p]}(r_n). \tag{9}$$

If n varies these affinoids form a countable increasing admissible open covering of $\hat{o}$. On the other hand, each one dimensional disc $\mathbf{B}(r_n)$ is an o-module object with respect to the induced Lubin-Tate group structure coming from $\mathbf{B}(r_n) \subseteq \mathbf{B}$. In this situation the isomorphism (5) induces for each n isomorphisms between affinoids over $\mathbb{C}_p$

$$\mathbf{B}(r_n)_{\mathbb{C}_p} \xrightarrow{\cong} \hat{o}_{n,\mathbb{C}_p} \tag{10}$$

according to [33], Thm. 3.6. Define a formal power series

$$h_n(Z) := \exp_{\mathbb{G}}((\omega_n/\omega)\log_{\mathbb{G}}(Z)) \in \mathbb{C}_p[[Z]]Z.$$

Since ω lies in the closure of L_∞ we may fix once and for all $\omega_n \in L_\infty$ such that

$$|\omega_n/\omega - 1| < p^{-n} \tag{11}$$

for all $n \geq 0$.

Lemma 4.1. *For all $n \in \mathbb{N}$ the power series h_n is a rigid analytic group automorphism of $\mathbf{B}(r_n)_{\mathbb{C}_p}$.*

Proof: We give the details of a proof sketched in the unpublished note [32]. Let $\mathbb{G}(X,Y) \in o[[X,Y]]$ be the formal group law underlying $\mathbb{G}$. Using basic properties of $\exp_{\mathbb{G}}$ and $\log_{\mathbb{G}}$, one computes that $h_n(Z)$ equals

$$\exp_{\mathbb{G}}(\mathbb{G}_a(\log_{\mathbb{G}}(Z), (\omega_n/\omega - 1)\log_{\mathbb{G}}(Z))) = \mathbb{G}(Z, \exp_{\mathbb{G}}((\omega_n/\omega - 1)\log_{\mathbb{G}}(Z))) \tag{12}$$

as formal power series over $\mathbb{C}_p$ where $\mathbb{G}_a$ denotes the formal additive group. By [33], Lem. 3.2 we have $[p^n].\mathbf{B}(r_n) = \mathbf{B}(r)$ whence

$$p^n \log_{\mathbb{G}}(\mathbf{B}(r_n)) = \log_{\mathbb{G}}([p^n].\mathbf{B}(r_n)) = \log_{\mathbb{G}}(\mathbf{B}(r)) = \mathbf{B}(r)$$

where the last identity follows from [23], Lem. §8.6.4. Hence, on $\mathbf{B}(r_n)$ we have the composite of the rigid analytic functions

$$\mathbf{B}(r_n)_{\mathbb{C}_p} \xrightarrow{\log_{\mathbb{G}}} p^{-n}\mathbf{B}(r)_{\mathbb{C}_p} \xrightarrow{(\omega_n/\omega)-1} \mathbf{B}(r)_{\mathbb{C}_p} \xrightarrow{\exp_{\mathbb{G}}} \mathbf{B}(r)_{\mathbb{C}_p}. \tag{13}$$

Using that the group law $\mathbb{G}$ is defined over o it follows that h_n is a rigid analytic function on $\mathbf{B}(r_n)_{\mathbb{C}_p}$. Applying the same reasoning to the formal inverse $h_n^{-1}(Z) = \exp_{\mathbb{G}}((\omega/\omega_n)\log_{\mathbb{G}}(Z))$ shows that h_n is a rigid automorphism of $\mathbf{B}(r_n)_{\mathbb{C}_p}$. It is clear from the definition of h_n that it respects the Lubin-Tate group structure on $\mathbf{B}(r_n)_{\mathbb{C}_p}$. $\square$

We fix once and for all a chain of finite Galois extensions

$$L := L_0 \subseteq L_1 \subseteq ... \subseteq L_\infty$$

of L with the property $\omega_n \in L_n$.

Lemma 4.2. *The group isomorphism*

$$\kappa \circ h_n : \mathbf{B}(r_n)\,\hat{\otimes}_L \mathbb{C}_p \xrightarrow{\cong} \hat{o}_n\,\hat{\otimes}_L \mathbb{C}_p$$

is already defined over the finite extension L_n of L.

Proof: Denote by B_n and A_n the affinoid algebras of $\mathbf{B}(r_n)$ and $\hat{o}_n$ respectively. We have obvious actions of the Galois group

$$G_n := Gal(\bar{L}/L_n)$$

on $\mathbf{B}(r_n)_{\mathbb{C}_p}, B_n \hat{\otimes}_L \mathbb{C}_p$ and $\mathrm{Aut}_{\mathbb{C}_p}(B_n \hat{\otimes}_L \mathbb{C}_p)$. Here, the latter refers to the group of $\mathbb{C}_p$-algebra automorphisms of $B_n \hat{\otimes} \mathbb{C}_p$. Given $\sigma \in G_n$ and a $\mathbb{C}_p$-valued point z of $\mathbf{B}(r_n)$ we find

$$h_n^{-1}(\sigma(h_n(\sigma^{-1}(z)))) = \exp_{\mathbb{G}}((\omega/\omega^\sigma)\log_{\mathbb{G}}(z)) = \exp_{\mathbb{G}}(\tau'(\sigma)^{-1}\log_{\mathbb{G}}(z)) = [\tau'(\sigma)^{-1}].z$$

where the middle identity and the final identity come from (8) and (7) respectively. Denoting by $h_n^\sharp$ the algebra automorphism associated to h_n it follows for each $\sigma \in G_n$ that

$$\sigma.h_n^\sharp = h_n^\sharp \circ [\tau'(\sigma)^{-1}]$$

in $\mathrm{Aut}_{\mathbb{C}_p}(B_n \hat{\otimes}_L \mathbb{C}_p)$. By (6) the cocycle

$$G_L \to \mathrm{Aut}_{\mathbb{C}_p}(B_n \hat{\otimes}_L \mathbb{C}_p),\ \sigma \mapsto [\tau'(\sigma)^{-1}]$$

gives the descent datum for the twisted form $\kappa^\sharp : A_n \hat{\otimes}_L \mathbb{C}_p \xrightarrow{\cong} B_n \hat{\otimes}_L \mathbb{C}_p$. According to the usual formalism of Galois descent (cf. [22], §9) we may therefore conclude that the $\mathbb{C}_p$-algebra automorphism

$$h_n^\sharp \circ \kappa^\sharp : A_n \hat{\otimes}_L \mathbb{C}_p \xrightarrow{\cong} B_n \hat{\otimes}_L \mathbb{C}_p$$

is G_n-equivariant. By the existence of topological L-bases for A_n and B_n, in the sense of [30], Prop. 10.1, taking G_n-invariants and applying Tate's theorem $\mathbb{C}_p^{G_n} = L_n$ (cf. [37], Prop. 3.1.8) yields the claim. □

Remark: Using that $F_{t'}(Z) = \exp(\omega \log_{\mathbb{G}}(Z))$ as power series over $\mathbb{C}_p$ (cf. [33], §4) the isomorphism $\kappa \circ h_n$ of the preceding proposition is given on $\mathbb{C}_p$-points z of $\mathbf{B}(r_n)$ via

$$(\kappa \circ h_n)_z(g) = \exp(g\omega_n \log_{\mathbb{G}}(z)),\ g \in o.$$

Proposition 4.3. *The Galois cocycle of the twisted form $\hat{o}_n$ of $\mathbf{B}(r_n)$ with respect to the finite extension L_n/L is given by*

$$\sigma \mapsto \exp_{\mathbb{G}}((\omega_n/\omega_n^\sigma)\log_{\mathbb{G}}(Z))$$

where $\sigma \in Gal(L_n/L)$.

Proof. Let $\sigma \in Gal(L_n/L)$ and let $\tilde{\sigma} \in G_L$ be any extension to $\bar{L}$. The value of the cocycle on σ equals the element

$$(h_n^\sharp\kappa^\sharp)\sigma(h_n^\sharp\kappa^\sharp)^{-1}\sigma^{-1} = h_n^\sharp\kappa^\sharp\sigma(\kappa^\sharp)^{-1}(h_n^\sharp)^{-1}\sigma^{-1} = h_n^\sharp(\kappa^\sharp\tilde{\sigma}(\kappa^\sharp)^{-1}\tilde{\sigma}^{-1})\tilde{\sigma}(h_n^\sharp)^{-1}\sigma^{-1}$$

in the automorphism group of the L_n-affinoid algebra $\mathcal{O}(\mathbf{B}(r_n)) \otimes_L L_n$, cf. [22],§9. As usual, we identify this automorphism with its value on the parameter and consequently, with the power series

$$\begin{aligned}
h_n^\sharp(\kappa^\sharp\tilde{\sigma}(\kappa^\sharp)^{-1}\tilde{\sigma}^{-1})\tilde{\sigma}(h_n^\sharp)^{-1}(Z) &\overset{(6)}{=} h_n^\sharp[\tau'(\sigma)^{-1}]\tilde{\sigma}(h_n^\sharp)^{-1}(Z) \\
&\overset{(7),(8)}{=} h_n^\sharp \exp_{\mathbb{G}}((\omega/\omega^{\tilde{\sigma}})\log_{\mathbb{G}})\tilde{\sigma}(h_n^\sharp)^{-1}(Z) \\
&\overset{(*)}{=} \exp_{\mathbb{G}}((\omega_n/\omega)(\omega/\omega^{\tilde{\sigma}})(\omega^{\tilde{\sigma}}/\omega_n^\sigma)\log_{\mathbb{G}}(Z)) \\
&= \exp_{\mathbb{G}}((\omega_n/\omega_n^\sigma)\log_{\mathbb{G}}(Z)).
\end{aligned}$$

Here, we used the identity

$$\tilde{\sigma}.(h_n^{\sharp})^{-1}(Z) = \exp_{\mathbb{G}}((\omega^{\tilde{\sigma}}/\omega_n^{\sigma})\log_{\mathbb{G}}(Z)) \in Z\mathbb{C}_p[[Z]]$$

in $(*)$. □

We now consider for each fixed n the base extension $\hat{o}_{K,n}$ of $\hat{o}_n$ to K. Again, if n varies these affinoids give a countable increasing open admissible covering of $\hat{o}_K$. As a result of our discussion each $\hat{o}_{K,n}$ is a twisted form of the Lubin-Tate group on $\mathbf{B}(r_n)_K$ trivialized over $K_n := KL_n$ by $\kappa \circ h_n$ and the Galois cocycle giving the descent datum is given by the preceding proposition.

4.2. **An étale covering.** We briefly like to indicate an alternative intrinsic characterization of the twisted form $\hat{o}$ and its affinoid subdomains $\hat{o}_n$. This builds on the analytic mapping properties of the logarithm

$$\lambda := \log_{\mathbb{G}}$$

associated to the Lubin-Tate group $\mathbb{G}$. It is best formulated in the language Berkovich analytic spaces [3], [4]. It also involves the beginnings of A.J. de Jong's theory of étale covering maps for Berkovich spaces [11].

Let $\mathbb{B}$ and $\mathbb{A}^1$ be the Berkovich analytic spaces over L equal to the one dimensional open unit disc around zero and the affine line respectively. The logarithm

$$\lambda : \mathbb{B} \longrightarrow \mathbb{A}^1$$

is an étale and surjective morphism, cf. [28], Lem. 6.1.1. Let $\bar{L}_\infty$ be the closure of $L_\infty \subseteq \mathbb{C}_p$ and $G_\infty := Gal(L_\infty/L)$. We endow the space $\mathbb{B}_{\bar{L}_\infty}$ with the semilinear Galois action associated to the cocycle (6). Similarly, we endow the space $\mathbb{A}^1_{\bar{L}_\infty}$ with the semilinear Galois action associated to the cocycle

$$\sigma \mapsto \tau'(\sigma)^{-1}Z.$$

Using (7) it is elementary to see that with these definitions the map $\lambda_{\bar{L}_\infty}$ becomes equivariant.

Lemma 4.4. *The Galois descent of $\mathbb{A}^1_{\bar{L}_\infty}$ is canonically isomorphic to $\mathbb{A}^1_L$.*

Proof. The semilinear Galois action on $\mathbb{A}^1_{\bar{L}_\infty}$ respects the natural increasing covering by closed discs around zero. Let s_n be a family of real numbers in $|\bar{L}^\times|$ tending towards infinity. Let B_n be the affinoid algebra of $\mathbb{B}(s_n)_{\bar{L}_\infty}$. Let $z \in B_n$ be a parameter, $b := \sum_{m\geq 0} a_m z^m \in B_n$ and $\sigma \in G_\infty$. By (8) we have

$$\sigma.(\sum_{m\geq 0} a_m z^m) = \sum_{m\geq 0} a_m^\sigma (\omega/\omega^\sigma)^m z^m$$

and thus $\sigma.b = b$ if and only if $a_m/\omega^m = (a_m/\omega^m)^\sigma$ for all $m \geq 0$. Consequently, b is Galois invariant if and only if $a_m/\omega^m \in L$ for all $m \geq 0$. It follows that the subring of Galois invariants in B_n is given by

$$B_n^{G_\infty} = \{\sum_{m\geq 0} c_m(\omega z)^m, c_m \in L, |c_m|(|\omega|s_n)^m \to 0 \text{ for } n \to \infty\}.$$

It is canonically isomorphic to the affinoid algebra of the closed disc of radius $|\omega|s_n$. For varying s_n these isomorphism glue to a canonical isomorphism between the descent of $\mathbb{A}^1_{\bar{L}_\infty}$ and $\mathbb{A}^1_L$. □

We point out here that

$$|\omega| = p^{\nu} \text{ with } \nu = \frac{1}{p-1} - \frac{1}{e(q-1)}$$

according to [33], Lemma 3.4b. Let $\hat{o}^{an}$ be the Galois descent of $\mathbb{B}_{\bar{L}_\infty}$. The Galois descent of the map $\lambda_{\bar{L}_\infty}$ is a morphism

$$\hat{o}^{an} \longrightarrow \mathbb{A}^1_L$$

between Berkovich analytic spaces over L. In the following we show that it is étale and surjective and therefore an étale covering of the affine line over L.

To do this we make use of the arguments in proof of [28], Lemma 6.1.1. For $n \geq 1$ we let

$$s_n := r|\pi|^{-en}$$

and let $\mathbb{B}(s_n) \subset \mathbb{A}^1$ be the closed disc of radius s_n. We let $E(s_n)$ be the connected component containing zero of $\lambda^{-1}(\mathbb{B}(s_n))$ so that the induced map

$$\lambda : E(s_n) \longrightarrow \mathbb{B}(s_n)$$

is finite étale and surjective. As in loc.cit. one obtains that $E(s_n)$ is the connected component containing zero of the inverse image of $\mathbb{B}(r)$ by $[\pi^{en}]$. Indeed, since $r < p^{-1/e(q-1)}$, the maps $\log_{\mathbb{G}}$ and $\exp_{\mathbb{G}}$ induce mutually inverse isomorphisms of $\mathbb{B}(r)$ ([23], §8.6 Lemma 4) so that $\exp_{\mathbb{G}}(\pi^{en}\mathbb{B}(s_n)) = \mathbb{B}(r)$. Next, we have $[\pi](Z) = \pi Z + Z^q$ and so, by the arguments given in the proof of [33], Lemma 3.2 one has

$$E(s_n) = [\pi^{en}]^{-1}(\mathbb{B}(r)) = \mathbb{B}(r^{1/q^{en}}) = \mathbb{B}(r_n).$$

As a result of this discussion we have a finite étale surjective morphism

$$\lambda : \mathbb{B}(r_n) \longrightarrow \mathbb{B}(s_n)$$

for any $n \geq 1$. Using the compatibility with λ and $[\pi]$ one finds that its degree equals q^{en}. The same properties hold for its base change from L to the field L_n. We endow the space $\mathbb{B}(r_n)_{L_n}$ with the semilinear $Gal(L_n/L)$-action associated to the cocycle from Proposition 4.3. Similarly, we endow the space $\mathbb{B}(s_n)_{L_n}$ with the semilinear $Gal(L_n/L)$-action associated to the cocycle

$$\sigma \mapsto (\omega_n/\omega_n^{\sigma})(Z).$$

This makes the map $\lambda_{L_n} : \mathbb{B}(r_n)_{L_n} \to \mathbb{B}(s_n)_{L_n}$ equivariant. The following lemma is proved in the same way as the preceding one.

Lemma 4.5. *The Galois descent of* $\mathbb{B}(s_n)_{L_n}$ *is canonically isomorphic to the closed disc* $\mathbb{B}(t_n)$ *where* $t_n := |\omega_n|s_n$.

We let $\hat{o}^{an}_n$ be the Galois descent of $\mathbb{B}(r_n)_{L_n}$. The above discussion yields a finite étale surjective morphism $\hat{o}^{an}_n \to \mathbb{B}(t_n)$ whose degree equals q^{en}. Taking the union of these maps over all n we find that the morphism

$$\hat{o}^{an} \longrightarrow \mathbb{A}^1_L$$

is étale and surjective.

4.3. **Cohomology and inverse limits.** To deal with the Picard group of the rigid analytic space $\hat{o}_K$ we need to establish a general result on sheaf cohomology and inverse limits.

Recall that a rigid analytic space X is called *quasi-Stein* if there is a countable increasing admissible open affinoid covering $\{X_n\}_{n\in\mathbb{N}}$ of X such that the restriction maps $\mathcal{O}(X_{n+1}) \longrightarrow \mathcal{O}(X_n)$ have dense image (cf. [21], Def. 2.3).

Let X be a fixed quasi-Stein space and let $\{X_n\}$ be a defining covering. Let $Ab(X)$ be the category of abelian sheaves on X.

Proposition 4.6. *For any $\mathcal{F} \in Ab(X)$ there is a short exact sequence*

$$0 \longrightarrow \varprojlim_n^{(1)} \mathcal{F}(X_n) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \varprojlim_n H^1(X_n, \mathcal{F}) \longrightarrow 0$$

where the right surjection is induced by the inclusions $X_n \to X$ for each $n \in \mathbb{N}$ and H^1 denotes sheaf cohomology.

Proof: Our argument is based on the ideas in [31], Prop. 2.4. Let $Proj_{\mathbb{N}}(Ab)$ be the category of $\mathbb{N}$-projective systems over the category of abelian groups Ab and let $F := \varprojlim_n$ be the projective limit viewed as an additive functor $Proj_{\mathbb{N}}(Ab) \to Ab$. Consider the additive functor $G : Ab(X) \to Proj_{\mathbb{N}}(Ab),\ \mathcal{F} \mapsto (\mathcal{F}(X_n))_n$. Both F and G are left exact. For any $\mathcal{F} \in Ab(X)$ there is a short exact sequence

$$0 \longrightarrow (R^1F)(G\mathcal{F}) \longrightarrow R^1(FG)(\mathcal{F}) \longrightarrow (FR^1G)(\mathcal{F}) \longrightarrow 0.$$

Indeed, an injective sheaf $\mathcal{I} \in Ab(X)$ is flasque whence the system $G(\mathcal{I})$ is surjective and therefore F-acyclic. The five-term exact sequence associated to the Grothendieck spectral sequence (e.g. [40] Thm. 5.8.3)

$$E_2^{pq} := (R^pF)(R^qG)(\mathcal{F}) \Rightarrow R^{p+q}(FG)(\mathcal{F})$$

yields the desired short exact sequence since $R^2F = 0$ (cf. [loc.cit.], Cor. 3.5.4).

After this observation let $res_n : Ab(X) \to Ab(X_n)$ be the restriction functor. It is almost immediate that we have a cohomological δ-functor $(T^i)_{i \geq 0} : Ab(X) \to Proj_{\mathbb{N}}(Ab)$ where

$$T^i := (H^i(X_n, res_n \circ (.)))_n.$$

On any G-ringed space the restriction of an injective sheaf to an admissible open subset remains injective (cf. [19], I.§2). Hence each res_n preserves injectives. Since $Ab(X)$ has enough injectives it follows that each $T^i,\ i > 0$ is effaceable and therefore $(T^i)_{i \geq 0}$ is universal. It now follows that the right-derived functors of the functor G are given by $(T^i)_{i \geq 0}$ and it remains to observe that the sheaf property yields $F \circ G = \Gamma$, the global section functor. □

4.4. **Picard groups.**

In this subsection we prove the structure result on the Picard group $Pic(\hat{o}_K)$.

Proposition 4.7. *There is an exact sequence*[*]

$$1 \longrightarrow \varprojlim_n^{(1)} \mathcal{O}(X_n)^\times \longrightarrow Pic(X) \longrightarrow \varprojlim_n Pic(X_n) \longrightarrow 1.$$

Proof: Let $\mathcal{O}_X$ be the structure sheaf of X. We apply the last result of the preceding subsection to $\mathcal{F} = \mathcal{O}_X^\times$ and use $H^1(X, \mathcal{O}_X^\times) = Pic(X)$ (cf. [17], Prop. 4.7.2). □

Remark: The vanishing of the $\lim^{(1)}$-term in the above sequence depends heavily on the base field of X. To give an example let for a moment $\mathbb{Q}_p \subseteq K \subseteq \mathbb{C}_p$ by an arbitrary complete field. The open unit disc $\mathbf{B}_K$ over K is quasi-Stein. A defining covering is provided by the closed subdiscs $\mathbf{B}(r)_K, r < 1$. In this situation $\varprojlim_r^{(1)} \mathcal{O}(\mathbf{B}(r)_K)^\times = 1$ is equivalent to K being spherically complete. This is a consequence of work of M. Lazard, cf. [24], Thm. 2 and Prop. 6.

Recall that the *dimension* of X at a point $x \in X$ equals the dimension of the local ring at x.

[*] We turn back to our earlier convention and denote the unit element in an abelian group by 1.

Lemma 4.8. *Let X be equidimensional with finitely many connected components. Taking global sections furnishes an equivalence of exact categories between vector bundles on X and finitely generated projective $\mathcal{O}(X)$-modules. In particular, $K_0(X) \cong K_0(\mathcal{O}(X))$ and $Pic(X) \cong Pic(\mathcal{O}(X))$ canonically.*

Proof: Following [18], Remarque V.1. one may apply *mutatis mutandis* the arguments appearing in the proof of [loc.cit.], Thm. V.1 on each connected component of X. □

The space $\hat{o}_K$ is a quasi-Stein space with respect to the covering given by the affinoids $\hat{o}_{K,n}$. This follows from the definition (9) and the fact that $\mathbf{B}^{[L:\mathbb{Q}_p]}$ is quasi-Stein with respect to the covering given by the $\mathbf{B}^{[L:\mathbb{Q}_p]}(r_n)$ (cf. remark above).

Proposition 4.9. *The canonical homomorphism*

$$Pic(\hat{o}_K) \xrightarrow{\cong} \varprojlim_n Pic(\hat{o}_{K,n})$$

is bijective.

Proof. We let $A = \mathcal{O}(\hat{o}_K)$ and $A_n := \mathcal{O}(\hat{o}_{K,n})$. By the preceding discussion it suffices to show that the natural homomorphism

$$Pic(A) \longrightarrow \varprojlim_n Pic(A_n)$$

is injective. To show this we use an adaption of the argument given in [18], Prop. V.3.2. Let therefore P be a projective rank 1 module over A whose class is in the kernel of this homomorphism. Let $P_n := P \otimes_A A_n$. The natural restriction map $A_{n+1} \to A_n$ is injective since this is true for its base change to $\mathbb{C}_p$ (cf. (10)). Applying $P \otimes_A (\cdot)$ we have an injective map $P_{n+1} \to P_n$ which we view as an inclusion. The first step now is to exhibit a well-chosen generator for each free A_n-module P_n. This is done by induction. Let x_1 be any generator for P_1. This starts the induction. To make the induction step suppose x_n is a well-chosen generator of the free A_n-module P_n. Let y_{n+1} be an arbitrary generator for P_{n+1}. Since the image of y_{n+1} under the map $P_{n+1} \to P_n$ generates $P_n = P_{n+1} \otimes_{A_{n+1}} A_n$ we may write $y_{n+1} = a_n x_n$ with $a_n \in A_n^\times$. Let

$$B_n := \mathcal{O}(\mathbf{B}(r_n)_{K_n}),\ U_n := \{b_0 + b_1 z + ... \in B_n^\times : b_0 = 1\},\ G_n := G(K_n/K).$$

Passing to a finite extension of K_n (if necessary) we have $r_n \in |K_n^\times|$ and so, as in the proof of Lemma 3.2, a direct product decomposition $B_n^\times = U_n K_n^\times$. Since the G_n-action on $\mathbf{B}(r_n)_{K_n}$ preserves the origin taking invariants gives a direct product decomposition

$$A_n^\times = W_n K^\times. \tag{14}$$

with $W_n := (U_n)^{G_n}$. Let $a_n = wv$ with $w \in W_n, v \in K^\times$. We now let $x_{n+1} := v^{-1} y_{n+1}$ which completes the induction step. Note that by construction $x_{n+1} \in W_n x_n$ for all n.

Now $U_n = 1 + U'_n$ with an o_{K_n}-module U'_n and $W_n = 1 + (U'_n)^{G_n}$ with the o_K-module $(U'_n)^{G_n}$. We see that

$$V_n := W_n x_n \subseteq P_n$$

is a convex subset of the L-Banach space P_n. Moreover, $B_{n+1} \subseteq B_n$ implies $U_{n+1} \subseteq U_n$ and hence $W_{n+1} \subseteq W_n$. Thus, $V_{n+1} = W_{n+1} x_{n+1} \subseteq W_n x_n = V_n$ and so

$$V_{n+1} \subseteq V_n$$

for all n. Now the map $P_{n+1} \to P_n$ is a compact linear map between L-Banach spaces (cf. [34], Lem. 6.1). This implies ([30], Remark 16.3) that the image of V_{n+1} in V_n is relatively compact. We may therefore argue as in [18], Prop. V.3.2

to obtain that $\cap_n V_n \neq \emptyset$. By Theorem B for quasi-Stein spaces ([21]) any nonzero element in this intersection generates P. Then P is free on this generator since $\mathrm{Ann}_A(x) \subseteq \mathrm{Ann}_{A_n}(x) = 0$. Hence the class of P in $Pic(A)$ is trivial. $\square$

Proposition 4.10. *The group $Pic(\hat{o}_{K,n}) = Cl(\hat{o}_{K,n})$ is finite for all $n \geq 1$.*

Proof. We pass to a finite extension of K_{n+1} (if necessary) to obtain $r_n, r_{n+1} \in |K_{n+1}^\times|$. Let $G = Gal(K_{n+1}/K)$. We equip $\mathbf{B}(r_n)_{K_{n+1}}$ and $\mathbf{B}(r_{n+1})_{K_{n+1}}$ with the semilinear Galois action coming from the cocycle

$$\sigma \mapsto \exp_{\mathbb{G}}((\omega_{n+1}/\omega_{n+1}^{\sigma}) \log_{\mathbb{G}}(Z))$$

for $\sigma \in G$ (Lemma 4.3). By the Lemmas 4.1 and 4.2 this is an action by group automorphisms with respect to the Lubin-Tate group structure on these discs. In particular, the origin of these discs remains fixed under this action. We let $B_i := \mathcal{O}(\mathbf{B}(r_i)_{K_{n+1}})$ and $A_i := \mathcal{O}(\hat{o}_{K,i})$ for $i = 1, 2$. The canonical inclusion $B_2 \to B_1$ is thus equivariant. Since ω_{n+1} also satisfies the defining condition (11) for ω_n we have $A_n = (B_n)^G$ (and of course $A_{n+1} = (B_{n+1})^G$). Thus, taking G-invariants of $B_2 \to B_1$ yields the restriction map $A_2 \to A_1$. Moreover, by the very definition of $\hat{o}_{K,n}$ the inclusion $\hat{o}_{K,n} \subseteq \hat{o}_{K,n+1}$ identifies the source with an affinoid subdomain in the target. Since K_{n+1} is locally compact we may therefore apply Prop. 3.4 which yields the assertion. $\square$

We now prove that, in case $L \neq \mathbb{Q}_p$, the group $Pic(\hat{o}_K)$ is not a finite group. We need two auxiliary lemmas.

Lemma 4.11. *Let R and R' be two normal o_K-algebras of topologically finite type which are flat over o_K. Suppose R is an integral domain. Let $R \to R'$ be a finite ring extension such that $R \otimes_{o_K} K \to R' \otimes_{o_K} K$ is a finite free extension of degree, say, m. Let $f : Spec(R') \to Spec(R)$ be the associated morphism. There is a closed subset $V \subset Spec(R)$ with $\mathrm{codim}_{Spec(R)} V \geq 2$ such that the induced morphism*

$$Spec(R') \setminus f^{-1}(V) \longrightarrow Spec(R) \setminus V$$

is finite flat of degree equal to m.

Proof. We argue along the lines of [10], Lemma 7.3.2. Since R is noetherian we may consider a finite presentation of the R-module R'

$$R^{m_1} \xrightarrow{\alpha} R^{m_0} \longrightarrow R' \longrightarrow 0.$$

We may assume here that $0 \leq m_0 - m \leq m_1$. We consider the m-th Fitting ideal $F_m(R')$ of the R-module R' ([15], III.20.2), i.e. the ideal of R generated by the $(m_0 - m, m_0 - m)$-minors of the matrix α. After inverting a prime element of o_K the R-module R' is generated by m elements. This implies $F_m(R') \neq 0$ (loc.cit. Cor. 20.5/Prop. 20.6). On the complement of $V := Spec(R/F_m(R'))$ the ideal $F_m(R')$ becomes invertible. According to [7], Lemma 3.14 the R-module R' is therefore locally free of rank m over the open set $Spec(R) \setminus V$. Finally, using the normality of R and R' it follows as in the proof of [10], Lemma 7.3.2 that $\mathrm{codim}_{Spec(R)} V \geq 2$. $\square$

For any rigid K-analytic space X we denote by $\mathcal{O}(X)^0 \subseteq \mathcal{O}(X)$ the K-subalgebra consisting of holomorphic functions which are bounded by 1.

Lemma 4.12. *Let $f : X \to \mathbf{B}_K$ be a finite morphism of rigid K-analytic spaces. Suppose the induced homomorphism $\mathcal{O}(\mathbf{B}_K)^0 \to \mathcal{O}(X)^0$ is an integral ring extension. Then any function $F \in \mathcal{O}(X)^0$ has only finitely many zeroes on X.*

Proof. Let z be a parameter on the disc $\mathbf{B}_K$. According to [10], Lem. 7.3.4, we have $\mathcal{O}(\mathbf{B}_K)^0 = o_K[[z]]$, the ring of formal power series over o_K in the variable z. Let $H \in \mathcal{O}(X)^0$ and consider an equation

$$H^m + b_1 H^{m-1} + ... + b_m = 0$$

with $b_i \in \mathcal{O}(\mathbf{B}_K)^0$. Since $\mathcal{O}(\mathbf{B}_K)^0$ is an integral domain we may assume $b_m \neq 0$. If $H(x) = 0$ for some $x \in X$ then $b_m(f(x)) = 0$. By the Weierstrass preparation theorem for $o_K[[X]]$, [9], VII, §3.8 Prop. 6, the power series b_m has at most finitely many zeroes. Since f has finite fibres according to [6], Cor. 9.6.3/6, we conclude that H has at most finitely many zeroes on X. □

Proposition 4.13. *Let $L \neq \mathbb{Q}_p$. The ideal sheaf defining the zero section of the rigid group $\hat{o}_K$ is not a torsion element in $Pic(\hat{o}_K)$.*

Proof: Abbreviate $A := \mathcal{O}(\hat{o}_K)$, $B := \mathcal{O}(\mathbf{B}_K)$ and $B_{\mathbb{C}_p} := \mathcal{O}(\mathbf{B}_{\mathbb{C}_p})$ and let $z \in B$ be a parameter. Assume for a contradiction that the ideal sheaf in question is a torsion element. If $I \subseteq A$ denotes the corresponding ideal of global sections there is $1 < m < \infty$ such that

$$I^m = (f)$$

with some $f \in A$. Since the trivialization κ preserves the origin we have $I \mapsto (z)$ via $Pic(A) \to Pic(B_{\mathbb{C}_p})$ whence $(f) \mapsto (z^m)$. Now consider f as a rigid K-analytic map $\hat{o}_K \to \mathbb{A}^1_K$ into the affine line over K. The composite

$$\mathbf{B}_{\mathbb{C}_p} \xrightarrow{\kappa} \hat{o}_{\mathbb{C}_p} \xrightarrow{f_{\mathbb{C}_p}} \mathbb{A}^1_{\mathbb{C}_p}$$

is then given by a power series F generating the ideal (z^m) of B whence

$$F(z) = az^m(1 + b_1 z + b_2 z^2 + ...)$$

with $a \in \mathbb{C}_p^\times$, $b_i \in o_{\mathbb{C}_p}$ according to [20], Prop. 18.7. We may pass to a finite extension of K (if necessary) and have an element $x \in K$ with $|x| = |a|$. Passing to $x^{-1}f$ we see that F induces a rigid $\mathbb{C}_p$-analytic map $\mathbf{B}_{\mathbb{C}_p} \to \mathbf{B}_{\mathbb{C}_p}$ being the union over $\mathbb{C}_p$-affinoid maps $\mathbf{B}(r_n)_{\mathbb{C}_p} \to \mathbf{B}(r_n)_{\mathbb{C}_p}$. Hence, f is in fact a rigid K-analytic map

$$f : \hat{o}_K \to \mathbf{B}_K$$

equal to the union of K-affinoid maps $f_n : \hat{o}_{K,n} \to \mathbf{B}(r_n)_K$. The latter are induced by functions $f_n \in \mathcal{O}(\hat{o}_{K,n})$ generating $I^m\mathcal{O}(\hat{o}_{K,n})$, the m-th power of the ideal defining the zero section $Sp\ K \to \hat{o}_{K,n}$.

Next, we show that the $\mathcal{O}_{\mathbf{B}_K}$-module $f_*(\mathcal{O}_{\hat{o}_K})$ is locally free. Denote by $f^\sharp$ and $f_n^\sharp$ the corresponding ring homomorphisms on global sections. Fix n and let

$$A_n := \mathcal{O}(\hat{o}_{K,n}),\ B_n := \mathcal{O}(\mathbf{B}(r_n)_{K_n}).$$

We pass to a finite extension of K_n (if necessary) and have an element $a_n \in K_n$ such that $|a_n| = r_n$. Identifying $A_n \otimes_K K_n \simeq B_n$ via the group isomorphism $\kappa \circ h_n$ the map

$$f_n^\sharp \otimes_K K_n : B_n \longrightarrow B_n$$

is given by a power series in B_n defining the m-th power of the zero section $Sp\ K_n \to \mathbf{B}(r_n)_{K_n}$. Hence $f_n^\sharp \otimes_K K_n$ equals the map $(a_n^{-1}z) \mapsto (a_n^{-1}z)^m\epsilon$ with suitable $\epsilon \in B_n^\times$. Since $|a_n^{-1}z| = 1$ this is an isometry with associated graded map

$$gr^\bullet\,(f_n^\sharp \otimes_K K_n) : \sigma(a_n^{-1}z) \mapsto \sigma(a_n^{-1}z)^m\sigma(\epsilon)$$

and $\sigma(\epsilon) \in (gr^\bullet K_n)^\times$. Here, $gr^\bullet$ denotes the reduction functor from K_n-affinoid algebras into graded $(gr^\bullet K_n)$-algebras introduced by M. Temkin [39]. Clearly, the homomorphism $gr^\bullet\,(f_n^\sharp \otimes_K K_n)$ is finite free of rank m on the homogeneous basis elements $\sigma(a_n^{-1}z)^0, ..., \sigma(a_n^{-1}z)^{m-1}$. Hence, $f_n^\sharp \otimes_K K_n$ is finite free of rank m according to [25], Lem. I.6.4. By faithfully flat descent A_n is therefore a finitely generated projective B_n-module of rank m via $f_n^\sharp$. This shows the $\mathcal{O}_{\mathbf{B}_K}$-module $f_*(\mathcal{O}_{\hat{o}_K})$ to be a vector bundle of rank m.

By [18], V.2 Remarque 3^o this vector bundle must be trivial and so $f^\sharp$ induces a finite free ring extension $B \to A$ of degree m. Let A^0 and B^0 be the holomorphic functions on $\hat{o}_K$ and $\mathbf{B}_K$ respectively that are bounded above by 1. We claim that the ring extension $f^\sharp : B^0 \to A^0$ is integral. To see this we argue along the lines of [10], Lem. 7.3.3. Let $H \in A^0$. It satisfies an integral equation

$$T^m + b_1 T^{m-1} + ... + b_m = 0$$

with $b_i \in B$. We consider the ring extension induced by $f_n^\sharp$

$$R := \mathcal{O}(\mathbf{B}(r_n)_K)^0 \xrightarrow{\subseteq} R' := \mathcal{O}(\hat{o}_{K,n})^0.$$

It is a finite extension by [6], Cor. 6.4.1/6. Since $\mathbf{B}_K$ and $\hat{o}_K$ are normal, R and R' are normal o_K-algebras of topologically finite type which are flat over o_K. Clearly, R is an integral domain. Applying the Lemma 4.11 we find a closed set $V \subset Spec(R)$ with $\mathrm{codim}_{Spec(R)} V \geq 2$ such that the extension $R \to R'$ is finite flat of degree d over $Spec(R) \setminus V$. Consider H as an element of R' via the natural restriction map $A \to \mathcal{O}(\hat{o}_{K,n})$. It then satisfies an equation

$$T^m + b'_1 T^{m-1} + ... + b'_m = 0$$

with $b'_i \in \Gamma(Spec(R) \setminus V, \mathcal{O}_{Spec(R)}) = \Gamma(Spec(R), \mathcal{O}_{Spec(R)}) = R$. Comparing these b'_i to the b_i above we see that they must be equal as elements of $\mathcal{O}(\mathbf{B}(r_n)_K)$. Consequently, each b_i is of norm ≤ 1 on $\mathbf{B}(r_n)_K \subset \mathbf{B}_K$ for all n which means $b_i \in B^0$. This shows $f^\sharp : B^0 \to A^0$ to be an integral extension. By the preceding lemma we see that any element of A^0 has at most finitely many zeroes on $\hat{o}_K$. But according to (the proof of) [33], Lem. 3.9 the nonzero holomorphic function on $\hat{o}_K$ given on $\mathbb{C}_p$-valued points via

$$\kappa_z \mapsto \kappa_z(1) - \kappa_z(0)$$

is bounded above by 1 and has infinitely many zeroes. So we have arrived at a contradiction. □

As a result of the above discussion we have the following theorem.

Theorem 4.14. *The group $Pic(\hat{o}_K)$ is a profinite group. In case $L \neq \mathbb{Q}_p$ the isomorphism class of the ideal sheaf defining the zero section $Sp\ K \to \hat{o}_K$ is an element of infinite order in $Pic(\hat{o}_K)$.*

We briefly explain the relation of $Pic(\hat{o}_K)$ to the Grothendieck group $K_0(\hat{o}_K)$. By the Lemma 4.8 the latter coincides with $K_0(A)$ where $A = \mathcal{O}(\hat{o}_K)$.

For the following basic notions from algebraic K-theory we refer to [2]. Let R be a commutative associative unital ring, $Pic(R)$ its Picard group and $K_0(R)$ its Grothendieck group. Mapping $1 \mapsto [R]$ induces an injective group homomorphism $\mathbb{Z} \to K_0(R)$ which factores through the kernel of the determinant $\det : K_0(R) \to Pic(R)$. Denoting by $H_0(R)$ the abelian group of continuous maps $Spec(R) \to \mathbb{Z}$ we have the rank mapping $\mathrm{rk} : K_0(R) \to H_0(R)$ and a surjection

$$\mathrm{rk} \oplus \det : K_0(R) \to H_0(R) \oplus Pic(R). \tag{15}$$

The kernel $SK_0(R)$ consists of classes $[P] - [R^n]$ where P has constant rank, say, n and $\wedge^n P \cong R$. If R is noetherian of dimension one or a Prüferian domain (i.e. an integral domain such that any finitely generated ideal is invertible) then $SK_0(R) = 0$. Indeed, in both cases Serre's theorem (loc.cit., Thm. IV.2.5) yields that any finitely generated projective module is isomorphic to a direct sum of an invertible and a free module (cf. also [18], V.2 Remarque 3^o).

By [33], final discussion in sect. 3, the algebra of global sections $\mathcal{O}(\hat{o}_K)$ is a Prüferian domain which implies the

Proposition 4.15. *The rank and determinant homomorphisms give a canonical isomorphism of abelian groups*

$$\mathrm{rk} \oplus \det : K_0(\hat{o}_K) \xrightarrow{\cong} \mathbb{Z} \oplus Pic(\hat{o}_K).$$

Remark: Let $gr^\bullet$ be the reduction functor for K-affinoids introduced by M. Temkin [39]. In [29] the author develops a method to compute the Picard group of twisted forms of the closed unit disc which are '$gr^\bullet$-smooth', i.e. whose reduction is smooth over the 'graded field' $gr^\bullet K$. A large class of such forms is given by the tamely ramified ones. Since the field L_∞ is generated by the torsion points of the p-divisible group $\mathbb{G}'$ it is not tamely ramified over L. Therefore, the form $\hat{o}_n$ is generally not tamely ramified. Even worse, the descent datum of the form $\hat{o}_n$ involves the logarithm series $\log_{\mathbb{G}}$ (Prop. 4.3). To compute the reduction of $\hat{o}_n$ seems to require therefore a detailed knowledge of the coefficients of $\log_{\mathbb{G}}$ which is not available. We have therefore not been able to prove that $\hat{o}_n$ is '$gr^\bullet$-smooth'. In fact, we are rather sceptical about $\hat{o}_n$ having this property.

4.5. **General character spaces.** As explained above the space $\hat{o}$ parametrizes the locally analytic characters of the additive group o. In the following we explain briefly why the problem of determining the Picard group of general character spaces essentially reduces to the case of (copies of) $\hat{o}$.

So consider an arbitrary locally L-analytic group Z which is abelian and topologically finitely generated. Let $d := \dim_L Z$. The following generalization of the rigid analytic character variety $\hat{o}$ has been introduced by M. Emerton in [16], (6.4). Let $Rig(K)$ be the category of rigid analytic spaces over K. For each $X \in Rig(K)$ let $\hat{Z}(X)$ be the group of abstract group homomorphisms $Z \to \mathcal{O}(X)^\times$ with the property that for each admissible open affinoid subspace $U \subseteq X$ the map

$$Z \to \mathcal{O}(X)^\times \xrightarrow{res} \mathcal{O}(U)$$

is a $\mathcal{O}(U)$-valued locally L-analytic function on Z. This defines a contravariant functor

$$\hat{Z}_K : Rig(K) \to Ab$$

which is in fact representable by a smooth rigid K-analytic group $\hat{Z}_K$ on a quasi-Stein space. Our character group $\hat{o}_K$ corresponds to the case $Z = o$. The association $Z \mapsto \hat{Z}_K$ is a contravariant functor that converts direct products into fibre products over K.

Examples: Let $\mathbb{G}_m$ and μ_n be the rigid K-analytic multiplicative group and the rigid K-analytic group of roots of unity of order $n \geq 1$ respectively. Mapping a locally analytic character to its value on 1 respectively on 1 mod m induces group isomorphisms

$$\hat{\mathbb{Z}}_K \xrightarrow{\cong} \mathbb{G}_m,\ (\widehat{\mathbb{Z}/m\mathbb{Z}})_K \xrightarrow{\cong} \mu_m$$

(loc.cit.).

To get a first impression of the space $\hat{Z}_K$ we look at dimension and number of connected components. Let $\mu \subseteq Z$ be the torsion subgroup of Z. By [16], Prop. 6.4.1 the inclusion of the unique maximal compact open subgroup Z_0 into Z induces a (noncanonical) isomorphism $Z_0 \times \mathbb{Z}^r \cong Z$ for some unique $r \geq 0$. Hence, there is a (noncanonical) isomorphism of rigid groups

$$\hat{Z}_K \xrightarrow{\cong} \hat{Z}_{0K} \times_K \mathbb{G}_m^r.$$

To proceed further we impose a mild condition on the group Z_0. First of all, being abelian profinite Z_0 contains a unique open pro-p-Sylow subgroup $Z_0(p)$. The torsion part $Z_0(p)^{\rm tor}$ of the latter group is finite and a direct factor so that μ is finite. Any complement $Z_0(p)^{\rm fl}$ (as $\mathbb{Z}_p$-module) in $Z_0(p)$ to $Z_0(p)^{\rm tor}$ has finite index in $Z_0(p)$ and hence is open according to [12], Thm. 1.17. It is therefore naturally endowed with a structure of abelian locally L-analytic group. As such is has an open subgroup which is isomorphic, as locally L-analytic group, to the standard group o^d (in the sense of [8], Thm. III. 7.3.4). We **assume** in the following that the torsion part in $Z_0(p)$ admits a complement $Z_0(p)^{\rm fl}$ which is isomorphic, as locally L-analytic group to o^d. We fix such an isomorphism.

Example: Let $\mathbb{T}$ be a linear algebraic torus over L and $Z = \mathbb{T}(L)$ its group of L-rational points. Then Z is topologically finitely generated. Indeed, this is immediate for the split part of Z and follows for the anisotropic part of Z by compactness, cf. [5], Cor. §9.4. Now assume that $\mathbb{T}$ is split over L so that we may identify $Z = (L^\times)^d$ and $Z_0 = (o^\times)^d$. Suppose the ramification index e of $L/\mathbb{Q}_p$ satisfies $1 > e/(p-1)$. A possible choice for $Z_0(p)^{\rm fl}$ is given by $(1+\pi o)^d$. Moreover, the usual logarithm series followed by multiplication with π^{-1} provides a locally L-analytic group isomorphism $Z_0(p)^{\rm fl} \xrightarrow{\cong} o^d$, e.g. [27], Prop. 5.5.

By assumption the inclusion $\mu \times Z_0(p)^{\rm fl} \subseteq Z$ induces a (noncanonical) isomorphism of locally L-analytic groups

$$\mu \times o^d \times \mathbb{Z}^r \xrightarrow{\cong} Z$$

with unique $r \geq 0$. Applying the functor $\hat{(\cdot)}_K$ gives a (noncanonical) isomorphism of rigid groups

$$\hat{Z}_K \xrightarrow{\cong} \hat{\mu}_K \times_K \hat{o}^d_K \times_K \mathbb{G}^r_m.$$

The space $\hat{\mu}_K$ is a finite disjoint union of points $Sp\ K_i,\ i = 1, ..., s$ (with finite field extensions K_i/K). Since each $\hat{o}^d_{K_i}$ is connected and $\mathbb{G}^r_{m,K_i}$ is geometrically connected, their fibre product over K_i remains connected by [13], Cor. 8.4. Hence,

$$\pi_0(\hat{Z}_K) = s \leq \#\mu \text{ and } \dim \hat{Z}_K = d + r.$$

We conclude with the remark that, in this situation, the natural projection morphism $\hat{\iota} : \hat{Z}_K \to \hat{\mu}_K \times_K \hat{o}^d_K$ induces an isomorphism

$$Pic(\hat{Z}_K) \xrightarrow{\cong} Pic(\hat{\mu}_K \times_K \hat{o}^d_K) = \oplus_{i=1,...,s} Pic(\hat{o}^d_{K_i}).$$

Indeed, using an argument with cohomology and inverse limits similar to subsect. 4.3 one is reduced to show that the natural projection $\hat{o}^d_{K,n} \times_K \mathbb{G}^r_{m,n} \to \hat{o}^d_{K,n}$ induces an isomorphism on Picard groups for all n. Here, $\mathbb{G}^r_{m,n} \subset \mathbb{G}^r_m$ is the annulus defined by $|p|^n \leq z_i \leq |p|^{-n}, i = 1, ..., r$. A finite induction reduces to the case $d = r = 1$. This case follows then by properties of the one dimensional affinoids $\hat{o}_{K,n}$ and the fact that $Pic(\mathbb{G}_{m,n}) = 1$. We leave the remaining details to the interested reader.

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