LOCALLY ANALYTIC REPRESENTATIONS OF $\mathrm{GL}(2,L)$ VIA SEMISTABLE MODELS OF $\mathbb{P}^1$

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ABSTRACT. In this paper we study certain sheaves of p-adically complete rings of differential operators on semistable models of the projective line over the ring of integers in a finite extension L of $\mathbb{Q}_p$. The global sections of these sheaves can be identified with (central reductions of) analytic distribution algebras of wide open congruence subgroups. It is shown that the global sections functor furnishes an equivalence between the categories of coherent module sheaves and finitely presented modules over the distribution algebras. Using the work of M. Emerton, we then describe admissible representations of $\mathrm{GL}_2(L)$ in terms of sheaves on the projective limit of these formal schemes. As an application, we show that representations coming from certain equivariant line bundles on Drinfeld's first étale covering of the p-adic upper half plane are admissible.

CONTENTS

1. Introduction

Let $L/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathfrak{o} = \mathfrak{o}_L$, and denote by $K \supset L$ a completely and discretely valued extension of L (whose absolute value extends that of L). This paper is about admissible locally L-analytic representations of the group $G = \mathrm{GL}_2(L)$ on K-vector spaces, cf. [38]. By the theory of Schneider and Teitelbaum, the category of these representations is (anti-)equivalent to the category of modules M over the locally analytic distribution algebra $D(G, K)$ of G which are coadmissible when considered as modules over the locally analytic distribution algebra $D(H, K)$, for any compact open subgroup $H \subset G$. Let V be an admissible locally analytic representation of G and put $M = V' := \mathrm{Hom}_K^{\mathrm{cont}}(V, K)$, which is then a $D(G, K)$-module. In [14] M. Emerton has highlighted the role played by the subspaces $V_{\mathbb{H}^\circ - \mathrm{an}} \subset V$ of *rigid analytic vectors* for certain rigid analytic subgroups $\mathbb{H}^\circ \subset \mathrm{GL}_{2,L}^{\mathrm{rig}}$, where the latter is the rigid analytic group associated to the algebraic group $\mathrm{GL}_{2,L}$ over L. The rigid analytic groups which we consider here are the *wide open principal congruence subgroups* $\mathbb{G}(k)^\circ$ which have the property that $\mathbb{G}(k)^\circ(\mathbb{C}_p) = 1 + \varpi^k M_2(\mathfrak{m}_{\mathbb{C}_p})$, where $\mathfrak{m}_{\mathbb{C}_p}$ is the maximal ideal of the valuation ring of $\mathbb{C}_p$ and $\varpi \in \mathfrak{o}$ is a uniformizer. Following the work of Emerton we show in 5.2.4 that the continuous dual $M_k = (V_{\mathbb{G}(k)^\circ - \mathrm{an}})'$ is then a finitely presented module over the *analytic distribution algebra* $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ, K) = (\mathcal{O}(\mathbb{G}(k)^\circ)\widehat{\otimes}_L K)'$.

The first main result of this paper is a localizaton theory for finitely presented modules over the rings $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ, K)$. For technical reasons we actually restrict our attention to modules on which the universal enveloping algebra of $\mathfrak{gl}_{2,L}$ acts via the character θ_0 which is the central character of the trivial one-dimensional representation. We therefore consider here the central reduction $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ, K)_{\theta_0}$. Now, in order to develop the localization theory, we make use of certain semistable integral models $\mathbb{X}_n$ of the projective line

$\mathbb{X}_0 = \mathbb{P}^1_{\mathfrak{o}}$ over $\mathfrak{o}^1$. Denote by $\mathfrak{X}_n$ the formal completion of $\mathbb{X}_n$ along its special fiber. In section 3, we define and study sheaves $\widetilde{\mathcal{D}}_{n,k}^{(m)}$ (for $k \geqslant n$) of differential operators on $\mathbb{X}_n$. When $m = 0$ the sheaf $\widetilde{\mathcal{D}}_{n,k}^{(0)}$ is generated by the pull-back of $p^k \mathcal{T}_{\mathbb{X}_0}$ and the structure sheaf, where $\mathcal{T}_{\mathbb{X}_0}$ is the relative tangent sheaf of $\mathbb{X}_0$ over $\mathbb{S} = \mathrm{Spec}(\mathfrak{o})$. By p-adic completion, the sheaf $\widetilde{\mathcal{D}}_{n,k}^{(m)}$ gives rise to a sheaf of p-adically complete differential operators on $\mathfrak{X}_n$ which we denote by $\widetilde{\mathscr{D}}_{n,k}^{(m)}$. Then we consider the inductive limit

$$\widetilde{\mathscr{D}}_{n,k}^\dagger = \varinjlim_m \widetilde{\mathscr{D}}_{n,k}^{(m)}$$

and we put $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^\dagger = \widetilde{\mathscr{D}}_{n,k}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}$. The main result of section 4 is that $\mathfrak{X}_n$ is $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^\dagger$-affine. By this we mean that every coherent $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^\dagger$-module $\mathscr{M}$ is generated (over $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^\dagger$) by its global sections, and that $H^i(\mathfrak{X}_n, \mathscr{M}) = 0$ for all $i > 0$, cf. [3]. In particular, the global sections functor $H^0(\mathfrak{X}_n, -)$ gives an equivalence of categories of coherent $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^\dagger$-modules and finitely presented modules over $H^0\big(\mathfrak{X}_n, \widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^\dagger\big)$. Moreover, and this is where the connection with representation theory is made, the latter ring is canonically isomorphic $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_{\theta_0}$.

This $\mathcal{D}$-affinity result for the sheaves $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^\dagger$ partly generalizes the work of C. Huyghe in [23], where she shows $\mathscr{D}_{\mathbb{Q}}^\dagger$-affinity for all projective spaces, and, moreover, considers differential operators with coefficients. The sheaf of arithmetic differential operators considered in [23] is identical to our sheaf $\widetilde{\mathscr{D}}_{0,0,\mathbb{Q}}^\dagger$ on the smooth model $\mathbb{X}_0$. The proofs which lead to $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^\dagger$-affinity of $\mathfrak{X}_n$ are in general easy variations of the proofs given by Huyghe in the case when $(n, k) = (0, 0)$. In many cases we follow her strategy and arguments word for word, and we do not claim any originality here. In some cases we have chosen to slightly expand her arguments. And we have decided to first discuss properties of the sheaves of rings $\widetilde{\mathscr{D}}_{n,k}^{(m)}$ (in sec. 3) and then discuss coherent modules over these sheaves (in sec. 4), whereas in [23] this is often done in a more parallel way.

In section 5 we pass to the limit over all n and come back to representations of the non-compact group $G = \mathrm{GL}_2(L)$. The sheaves $\widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^\dagger$ assemble to a sheaf of infinite order differential operators $\widetilde{\mathscr{D}}_{\infty,\mathbb{Q}}^\dagger$ on the space $\mathfrak{X}_\infty = \varprojlim_n \mathfrak{X}_n$. The space $\mathfrak{X}_\infty$ has a natural $\mathrm{GL}_2(L)$-action extending the $\mathrm{GL}_2(\mathfrak{o})$-action on any of the formal schemes $\mathfrak{X}_n$, and $\widetilde{\mathscr{D}}_{\infty,\mathbb{Q}}^\dagger$ is equivariant with respect to this action. In this situation, the above equivalences of categories for each $\mathfrak{X}_n$ assemble to an equivalence of categories between admissible locally analytic $\mathrm{GL}_2(L)$-representations with trivial infinitesimal character and a full subcategory of equivariant $\widetilde{\mathscr{D}}_{\infty,\mathbb{Q}}^\dagger$-modules.

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[1]When $L = \mathbb{Q}_p$ these schemes had already been studied in [33].

We remark at this point that some of our constructions are related – in the case of the *smooth* formal model – to the work of K. Ardakov and S. Wadsley, cf. [2], and the work of one of us, [34]. Indeed, the family of sheaves of completed differential operators $\widetilde{\mathscr{D}}^{(0)}_{0,k,\mathbb{Q}}$, as defined here, is essentially the same family of sheaves as considered in these papers. Furthermore, the sheaf $\widetilde{\mathscr{D}}^{\dagger}_{\infty,\mathbb{Q}}$ is a kind of Arens-Michael completion of the sheaf of algebraic differential operators, and as such closely related to the work of Ardakov and Wadsley, cf. [1]. We would also like to point out that our constructions made here, and the arguments used in proofs of the main results, carry over to more general, not necessarily semistable, models, and also to general reductive groups and their flag varieties [24].

In section 6 we compute the equivariant $\widetilde{\mathscr{D}}^{\dagger}_{\infty,\mathbb{Q}}$-modules for a class of examples including smooth representations, principal series representations and representations coming from the p-adic upper half plane.

In section 7 we give an application of our methods and show that representations coming from certain equivariant line bundles (the structure sheaf and the sheaf of differential forms) on Drinfeld's first étale covering $\Sigma_1 \xrightarrow{f} \Sigma_0$, cf. [13], of the p-adic upper half plane Σ_0 are admissible. Here is a sketch of the arguments for the case of the structure sheaf. Our aim is to show that $M := H^0(\Sigma_1, \mathcal{O})$ is a coadmissible $D(\mathrm{GL}_2(\mathfrak{o}_L))$-module. A crucial step towards this goal is to show $M = \varprojlim M_n$ where M_n is a finitely generated $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^{\circ})$-module. Together with some additional compatibility property, cf. 7.5, Emerton's reformulation of 'admissibility' in terms of analytic distribution algebras, cf. [14], will then ensure that M is a coadmissible $D(\mathrm{GL}_2(\mathfrak{o}_L))$-module.

There is a natural admissible covering $\Sigma_0(0) \subset \Sigma_0(1) \subset \Sigma_0(2) \subset \ldots$ of Σ_0 by affinoids $\Sigma_0(n)$, and an induced covering of Σ_1 by the preimages $\Sigma_1(n) = f^{-1}(\Sigma_0(n))$. We let $M_n = H^0(\Sigma_1(n), \mathcal{O})^{\dagger}$ be the space of overconvergent rigid-analytic functions on $\Sigma_1(n)$. Our aim is to 'sheafify' this space in a suitable way. A key role is played by specialization maps $\mathrm{sp}_{\widehat{\Sigma}_i} : \Sigma_i \to \widehat{\Sigma}_i$ $(i = 0, 1)$ to formal models $\widehat{\Sigma}_0$ and $\widehat{\Sigma}_1$ of Σ_0 and Σ_1, respectively, and the fact that the covering map f extends to a morphism $\hat{f}$ between the formal models. If $v_n : \widehat{\Sigma}_1(n) \hookrightarrow \widehat{\Sigma}_1$ is the open embedding, then Berthelot has defined in such a setting a sheaf $v_n^{\dagger}\mathcal{O}$ on Σ_1, cf. [6, sec. 4], which is supported on $\Sigma_1(n)$. We then consider the sheaf $\mathscr{M}_n = \hat{f}_*\left((\mathrm{sp}_{\widehat{\Sigma}_1})_* v_n^{\dagger}\mathcal{O}\right)$. This sheaf is supported on the closure $\widehat{\Sigma}_0(n)_s^c$ of $\widehat{\Sigma}_0(n)_s = \mathrm{sp}_{\widehat{\Sigma}_0}(\Sigma_0(n))$. This closed subset, with its induced reduced subscheme structure, is isomorphic to the special fiber of $\mathfrak{X}_n$ (after base change to $\overline{\mathbb{F}}_p$). More is true: we show that $\mathscr{M}_n$ carries the structure of a coherent $\widetilde{\mathscr{D}}^{\dagger}_{n,n,\mathbb{Q}}$-module. This is proved by making use of results of Caro [8], Noot-Huyghe/Trihan [28], Shiho [39] and Tsuzuki [43] on the coherence of the direct image in crystalline cohomology. The upshot is then that, by our localization theory, $H^0(\mathfrak{X}_n, \mathscr{M}_n) = H^0(\Sigma_1(n), \mathcal{O})^{\dagger} = M_n$ is a finitely generated $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^{\circ})$-module.

We therefore see that, in this instance, techniques and results from the theory of crystalline cohomology and arithmetic $\mathscr{D}$-modules can be successfully applied in the context of locally analytic representation theory, and we hope that this develops into a systematic and fruitful interaction between these fields.

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Notation. L is a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathfrak{o}$, uniformizer ϖ. We let q denote the cardinality of the residue field $\mathfrak{o}/(\varpi)$ which we also denote by $\mathbb{F}_q$.

2. The integral models $\mathbb{X}_n$ and their formal completions $\mathfrak{X}_n$

2.1. **Construction via blowing up.** The integral models $\mathbb{X}_n$ have been discussed in detail in [33] in the case when $\mathfrak{o} = \mathbb{Z}_p$. Since the generalization to the ring of integers $\mathfrak{o}$ of the finite extension $L/\mathbb{Q}_p$ is straightforward, we will only briefly recall the construction. Let $\Lambda \subset L^{\oplus 2}$ be a lattice, and let $\mathbb{X} = \mathbb{X}_0 = \mathbb{P}\mathrm{roj}(\Lambda)$ be the smooth model of $\mathbb{P}^1_L = \mathbb{P}\mathrm{roj}(L^{\oplus 2})$ associated to Λ. While it is eventually useful to note that all constructions in sections 2-4 can be done canonically for an arbitrary lattice Λ we will henceforth consider only the case when $\Lambda = \mathfrak{o} \oplus \mathfrak{o}$ is the standard lattice.

Blowing up $\mathbb{X}_0$ in the $\mathbb{F}_q$-rational points of its special fiber (i.e., blowing up the corresponding ideal sheaf) gives a model $\mathbb{X}_1$ whose special fiber has $q + 2$ irreducible components, all of which are isomorphic to $\mathbb{P}^1_{\mathbb{F}_q}$. Exactly one of these components intersects transversely every other irreducible component, and we call it the 'central' component, whereas the other components are called the 'outer' components. The smooth $\mathbb{F}_q$-rational points lie only on the outer components, and blowing up $\mathbb{X}_1$ in these smooth $\mathbb{F}_q$-valued points of its special fiber produces the model $\mathbb{X}_2$. In general, $\mathbb{X}_{n+1}$ is obtained by blowing up the smooth $\mathbb{F}_q$-rational points of the special fiber of $\mathbb{X}_n$. For $n \geqslant n'$ we denote the blow-up morphism from $\mathbb{X}_n$ to $\mathbb{X}_{n'}$ by

$$\mathrm{pr}_{n,n'} : \mathbb{X}_n \longrightarrow \mathbb{X}_{n'}.$$

The morphism $\mathrm{pr}_{n,0} : \mathbb{X}_n \to \mathbb{X} = \mathbb{X}_0$ is often denoted by pr when n is clear from the context.

Let $\mathfrak{X}_n$ be the completion of $\mathbb{X}_n$ along its special fiber. $\mathfrak{X}_n$ has an open affine covering by 'dumbbell' shaped formal schemes of the form

$$\mathrm{Spf}\left(\mathfrak{o}\langle x, z\rangle\Big[\frac{1}{x^{q-1}-1}, \frac{1}{z^{q-1}-1}\Big]^{\wedge}\Big/\Big(xz - \varpi\Big)\right),$$

(where $(.)^{\wedge}$ denotes the ϖ-adic completion) together with formal affine lines $\widehat{\mathbb{A}}^1_{\mathfrak{o}} = \mathrm{Spf}(\mathfrak{o}\langle t \rangle)$ (which are only needed for the 'outermost' components). As above we denote the blow-up morphisms by $\mathrm{pr}_{n,n'}$. Removing the smooth $\mathbb{F}_q$-rational points from $\mathfrak{X}_n$ gives an open subscheme $\mathfrak{X}^\circ_n \subset \mathfrak{X}_n$. The morphism $\mathrm{pr}_{n+1,n} : \mathfrak{X}_{n+1} \to \mathfrak{X}_n$ induces then an isomorphism from the preimage $\mathrm{pr}^{-1}_{n+1,n}(\mathfrak{X}^\circ_n)$ to $\mathfrak{X}^\circ_n$, so that we can consider $\mathfrak{X}^\circ_n$ as an open subscheme of $\mathfrak{X}_{n+1}$, which is actually contained in $\mathfrak{X}^\circ_{n+1}$. The inductive limit $\varinjlim_n \mathfrak{X}^\circ_n$ is a formal model of the p-adic upper half plane over L, cf. section 6.4.1.

2.2. Group actions on $\mathbb{X}_n$.

2.2.1. *The group schemes* $\mathbb{G}(k)$. Put

$$\mathbb{G}(0) = \mathbb{G} = \mathrm{GL}_{2,\mathfrak{o}} = \mathrm{Spec}\left(\mathfrak{o}\left[a, b, c, d, \frac{1}{\Delta}\right]\right) ,$$

where $\Delta = ad - bc$, and the co-multiplication is the one given by the usual formulas. For $k \geqslant 1$ let a_k, b_k, c_k, and d_k denote indeterminates. Define an affine group scheme $\mathbb{G}(k)$ over $\mathfrak{o}$ by setting

$$\mathcal{O}(\mathbb{G}(k)) = \mathfrak{o}\left[a_k, b_k, c_k, d_k, \frac{1}{\Delta_k}\right] , \quad \text{where} \quad \Delta_k = (1 + \varpi^k a_k)(1 + \varpi^k d_k) - \varpi^{2k} b_k c_k ,$$

and let the co-multiplication

$$\mathcal{O}(\mathbb{G}(k)) \longrightarrow \mathcal{O}(\mathbb{G}(k)) \otimes_{\mathfrak{o}} \mathcal{O}(\mathbb{G}(k)) = \mathfrak{o}\left[a_k, b_k, c_k, d_k, a'_k, b'_k, c'_k, d'_k, \frac{1}{\Delta_k}, \frac{1}{\Delta'_k}\right]$$

be given by the formulas

$$
\begin{array}{ccccccccc}
a_k & \mapsto & a_k & + & a'_k & + & \varpi^k a_k a'_k & + & \varpi^k b_k c'_k , \\
b_k & \mapsto & b_k & + & b'_k & + & \varpi^k a_k b'_k & + & \varpi^k b_k d'_k , \\
c_k & \mapsto & c_k & + & c'_k & + & \varpi^k c_k a'_k & + & \varpi^k d_k c'_k , \\
d_k & \mapsto & d_k & + & d'_k & + & \varpi^k d_k d'_k & + & \varpi^k c_k b'_k .
\end{array}
$$

These group schemes are connected by homomorphisms $\mathbb{G}(k) \to \mathbb{G}(k-1)$ given on the level of algebras as follows:

$$a_{k-1} \mapsto \varpi a_k , \quad b_{k-1} \mapsto \varpi b_k , \quad c_{k-1} \mapsto \varpi c_k , \quad d_{k-1} \mapsto \varpi d_k ,$$

if $k > 1$. For $k = 1$ we put

$$a \mapsto 1 + \varpi a_1 , \quad b \mapsto \varpi b_1 , \quad c \mapsto \varpi c_1 , \quad d \mapsto 1 + \varpi d_1 .$$

For a flat $\mathfrak{o}$-algebra R the homomorphism $\mathbb{G}(k) \to \mathbb{G}(0) = \mathbb{G}$ induces an isomorphism of $\mathbb{G}(k)(R)$ with a subgroup of $\mathbb{G}(R)$, namely

$$\mathbb{G}(k)(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{G}(R) \;\middle|\; a - 1, b, c, d - 1 \in \varpi^k R \right\} .$$

Of course, the preceding formulas defining the group schemes are derived formally from this description by setting $a = 1 + \varpi^k a_k$, $b = \varpi^k b_k$, $c = \varpi^k c_k$, and $d = 1 + \varpi^k d_k$.

2.2.2. *The rigid-analytic groups* $\mathbb{G}(k)^{\mathrm{rig}}$ *and* $\mathbb{G}(k)^\circ$. Let $\widehat{\mathbb{G}}(k)$ be the completion of $\mathbb{G}(k)$ along its special fiber $\mathbb{G}(k)_{\mathbb{F}_q}$. This is a formal group scheme over $\mathrm{Spf}(\mathfrak{o})$. Its generic fiber in the sense of rigid geometry is an affinoid rigid-analytic group over L which we denote by $\mathbb{G}(k)^{\mathrm{rig}}$. We have for any completely valued field L'/L (whose valuation extends the p-adic valuation)

$$\mathbb{G}(k)^{\mathrm{rig}}(L') = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{G}(\mathfrak{o}_{L'}) \;\middle|\; a - 1, b, c, d - 1 \in \varpi^k \mathfrak{o}_{L'} \right\} .$$

Furthermore, we let $\widehat{\mathbb{G}}(k)^\circ$ be the completion of $\mathbb{G}(k)$ in the closed point corresponding to the unit element in $\mathbb{G}(k)_{\mathbb{F}_q}$. This is a formal group scheme over $\mathrm{Spf}(\mathfrak{o})$ (not of topologically finite type). Its generic fiber in the sense of Berthelot, cf. [11, sec. 7.1], is a so-called 'wide open' rigid-analytic group over L which we denote by $\mathbb{G}(k)^\circ$. We have for any completely valued field L'/L (whose valuation extends the p-adic valuation)

$$\mathbb{G}(k)^\circ(L') = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{G}(\mathfrak{o}_{L'}) \;\middle|\; a - 1, b, c, d - 1 \in \varpi^k \mathfrak{m}_{\mathfrak{o}_{L'}} \right\} .$$

Proposition 2.2.3. *(i) For $k \geqslant n$ the natural action of $\mathrm{GL}_{2,L}$ on $\mathbb{P}^1_L$ extends to an action of the group scheme $\mathbb{G}(k)$ on $\mathbb{X}_n$.*

(ii) For $k \geqslant n$ the formal group scheme $\widehat{\mathbb{G}}(k)$ acts on $\mathfrak{X}_n$. The action of the special fiber $\widehat{\mathbb{G}}(k)_s$ on the special fiber $\mathfrak{X}_{n,s}$ of $\mathfrak{X}_n$ is trivial if $k > n$.

Proof. (i) Since there is a morphism of group schemes $\mathbb{G}(k) \to \mathbb{G}(n)$ for $k \geqslant n$, we may restrict to $k = n$. We use induction on n, the case of $n = 0$ being obvious. Suppose the claim holds for $n - 1$. Let $\pi : \mathbb{X}_n \to \mathbb{X}_{n-1}$ be the blow-up morphism and let $Y \subset \mathbb{X}_{n-1}$ be an arbitrary open affine subscheme stable under $\mathbb{G}(n-1)$. Let $I \subset \mathcal{O}(Y)$ be the ideal whose blow-up gives rise to $\pi^{-1}(Y)$. Then $\varpi \in I$. Let $A = \mathcal{O}(\mathbb{G}(n))$ and $B = \mathcal{O}(Y)$. Denote by I_s, A_s and B_s the reductions mod ϖ. Let $c : B \to B \otimes A$ be the comorphism corresponding to the $\mathbb{G}(n-1)$-action on Y and let c' be its reduction mod ϖ. The special fibre of the natural $\mathfrak{o}$-morphism $\mathbb{G}(n) \to \mathbb{G}(n-1)$ factors through the unit section $\mathrm{Spec}\, \mathbb{F}_q \to \mathbb{G}(n-1)_s$ of the group scheme $\mathbb{G}(n-1)_s$. This means $c'(b') = b' \otimes 1$ for all $b' \in B'$ and hence

$$c(I) \subset I \otimes A + \varpi(B \otimes A) = I \otimes A \ .$$

Since c is a ring homorphism, this implies inductively $c(I^m) \subset I^m \otimes A$ for all m. This yields an induced comorphism $c : R(I) \to R(I) \otimes A$ where $R(I) := \oplus_{m \geqslant 0} I^m$ denotes the blow-up algebra. Hence, the group scheme $\mathbb{G}(n)$ acts naturally on $\pi^{-1}(Y) = \mathrm{Proj}(R(I))$. This implies the claim.

(ii) This follows from (i) by passing to formal completions. $\qquad\qquad\square$

2.3. **A very ample line bundle on $\mathbb{X}_n$.** Let $\mathrm{pr} : \mathbb{X}_n \to \mathbb{X} = \mathbb{X}_0$ be the blow-up morphism, and let $\mathcal{I}_n \subset \mathcal{O}_\mathbb{X}$ be the ideal sheaf that is blown up. We have $\mathcal{I}_n \otimes_\mathbb{Z} \mathbb{Q} = \mathcal{O}_\mathbb{X} \otimes_\mathbb{Z} \mathbb{Q}$ from which we deduce the existence of some $N = N(n) \in \mathbb{Z}_{>0}$ such that[2]

$$(2.3.1) \qquad\qquad\qquad\qquad p^N \mathcal{O}_\mathbb{X} \subset \mathcal{I}_n \subset \mathcal{O}_\mathbb{X} \ .$$

Put $\mathcal{S} = \bigoplus_{s \geqslant 0} \mathcal{I}_n^s$, then $\mathbb{X}_n$ is glued together from schemes $\mathbf{Proj}(\mathcal{S}(U))$ for affine open subsets $U \subset \mathbb{X}$. On each $\mathbf{Proj}(\mathcal{S}(U))$ there is an invertible sheaf $\mathcal{O}(1)$, and these glue together to give an invertible sheaf $\mathcal{O}(1)$ on $\mathbb{X}_n$ which we will denote $\mathcal{O}_{\mathbb{X}_n/\mathbb{X}}(1)$ (cf. the discussion in [21, ch. II,§7].) This invertible sheaf is in fact the inverse image ideal sheaf $\mathrm{pr}^{-1}(\mathcal{I}_n) \cdot \mathcal{O}_{\mathbb{X}_n}$, cf. [21, ch. II, 7.13]. From 2.3.1 conclude that

$$(2.3.2) \qquad\qquad p^N \mathcal{O}_{\mathbb{X}_n} \subset \mathcal{O}_{\mathbb{X}_n/\mathbb{X}}(1) \subset \mathcal{O}_{\mathbb{X}_n} \quad \text{and} \quad \mathcal{O}_{\mathbb{X}_n} \subset \mathcal{O}_{\mathbb{X}_n/\mathbb{X}}(-1) \subset p^{-N} \mathcal{O}_{\mathbb{X}_n} \ .$$

And for any $r \geqslant 0$ we get

$$(2.3.3) \qquad\quad p^{rN} \mathcal{O}_{\mathbb{X}_n} \subset \mathcal{O}_{\mathbb{X}_n/\mathbb{X}}(1)^{\otimes r} \subset \mathcal{O}_{\mathbb{X}_n} \quad \text{and} \quad \mathcal{O}_{\mathbb{X}_n} \subset \mathcal{O}_{\mathbb{X}_n/\mathbb{X}}(-1)^{\otimes r} \subset p^{-rN} \mathcal{O}_{\mathbb{X}_n} \ .$$

Lemma 2.3.4. *Let $0 \leqslant n' \leqslant n$. (i) $(\mathrm{pr}_{n,n'})_*(\mathcal{O}_{\mathbb{X}_n}) = \mathcal{O}_{\mathbb{X}_{n'}}$.*

(ii) For all $i > 0$ one has $R^i(\mathrm{pr}_{n,n'})_*(\mathcal{O}_{\mathbb{X}_n}) = 0$.

Proof. (i) The morphism $\mathrm{pr}_{n,n'} : \mathbb{X}_n \to \mathbb{X}_{n'}$ is a birational projective morphism of noetherian integral schemes, and $\mathbb{X}_{n'}$ is normal. The assertion then follows exactly as in the proof of Zariski's Main Theorem as given in [21, ch. III, Cor. 11.4].

(ii) We note that $R^i(\mathrm{pr}_{n,n'})_*(\mathcal{O}_{\mathbb{X}_n})$ vanishes when $i > 1$, by [21, ch. III, 11.2]. So we will only consider the case $i = 1$ now. Using the Leray spectral sequence for the higher direct images under the composite maps $\mathrm{pr}_{n-1,n-2} \circ \mathrm{pr}_{n,n-1}$, together with (i), we see that it

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[2]In fact, these inclusions hold for $N(n) = n$, but we will not need this.

suffices to show that $R^1(\mathrm{pr}_{n,n-1})_*(\mathcal{O}_{\mathbb{X}_n}) = 0$. The sheaf in question certainly vanishes on $\mathbb{X}_{n-1}^\circ$, because the restriction of $\mathrm{pr}_{n,n-1}$ is an isomorphism

$$(\mathrm{pr}_{n,n-1})^{-1}\left(\mathbb{X}_{n-1}^\circ\right) \xrightarrow{\simeq} \mathbb{X}_{n-1}^\circ .$$

The question is local, and by what we have just seen, it suffices to consider the stalk of the sheaf $R^1(\mathrm{pr}_{n,n-1})_*(\mathcal{O}_{\mathbb{X}_n})$ at a smooth $\mathbb{F}_q$-rational point P of the special fiber of $\mathbb{X}_{n-1}$. Let x be a local coordinate at P, so that P corresponds to the ideal $(x, \varpi) \subset \mathfrak{o}[x]$. Put

$$R = \mathfrak{o}[x] , \quad R' = \mathfrak{o}[x, z]/(xz - \varpi) , \quad R'' = \mathfrak{o}[x, t]/(x - \varpi t) .$$

Blowing up $\mathrm{Spec}(R)$ in the ideal (x, ϖ) gives a scheme which is covered by

$$V_0 = \mathrm{Spec}(R') \quad \text{and} \quad V_1 = \mathrm{Spec}(R'') ,$$

where these schemes are glued together via the relation $zt = 1$. For $f \in \mathfrak{o}[x] \setminus (x, \varpi)$ we consider the open affine subset $D_f = \mathrm{Spec}(R_f) \subset \mathrm{Spec}(R)$, which is an open neighborhood of P. The family of all D_f forms a fundamental system of open neighborhoods of P. The preimage $(\mathrm{pr}_{n,n-1})^{-1}\left(D_f\right)$ is then the same as the blow-up of D_f in P, i.e.,

$$(\mathrm{pr}_{n,n-1})^{-1}\left(D_f\right) = \mathrm{Spec}(R_f') \cup \mathrm{Spec}(R_f'') .$$

We now compute $H^1\left((\mathrm{pr}_{n,n-1})^{-1}\left(D_f\right), \mathcal{O}_{\mathbb{X}_n}\right)$ as Čech cohomology. What we then have to show is that the canonical map

$$R_f' \oplus R_f'' \longrightarrow R_f'\left[\frac{1}{z}\right] = R_f''\left[\frac{1}{t}\right] , \quad (f_1, f_2) \mapsto f_1 - f_2 ,$$

is surjective. An element in the group on the right hand side can be written as a finite sum $\sum_{j \in \mathbb{Z}} a_j t^j$ with $a_j \in R_f$. Then we have $f_1 \overset{\text{def}}{=} \sum_{j<0} a_j t^j = \sum_{j>0} a_{-j} z^j \in R_f'$ and $f_2 \overset{\text{def}}{=} -\sum_{j \geqslant 0} a_j t^j \in R_f''$. This shows the surjectivity of the map in question and the vanishing of $H^1\left((\mathrm{pr}_{n,n-1})^{-1}\left(D_f\right), \mathcal{O}_{\mathbb{X}_n}\right)$. This in turn means that the stalk of $R^1(\mathrm{pr}_{n,n-1})_*(\mathcal{O}_{\mathbb{X}_n})$ at P vanishes, and hence $R^1(\mathrm{pr}_{n,n-1})_*(\mathcal{O}_{\mathbb{X}_n}) = 0$. $\square$

Lemma 2.3.5. *There are $a_0, r_0 \in \mathbb{Z}_{>0}$ such that the line bundle*

$$(2.3.6) \qquad \mathcal{L}_n = \mathcal{O}_{\mathbb{X}_n/\mathbb{X}}(1) \otimes \mathrm{pr}^*\left(\mathcal{O}_{\mathbb{X}}(a_0)\right)$$

on $\mathbb{X}_n$ has the following properties:

(i) $\mathcal{L}_n$ is very ample over $\mathrm{Spec}(\mathfrak{o})$, and it is very ample over $\mathbb{X}$;

(ii) for all $r \geqslant r_0$, all $d \in \mathbb{Z}$, and all $i > 0$

$$R^i \mathrm{pr}_* \left(\mathcal{L}_n^{\otimes r} \otimes \mathrm{pr}^*(\mathcal{O}_{\mathbb{X}}(d)) \right) = 0 \; ;$$

(iii) for all $r \geqslant r_0$, all $d \geqslant 0$, and all $i > 0$

$$H^i \left(\mathbb{X}_n, \mathcal{L}_n^{\otimes r} \otimes \mathrm{pr}^*(\mathcal{O}_{\mathbb{X}}(d)) \right) = 0 \;.$$

Proof. (i) *Choosing a_0.* By [21, ch. II, ex. 7.14 (b)], the sheaf

$$(2.3.7) \qquad \mathcal{L}_n = \mathcal{O}_{\mathbb{X}_n/\mathbb{X}}(1) \otimes \mathrm{pr}^* \left(\mathcal{O}_{\mathbb{X}}(a_0) \right)$$

is very ample on $\mathbb{X}_n$ over $\mathrm{Spec}(\mathfrak{o})$ for suitable $a_0 > 0$. We fix such an a_0. By [19, 4.4.10 (v)] it is then also very ample over $\mathbb{X}$, and hence (i) is fulfilled.

(ii) *Finding r_0.* By [21, ch. III, 5.2, 8.8] there is $r_0 > 0$ such that for all $r \geqslant r_0$, and all $i > 0$, one has

$$(2.3.8) \qquad H^i \left(\mathbb{X}_n, \mathcal{L}_n^{\otimes r} \right) = H^i \left(\mathbb{X}_n, \mathcal{L}_n^{\otimes r} \otimes \mathrm{pr}^*(\mathcal{O}_{\mathbb{X}}(-1)) \right) = 0$$

and

$$(2.3.9) \qquad R^i \mathrm{pr}_* \left(\mathcal{L}_n^{\otimes r} \right) = 0 \;.$$

We fix an r_0 with the properties 2.3.8 and 2.3.9. By [21, ch. III, ex. 8.3] we have for all $i \geqslant 0$

$$(2.3.10) \qquad R^i \mathrm{pr}_* \left(\mathbb{X}_n, \mathcal{L}_n^{\otimes r} \otimes \mathrm{pr}^*(\mathcal{O}_{\mathbb{X}}(d)) \right) = R^i \mathrm{pr}_* \left(\mathcal{L}_n^{\otimes r} \right) \otimes \mathcal{O}_{\mathbb{X}}(d) \;.$$

Hence 2.3.9 and 2.3.10 together give (ii) when $i > 0$.

(iii) Consider the exact sequence

$$(2.3.11) \qquad 0 \to \mathcal{O}_{\mathbb{X}}(-1) \to \mathcal{O}_{\mathbb{X}}^{\oplus 2} \to \mathcal{O}_{\mathbb{X}}(1) \to 0 \;,$$

cf. [21, ch. II, example 8.20.1]. Tensoring with $\mathcal{O}_{\mathbb{X}}(d)$ gives

$$(2.3.12) \qquad 0 \to \mathcal{O}_{\mathbb{X}}(d-1) \to \left(\mathcal{O}_{\mathbb{X}}(d) \right)^{\oplus 2} \to \mathcal{O}_{\mathbb{X}}(d+1) \to 0 \;,$$

Because the sheaves in this sequence are locally free, taking the pull-back to $\mathbb{X}_n$ gives the exact sequence

$$(2.3.13) \qquad 0 \to \mathrm{pr}^* \Big(\mathcal{O}_\mathbb{X}(d-1) \Big) \to \Big(\mathrm{pr}^* (\mathcal{O}_\mathbb{X}(d)) \Big)^{\oplus 2} \to \mathrm{pr}^* \Big(\mathcal{O}_\mathbb{X}(d+1) \Big) \to 0 \ .$$

Tensoring with $\mathcal{L}_n^{\otimes r}$ gives the exact sequence

$$(2.3.14)$$
$$0 \to \mathcal{L}_n^{\otimes r} \otimes \mathrm{pr}^* \Big(\mathcal{O}_\mathbb{X}(d-1) \Big) \to \Big(\mathcal{L}_n^{\otimes r} \otimes \mathrm{pr}^* (\mathcal{O}_\mathbb{X}(d)) \Big)^{\oplus 2} \to \mathcal{L}_n^{\otimes r} \otimes \mathrm{pr}^* \Big(\mathcal{O}_\mathbb{X}(d+1) \Big) \to 0 \ .$$

The proof of (iii) is now by induction on d, starting with $d=0$ and using 2.3.8. $\qquad\square$

3. THE SHEAVES $\widetilde{\mathscr{D}}_{n,k}^{(m)}$ ON $\mathfrak{X}_n$

3.1. **The sheaves $\widetilde{\mathcal{T}}_{n,k}^{(m)}$.** Let $\mathfrak{g}_\mathfrak{o}$ denote the Lie algebra of the group scheme $\mathbb{G}(0) = \mathrm{GL}_{2,\mathfrak{o}}$ over $\mathfrak{o}$. For $k \geqslant n$ the scheme $\mathbb{X}_n$ is equipped with a right action of the group scheme $\mathbb{G}(k)$. This gives rise to a map

$$(3.1.1) \qquad\qquad \mathrm{Lie}(\mathbb{G}(k)) = \varpi^k \mathfrak{g}_\mathfrak{o} \to H^0(\mathbb{X}_n, \mathcal{T}_{\mathbb{X}_n^{\mathrm{log}}})$$

from the Lie algebra $\varpi^k \mathfrak{g}_\mathfrak{o}$ of $\mathbb{G}(k)$ to the global sections of the relative logarithmic tangent sheaf $\mathcal{T}_{\mathbb{X}_n^{\mathrm{log}}}$ of $\mathbb{X}_n$ [33]. Denote by $\widetilde{\mathcal{T}}_{n,k}$ the $\mathcal{O}_{\mathbb{X}_n}$-submodule of $\mathcal{T}_{\mathbb{X}_n^{\mathrm{log}}}$ generated by the image of 3.1.1. Denote by $\mathcal{T}_\mathbb{X}$ the relative tangent sheaf on $\mathbb{X} = \mathbb{X}_0$ over $\mathrm{Spec}(\mathfrak{o})$.

Lemma 3.1.2. *(i)* $\widetilde{\mathcal{T}}_{n,k} = \varpi^k (\mathrm{pr}_{n,0})^* (\mathcal{T}_\mathbb{X})$. *More generally, for $n' \leqslant n$ one has*

$$\widetilde{\mathcal{T}}_{n,k} = \varpi^{k-k'} (\mathrm{pr}_{n,n'})^* (\widetilde{\mathcal{T}}_{n',k'}) \ ,$$

where the latter is to be considered as a subsheaf of $\mathcal{T}_{\mathbb{X}_n^{\mathrm{log}}} \otimes_\mathfrak{o} L$.
(ii) $\widetilde{\mathcal{T}}_{n,k}$ *is locally free of rank one over $\mathcal{O}_{\mathbb{X}_n}$.*

Proof. (i) We note that $(\mathrm{pr}_{n,0})_* (\mathcal{T}_{\mathbb{X}_n^{\mathrm{log}}})$ is a subsheaf of $\mathcal{T}_\mathbb{X}$, and, in fact, we have inclusions

$$\varpi^n \mathcal{T}_\mathbb{X} \subset (\mathrm{pr}_{n,0})_* (\mathcal{T}_{\mathbb{X}_n^{\mathrm{log}}}) \subset \mathcal{T}_\mathbb{X} \ ,$$

where the inclusion on the left follows from [33, 5.2]. For $k \geqslant n$ there is a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Lie}(\mathbb{G}(k)) & = & \varpi^k \mathfrak{g}_\mathfrak{o} & \longrightarrow & H^0(\mathbb{X}_n, \mathcal{T}_{\mathbb{X}_n^{\log}}) \\
& & \downarrow & & \downarrow \\
\mathrm{Lie}(\mathbb{G}(0)) & = & \mathfrak{g}_\mathfrak{o} & \rightarrow & H^0(\mathbb{X}, \mathcal{T}_{\mathbb{X}})
\end{array}
$$

with injective vertical arrows. The assertion follows.

(ii) Immediate consequence of (i). $\hfill \square$

3.2. The sheaves $\widetilde{\mathcal{D}}_{n,k}^{(m)}$.

Let $\widetilde{\mathcal{D}}_{0,0}^{(m)}$ be the sheaf of level m differential operators on the scheme $\mathbb{X}_0$ which is smooth over $\mathfrak{o}$, cf. [5, 2.2.1]. This sheaf is filtered by the subsheaves $\widetilde{\mathcal{D}}_{0,0;d}^{(m)}$ of differential operators of order less or equal to d. We then let t be a variable and consider the Rees algebra $R_t^{(m)} = \bigoplus_{d \geqslant 0} \widetilde{\mathcal{D}}_{0,0;d}^{(m)} t^d$ which is a sheaf of rings on $\mathbb{X}_0$ with its obvious multiplication. For any $a \in \mathfrak{o}$, sending t to a gives a homomorphism of sheaves of $\mathfrak{o}$-algebras $R_t^{(m)} \to \widetilde{\mathcal{D}}_{0,0}^{(m)}$. Given $k \in \mathbb{Z}_{\geqslant 0}$ and setting $a = \varpi^k$, we denote the image of this homomorphism by $\widetilde{\mathcal{D}}_{0,k}^{(m)}$. In order to describe this and the sheaves to be introduced below explicitly, we fix an open affine covering $\mathbb{X}_0 = U_0 \cup U_1$ with $U_0 = \mathrm{Spec}(\mathfrak{o}[x])$, $U_1 = \mathrm{Spec}(\mathfrak{o}[y])$, and where $xy = 1$ on the intersection $U_0 \cap U_1 = \mathrm{Spec}(\mathfrak{o}[x, x^{-1}]) = \mathrm{Spec}(\mathfrak{o}[y, y^{-1}])$. Then, if ∂_x and ∂_y denote the derivatives with respect to x and y, respectively, then local sections of $\widetilde{\mathcal{D}}_{0,k}^{(m)}$ over $U \subset U_0$ are of the form $\sum_{d=0}^N a_d \frac{q_d^{(m)}!}{d!} (\varpi^k \partial_x)^d$ with $a_d \in \mathcal{O}_{\mathbb{X}_0}(U)$. Here, $q_d^{(m)}$ denotes the quotient of the euclidean division of d by p^m, cf. [5]. The same holds for local sections over $V \subset U_1$ with ∂_x replaced by ∂_y.

For any $n \in \mathbb{Z}_{\geqslant 0}$ and $k \in \mathbb{Z}_{\geqslant n}$ we put

$$
\widetilde{\mathcal{D}}_{n,k}^{(m)} = \mathrm{pr}_{n,0}^* \left(\widetilde{\mathcal{D}}_{0,k}^{(m)} \right) = \mathcal{O}_{\mathbb{X}_n} \otimes_{\mathrm{pr}_{n,0}^{-1}(\mathcal{O}_{\mathbb{X}_0})} \mathrm{pr}_{n,0}^{-1} \left(\widetilde{\mathcal{D}}_{0,k}^{(m)} \right) .
$$

While this is even defined – as an $\mathcal{O}_{\mathbb{X}_n}$-module – when $0 \leqslant k \leqslant n$, we claim that this sheaf is a subsheaf of *rings* of $\mathrm{pr}_{n,0}^* \left(\mathcal{D}_{\mathbb{X}_0} \otimes \mathbb{Q} \right)$ when $k \geqslant n$. To see this we use the formulas for local coordinates on $\mathbb{X}_n$ as given in [33, 4.3.5]. Arguing inductively and using the fact that the blow-up morphism $\mathrm{pr}_{n,n-1} : \mathbb{X}_n \to \mathbb{X}_{n-1}$ is an isomorphism over the open subset $\mathbb{X}_{n-1}^\circ$, cf. loc.cit., it suffices to consider a local coordinate $x_{\underline{a}}^{(n)}$ on one of the "outermost" components $\mathbb{D}_{\underline{a}}^{(n)}$. The special fiber of the latter scheme is an affine line over $\mathbb{F}_q$ all of whose $\mathbb{F}_q$-rational points have been removed except one, and this is the point where the coordinate $x_{\underline{a}}^{(n)}$ vanishes. Because of the action of the group $\mathrm{GL}_2(\mathfrak{o})$ we may assume that $x_{\underline{a}}^{(n)}$ is related to the coordinate x introduced above by the equation $x = \varpi^n x_{\underline{a}}^{(n)}$. (The prime p has to be replaced by ϖ in the formulas in [33, 4.3.5].) It then follows that

$$
\frac{q_d^{(m)}!}{d!} (\varpi^k \partial_x)^d . (x_{\underline{a}}^{(n)})^i = \begin{cases} q_d^{(m)}! \binom{i}{d} \varpi^{(k-n)d} (x_{\underline{a}}^{(n)})^{i-d} & , \quad d \leqslant i \\ 0 & , \quad d > i \end{cases}
$$

and this shows that $\mathcal{O}_{\mathbb{X}_n}$ is stable under all differential operators in $\mathrm{pr}_{n,0}^{-1}\left(\widetilde{\mathcal{D}}_{0,k}^{(m)}\right)$ when $k \geqslant n$. This in turn proves that $\widetilde{\mathcal{D}}_{n,k}^{(m)}$ is indeed a sheaf of rings when $k \geqslant n$.

The sheaf $\widetilde{\mathcal{D}}_{n,k}^{(m)}$ is filtered by the subsheaves $\widetilde{\mathcal{D}}_{n,k;d}^{(m)}$ of operators of order less than or equal to d, and we denote the corresponding graded sheaf $\bigoplus_d \widetilde{\mathcal{D}}_{n,k;d}^{(m)}/\widetilde{\mathcal{D}}_{n,k;d-1}^{(m)}$ by $\mathrm{gr}\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right)$.

Proposition 3.2.1. *(i) For $n' \leqslant n$ one has $(\mathrm{pr}_{n,n'})^*\left(\widetilde{\mathcal{D}}_{n',k}^{(m)}\right) = \widetilde{\mathcal{D}}_{n,k}^{(m)}$.*

(ii) For $n' \leqslant n$ one has $(\mathrm{pr}_{n,n'})_*\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right) = \widetilde{\mathcal{D}}_{n',k}^{(m)}$.

(iii) There is a canonical isomorphism of graded sheaves of algebras

$$\mathrm{gr}\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right) \simeq \mathrm{Sym}(\widetilde{\mathcal{T}}_{n,k})^{(m)} = \bigoplus_{d \geqslant 0} \frac{q_d^{(m)}!}{d!} \widetilde{\mathcal{T}}_{n,k}^{\otimes d} .$$

(iv) The canonical homomorphism of graded rings

$$\mathrm{gr}\left(H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})\right) \longrightarrow H^0\left(\mathbb{X}_n, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right)\right)$$

is an isomorphism.

(v) The rings $\mathrm{gr}\left(H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})\right)$ and $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$ are noetherian.

(vi) There is a basis of the topology $\mathfrak{U}$ of $\mathbb{X}_n$, consisting of open affine subsets, such that for any $U \in \mathfrak{U}$, the ring $H^0(U, \widetilde{\mathcal{D}}_{n,k}^{(m)})$ is noetherian.

Proof. (i) This follows directly from the definition of $\widetilde{\mathcal{D}}_{n,k}^{(m)}$.

(ii) The sheaves $\widetilde{\mathcal{D}}_{0,k;d}^{(m)}$ are locally free of finite rank, and so are the sheaves $\widetilde{\mathcal{D}}_{n',k;d}^{(m)}$, by (i). We can thus apply the projection formula, together with 2.3.4, and get

$$(\mathrm{pr}_{n,n'})_*\left(\widetilde{\mathcal{D}}_{n,k;d}^{(m)}\right) = \widetilde{\mathcal{D}}_{n',k;d}^{(m)} .$$

The assertion follows because the direct image commutes with inductive limits on a noetherian space.

(iii) We consider the exact sequence on $\mathbb{X}$

(3.2.2) $$0 \longrightarrow \widetilde{\mathcal{D}}_{0,k;d-1}^{(m)} \longrightarrow \widetilde{\mathcal{D}}_{0,k;d}^{(m)} \longrightarrow \mathrm{Sym}^d(\widetilde{\mathcal{T}}_{0,k})^{(m)} \longrightarrow 0 .$$

(Exactness is straightforward in this case.) Now we apply $(\mathrm{pr}_{n,0})^*$ and use (i).

(iv) We start again with the case $n = 0$ and note that

$$(3.2.3) \qquad \mathrm{Sym}^d(\widetilde{\mathcal{T}}_{0,k})^{(m)} = \frac{q_d^{(m)}!}{d!}\widetilde{\mathcal{T}}_{0,k}^{\otimes d} = \frac{q_d^{(m)}!}{d!}\varpi^{kd}\mathcal{T}_{\mathbb{X}}^{\otimes d} \simeq \mathcal{O}_{\mathbb{X}}(2d) \ .$$

Now we use the exact sequence 3.2.2 and induction on d to conclude that the sheaves $\widetilde{\mathcal{D}}_{0,k;d}^{(m)}$ have vanishing higher cohomology. Hence applying $H^0(-)$ to the sequence 3.2.2 gives an exact sequence and therefore the assertion when $n = 0$. Using point (ii) we reduce the case for arbitrary n to the case when $n = 0$.

(v) We note that $H^0\left(\mathbb{X}_n, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right)\right) = H^0\left(\mathbb{X}, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{0,k}^{(m)}\right)\right)$. Furthermore, by 3.2.3 we have $\mathrm{gr}\left(\widetilde{\mathcal{D}}_{0,k}^{(m)}\right) \simeq \mathrm{gr}\left(\widetilde{\mathcal{D}}_{0,0}^{(m)}\right)$. Because it is well-known that $H^0\left(\mathbb{X}, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{0,0}^{(m)}\right)\right)$ is a finitely generated $H^0\left(\mathbb{X}, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{0,0}^{(0)}\right)\right)$-algebra, and because the latter ring is known to be noetherian (isomorphic to $\mathfrak{o}[x, y, z]/(xz - y^2)$), the ring $H^0\left(\mathbb{X}, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{0,0}^{(m)}\right)\right)$ is noetherian, and so is $H^0\left(\mathbb{X}, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{0,k}^{(m)}\right)\right)$. Now we apply (iv) and can conclude that $\mathrm{gr}\left(H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})\right)$ is noetherian. But then $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$ is noetherian as well.

(vi) Take U small enough such that $\widetilde{\mathcal{T}}_{n,k}|_U$ is free over $\mathcal{O}_U$. Then $\mathrm{gr}\left(H^0(U, \widetilde{\mathcal{D}}_{n,k}^{(m)})\right) \simeq \mathrm{Sym}_{\mathcal{O}(U)}(\mathcal{O}(U)[T])^{(m)}$, which is known to be noetherian [23, Prop. 1.3.6]. It follows that $H^0(U, \widetilde{\mathcal{D}}_{n,k}^{(m)})$ is noetherian too. $\qquad\square$

3.2.4. *Twisting by $\mathcal{L}_n$.* Recall the very ample line bundle $\mathcal{L}_n$ from 2.3.5. In the following we will always use this line bundle to 'twist' $\mathcal{O}_{\mathbb{X}_n}$-modules. If $\mathcal{F}$ is a $\mathcal{O}_{\mathbb{X}_n}$-module and $r \in \mathbb{Z}$ we thus put $\mathcal{F}(r) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{X}_n}} \mathcal{L}_n^{\otimes r}$.

Some caveat is in order when we deal with sheaves which are equipped with both a left and a right $\mathcal{O}_{\mathbb{X}_n}$-module structure (which may not coincide). For instance, if $\mathcal{F}_d = \widetilde{\mathcal{D}}_{n,k;d}^{(m)}$ then we let

$$\mathcal{F}_d(r) = \widetilde{\mathcal{D}}_{n,k;d}^{(m)}(r) = \widetilde{\mathcal{D}}_{n,k;d}^{(m)} \otimes_{\mathcal{O}_{\mathbb{X}_n}} \mathcal{L}_n^{\otimes r} \ ,$$

where we consider $\mathcal{F}_d = \widetilde{\mathcal{D}}_{n,k;d}^{(m)}$ as a *right $\mathcal{O}_{\mathbb{X}_n}$-module*. Similarly we put $\widetilde{\mathcal{D}}_{n,k}^{(m)} \otimes_{\mathcal{O}_{\mathbb{X}_n}} \mathcal{L}_n^{\otimes r}$, where we consider $\widetilde{\mathcal{D}}_{n,k}^{(m)}$ as a *right $\mathcal{O}_{\mathbb{X}_n}$-module*. Then we have $\widetilde{\mathcal{D}}_{n,k}^{(m)}(r) = \varinjlim_d \mathcal{F}_d(r)$. When we consider the associated graded sheaf of $\widetilde{\mathcal{D}}_{n,k}^{(m)}(r)$, it is with respect to the filtration by the $\mathcal{F}_d(r)$. The sheaf $\widetilde{\mathcal{D}}_{n,k}^{(m)}(r)$ is a coherent left $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module.

Proposition 3.2.5. *(i)* $H^0\left(\mathbb{X}_n, \mathrm{Sym}\left(\mathrm{pr}_{n,0}^*(\varpi^k\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2})\right)^{(m)}\right)$ *is a finitely generated $\mathfrak{o}$-algebra, and in particular noetherian.*

(ii) For any $r \in \mathbb{Z}$ the $H^0\left(\mathbb{X}_n, \mathrm{Sym}\left(\mathrm{pr}_{n,0}^*(\varpi^k\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2})\right)^{(m)}\right)$-module

$$H^0\left(\mathbb{X}_n, \mathrm{Sym}\left(\mathrm{pr}_{n,0}^*(\varpi^k\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2})\right)^{(m)}(r)\right)$$

is finitely generated.

(iii) For any $r \in \mathbb{Z}$ the $H^0\left(\mathbb{X}_n, \mathrm{Sym}(\widetilde{\mathcal{T}}_{n,k})^{(m)}\right)$-module $H^0\left(\mathbb{X}_n, \mathrm{Sym}(\widetilde{\mathcal{T}}_{n,k})^{(m)}(r)\right)$ is finitely generated.

Proof. (i) We note first that by 3.2.1

$$H^0\left(\mathbb{X}_n, \mathrm{Sym}\left(\mathrm{pr}_{n,0}^*(\varpi^k\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2})\right)^{(m)}\right) = H^0\left(\mathbb{X}, \mathrm{Sym}\left(\varpi^k\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)}\right) \ .$$

We also note that

$$\mathrm{Sym}\left(\varpi^k\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)} \simeq \mathrm{Sym}\left(\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)} \ .$$

The remaining arguments are exactly as in the proof of [23, 4.2.1].

(ii) We first note that

$$H^0\left(\mathbb{X}_n, \mathrm{Sym}\left(\mathrm{pr}_{n,0}^*(\varpi^k\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2})\right)^{(m)}(r)\right)$$

is isomorphic to

$$H^0\left(\mathbb{X}, \mathrm{Sym}\left(\varpi^k\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)} \otimes (\mathrm{pr}_{n,0})_*\left(\mathcal{L}^{\otimes r}\right)\right)$$

which is in turn isomorphic to

$$H^0\left(\mathbb{X}, \mathrm{Sym}\left(\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)} \otimes (\mathrm{pr}_{n,0})_*\left(\mathcal{L}^{\otimes r}\right)\right)$$

It follows from 2.3.3 that

$$(\mathrm{pr}_{n,0})_*\left(\mathcal{L}^{\otimes r}\right) \subset \mathcal{O}_{\mathbb{X}}(r \cdot a_0) \ .$$

Therefore,

$$H^0\left(\mathbb{X}, \mathrm{Sym}\left(\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)} \otimes (\mathrm{pr}_{n,0})_*\left(\mathcal{L}^{\otimes r}\right)\right)$$

can be considered as a submodule of

$$H^0\left(\mathbb{X}, \mathrm{Sym}\left(\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)} \otimes \mathcal{O}_{\mathbb{X}}(r \cdot a_0)\right) .$$

By [23, 4.2.1], the latter is a finitely generated module over $H^0\left(\mathbb{X}, \mathrm{Sym}\left(\mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)}\right)$. As this ring is noetherian, cf. (i), the submodule considered before is finitely generated too.

(iii) We note first that by 3.2.1

$$H^0\left(\mathbb{X}_n, \mathrm{Sym}(\widetilde{\mathcal{T}}_{n,k})^{(m)}\right) = H^0\left(\mathbb{X}, \mathrm{Sym}(\widetilde{\mathcal{T}}_{0,k})^{(m)}\right)$$

and

$$H^0\left(\mathbb{X}_n, \mathrm{Sym}(\widetilde{\mathcal{T}}_{n,k})^{(m)}(r)\right) = H^0\left(\mathbb{X}, \mathrm{Sym}(\widetilde{\mathcal{T}}_{0,k})^{(m)} \otimes (\mathrm{pr}_{n,0})_*\left(\mathcal{L}^{\otimes r}\right)\right) .$$

We have obvious quotient maps coming from taking $d = 1$ in (2.3.12)

$$H^0\left(\mathbb{X}, \mathrm{Sym}\left(\varpi^k \mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)}\right) \twoheadrightarrow H^0\left(\mathbb{X}, \mathrm{Sym}(\widetilde{\mathcal{T}}_{0,k})^{(m)}\right)$$

and

$$H^0\left(\mathbb{X}, \mathrm{Sym}\left(\varpi^k \mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)} \otimes (\mathrm{pr}_{n,0})_*\left(\mathcal{L}^{\otimes r}\right)\right) \twoheadrightarrow H^0\left(\mathbb{X}, \mathrm{Sym}(\widetilde{\mathcal{T}}_{0,k})^{(m)} \otimes (\mathrm{pr}_{n,0})_*\left(\mathcal{L}^{\otimes r}\right)\right) ,$$

and the module structure of $H^0\left(\mathbb{X}, \mathrm{Sym}(\widetilde{\mathcal{T}}_{0,k})^{(m)} \otimes (\mathrm{pr}_{n,0})_*\left(\mathcal{L}^{\otimes r}\right)\right)$ over $H^0\left(\mathbb{X}, \mathrm{Sym}(\widetilde{\mathcal{T}}_{0,k})^{(m)}\right)$ comes from the module structure of

$$H^0\left(\mathbb{X}, \mathrm{Sym}\left(\varpi^k \mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)} \otimes (\mathrm{pr}_{n,0})_*\left(\mathcal{L}^{\otimes r}\right)\right)$$

over $H^0\left(\mathbb{X}, \mathrm{Sym}\left(\varpi^k \mathcal{O}_{\mathbb{X}}(1)^{\oplus 2}\right)^{(m)}\right)$ via these quotient maps. Assertion (iii) now follows from assertion (ii). $\qquad\square$

3.3. Global sections of $\widetilde{\mathcal{D}}_{n,k}^{(m)}$, $\widetilde{\mathscr{D}}_{n,k}^{(m)}$, and $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{\dagger}$.

3.3.1. *Divided power enveloping algebras.* We denote by $U_{\varpi^k}^{(m)}(\mathfrak{g}_{\mathfrak{o}})$ the $\mathfrak{o}$-subalgebra of $U(\mathfrak{g}) = U(\mathfrak{g}_{\mathfrak{o}}) \otimes_{\mathfrak{o}} L$ generated by the elements

$$(3.3.2) \qquad q_{\nu_1}^{(m)}! \frac{(\varpi^k e)^{\nu_1}}{\nu_1!} \cdot q_{\nu_2}^{(m)}! \varpi^{k\nu_2} \binom{h_1}{\nu_2} \cdot q_{\nu_3}^{(m)}! \varpi^{k\nu_3} \binom{h_2}{\nu_3} \cdot q_{\nu_4}^{(m)}! \frac{(\varpi^k f)^{\nu_4}}{\nu_4!} \ .$$

We considered the same algebra in [33, 2.7] where it is denoted by $U^{(m)}(p^k \mathfrak{g}_{\mathbb{Z}_p})$ (there we have $\varpi = p$). Because this latter notation is potentially misleading, we have changed it here. We denote by $\widehat{U}_{\varpi^k}^{(m)}(\mathfrak{g}_{\mathfrak{o}})$ the p-adic completion of $U_{\varpi^k}^{(m)}(\mathfrak{g}_{\mathfrak{o}})$. In the following the subscript θ_0 indicates the same central reduction as in [33, sec. 3].

Proposition 3.3.3. *There is a canonical injective homomorphism of $\mathfrak{o}$-algebras*

$$(3.3.4) \qquad \xi_{n,k}^{(m)} : U_{\varpi^k}^{(m)}(\mathfrak{g}_{\mathfrak{o}})_{\theta_0} \to H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}) \ ,$$

and $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$ is a finitely generated module over $U_{\varpi^k}^{(m)}(\mathfrak{g}_{\mathfrak{o}})_{\theta_0}$ via $\xi_{n,k}^{(m)}$. In particular, the ring $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$ is noetherian. Moreover, $\mathrm{coker}(\xi_{n,k}^{(m)})$ is annihilated by a finite power of p.

Proof. By 3.2.1 we have $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}) = H^0(\mathbb{X}_0, \widetilde{\mathcal{D}}_{0,k}^{(m)})$. The remaining arguments are as in [33, 3.7]. $\qquad\square$

Now let $\widetilde{\mathscr{D}}_{n,k}^{(m)}$ be the p-adic completion of $\widetilde{\mathcal{D}}_{n,k}^{(m)}$, which we will always consider as a sheaf on the formal scheme $\mathfrak{X}_n$. We also put $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m)} = \widetilde{\mathscr{D}}_{n,k}^{(m)} \otimes_{\mathbb{Z}} \mathbb{Q}$, and

$$(3.3.5) \qquad \widetilde{\mathscr{D}}_{n,k}^{\dagger} = \varinjlim_m \widetilde{\mathscr{D}}_{n,k}^{(m)} \ ,$$

and

$$(3.3.6) \qquad \widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{\dagger} = \widetilde{\mathscr{D}}_{n,k}^{\dagger} \otimes_{\mathbb{Z}} \mathbb{Q} = \varinjlim_m \widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m)} \ .$$

Recall the analytic distribution algebra $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ})$ of the rigid analytic group $\mathbb{G}(k)^{\circ}$ in the sense of Emerton, cf. [14, ch. 5]: the space $\mathbb{G}(k)^{\circ}$ admits a countable increasing admissible covering by affinoids and so the vector space of global sections $\mathcal{O}(\mathbb{G}(k)^{\circ})$ has a natural Fréchet topology. The continuous linear dual

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^{\circ}) := \mathrm{Hom}_L^{\mathrm{cont}}(\mathcal{O}(\mathbb{G}(k)^{\circ}), L),$$

equipped with the strong topology and the algebra structure coming from dualising the comultiplication in $\mathcal{O}(\mathbb{G}(k)^\circ)$, is then a topological L-algebra. For example, a group element $g \in \mathbb{G}(k)^\circ(L)$ gives rise to the linear form $f \mapsto f(g)$ in $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)$ as does any Lie algebra element $\mathfrak{x} \in \mathfrak{g}$ via $f \mapsto \frac{d}{dt} f(\exp(t\mathfrak{x}))|_{t=0}$. In fact, the latter map identifies the distribution algebra with the direct limit of noetherian rings

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ) = \varinjlim_m \widehat{U}^{(m)}_{\varpi^k}(\mathfrak{g}_\mathfrak{o})_{\mathbb{Q}}$$

with flat transition maps. The ring $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)$ is therefore coherent (cf. [14, Prop. 5.2.3/Cor. 5.3.12] where our $U^{(m)}_{\varpi^k}(\mathfrak{g}_\mathfrak{o})$ is denoted by $A^{(m)}$).

Proposition 3.3.7. *(i) There is a basis of the topology $\mathfrak{U}$ of $\mathbb{X}_n$, consisting of open affine subsets, such that for any $U \in \mathfrak{U}$, the ring $H^0(U, \widetilde{\mathscr{D}}^{(m)}_{n,k})$ is noetherian.*

(ii) The transition map $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}} \to \widetilde{\mathscr{D}}^{(m+1)}_{n,k,\mathbb{Q}}$ is flat.

(iii) The sheaf $\widetilde{\mathscr{D}}^\dagger_{n,k,\mathbb{Q}}$ is coherent.

(iv) The ring $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}})$ is canonically isomorphic to the noetherian L-algebra $\widehat{U}^{(m)}_{\varpi^k}(\mathfrak{g}_\mathfrak{o})_{\mathbb{Q},\theta_0}$.

(v) The ring $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^\dagger_{n,k,\mathbb{Q}})$ is canonically isomorphic to the coherent L-algebra $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(k)^\circ)_{\theta_0}$.

Proof. (i) Let U be open affine and such that $\widetilde{\mathcal{T}}_{n,k}|_U$ is free over $\mathcal{O}_U$. Then use that $H^0(U, \widetilde{\mathscr{D}}^{(m)}_{n,k})$ is the p-adic completion of $H^0(U, \widetilde{\mathcal{D}}^{(m)}_{n,k})$. (The proof is similar to the one given in 4.2.1.) Because the latter is noetherian, cf. 3.2.1, the former is noetherian as well.

(ii) Can be proved as in [5, sec. 3.5].

(iii) Follows from (i) and (ii).

(iv) Follows from 4.2.1 (iii) and 3.3.3. (Note that using 4.2.1 here does not introduce a circular argument.)

(v) Follows from (iv) and the discussion preceding the proposition. $\qquad\square$

4. Localization on $\mathfrak{X}_n$ via $\widetilde{\mathscr{D}}^\dagger_{n,k,\mathbb{Q}}$

The general line of arguments follows fairly closely [23]. The numbers $k \geqslant n \geqslant 0$ are fixed throughout this section, and $\mathrm{pr} = \mathrm{pr}_{n,0} : \mathbb{X}_n \to \mathbb{X} = \mathbb{X}_0$ is the blow-up morphism.

4.1. Cohomology of coherent $\widetilde{\mathcal{D}}^{(m)}_{n,k}$-modules.

Lemma 4.1.1. *Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_{\mathbb{X}_n}$-module on $\mathbb{X}_n$. Then, for all $i \geqslant 2$ one has $H^i(\mathbb{X}_n, \mathcal{E}) = 0$.*

Proof. Since $\mathcal{E}$ is the inductive limit of its coherent $\mathcal{O}_{\mathbb{X}_n}$-submodules, and because cohomology commutes with inductive limits on a noetherian space, we may assume that $\mathcal{E}$ is coherent. Let $f : \mathbb{X}_n \to \mathrm{Spec}(\mathfrak{o})$ be the structure morphism. The sheaves $R^j f_*(\mathcal{E})$ are coherent on $\mathrm{Spec}(\mathfrak{o})$, cf. [21, ch. III, 8.8], and have thus vanishing cohomology in positive degrees. But since the fibers of f are one-dimensional one has $R^j f_*(\mathcal{E}) = 0$ for $j \geqslant 2$, cf. [21, ch. III, 11.2]. We can now conclude by the Leray spectral sequence for f. $\qquad\square$

Proposition 4.1.2. *Let r_0 be as in 2.3.5. Then, for all $r \geqslant r_0$ and all $i \geqslant 1$ one has*

$$(4.1.3) \qquad\qquad H^i\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}(r)\right) = 0 \ .$$

Proof. Note that

$$(4.1.4) \qquad\qquad \mathrm{gr}_0\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(r)\right) = \mathcal{F}_0(r) = \mathcal{L}_n^{\otimes r} \ .$$

For $d > 0$ we consider the exact sequence

$$(4.1.5) \qquad\qquad 0 \to \mathcal{F}_{d-1} \to \mathcal{F}_d \to \mathrm{gr}_d\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right) \to 0$$

from which we deduce the exact sequence

$$(4.1.6) \qquad\qquad 0 \to \mathcal{F}_{d-1}(r) \to \mathcal{F}_d(r) \to \mathrm{gr}_d\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right)(r) \to 0$$

because tensoring with a line bundle is an exact functor. It thus follows that

$$\mathrm{gr}_d\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(r)\right) \simeq \frac{q_d^{(m)}!}{d!}\varpi^{dk}\mathrm{pr}^*(\mathcal{T}_{\mathbb{X}}) \otimes \mathcal{L}_n^{\otimes r} \simeq \mathrm{pr}^*\left(\mathcal{O}_{\mathbb{X}}(2d)\right) \otimes \mathcal{L}_n^{\otimes r} \ .$$

Hence by 2.3.5 we find that the higher cohomology of this sheaf vanishes for all $d \geqslant 0$. Using the sequence 4.1.6 we can then deduce by induction on d that for all $i > 0$

$$H^i(\mathbb{X}_n, \mathcal{F}_d(r)) = 0 \ .$$

Because cohomology commutes with inductive limits on a noetherian scheme we obtain the asserted vanishing result. $\qquad\square$

Proposition 4.1.7. *Let $\mathcal{E}$ be a coherent $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module.*

(i) There is $r_1(\mathcal{E}) \in \mathbb{Z}$ such that for all $r \geqslant r_1(\mathcal{E})$ there is $s \in \mathbb{Z}_{\geqslant 0}$ and an epimorphism of $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-modules

$$\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(-r)\right)^{\oplus s} \twoheadrightarrow \mathcal{E} \ .$$

(ii) There is $r_2(\mathcal{E}) \in \mathbb{Z}$ such that for all $r \geqslant r_2(\mathcal{E})$ and all $i > 0$

$$H^i\left(\mathbb{X}_n, \mathcal{E}(r)\right) = 0 \ .$$

Proof. (i) As $\mathbb{X}_n$ is a noetherian scheme, $\mathcal{E}$ is the inductive limit of its coherent subsheaves. There is thus a coherent $\mathcal{O}_{\mathbb{X}_n}$-submodule $\mathcal{F} \subset \mathcal{E}$ which generates $\mathcal{E}$ as a $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module, i.e., an epimorphism of sheaves $\widetilde{\mathcal{D}}_{n,k}^{(m)} \otimes_{\mathcal{O}_{\mathbb{X}_n}} \mathcal{F} \xrightarrow{\alpha} \mathcal{E}$, where $\widetilde{\mathcal{D}}_{n,k}^{(m)}$ is considered with its right $\mathcal{O}_{\mathbb{X}_n}$-module structure. Next, there is $r_1(\mathcal{E})$ such that for all $r \geqslant r_1(\mathcal{E})$ the sheaf $\mathcal{F}(r) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{X}_n}} \mathcal{L}_n^{\otimes r}$ is generated by its global sections. Hence there is $s > 0$ and an epimorphism $\mathcal{O}_{\mathbb{X}_n}^{\oplus s} \twoheadrightarrow \mathcal{F}(r)$, and thus an epimorphism of $\mathcal{O}_{\mathbb{X}_n}$-modules $(\mathcal{O}_{\mathbb{X}_n}(-r))^{\oplus s} \twoheadrightarrow \mathcal{F}$. From this morphism get an epimorphism of $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-modules

$$\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(-r)\right)^{\oplus s} = \widetilde{\mathcal{D}}_{n,k}^{(m)} \otimes_{\mathcal{O}_{\mathbb{X}_n}} (\mathcal{O}_{\mathbb{X}_n}(-r))^{\oplus s} \twoheadrightarrow \widetilde{\mathcal{D}}_{n,k}^{(m)} \otimes_{\mathcal{O}_{\mathbb{X}_n}} \mathcal{F} \xrightarrow{\alpha} \mathcal{E} \ .$$

(ii) We prove this by descending induction on i. When $i > 1$ there is nothing to show, cf. 4.1.1. We will now deduce it for $i = 1$. Using (i) we find an epimorphism of $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-modules

$$\beta : \left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right)^{\oplus s} \twoheadrightarrow \mathcal{E}\left(r_1(\mathcal{E})\right) \ .$$

By 3.2.1, the kernel $\mathcal{R} = \ker(\beta)$ is a coherent $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module and we have an exact sequence

$$0 \to \mathcal{R} \to \left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right)^{\oplus s} \to \mathcal{E}\left(r_1(\mathcal{E})\right) \to 0 \ ,$$

which gives for any $t \in \mathbb{Z}$ the exact sequence

$$(4.1.8) \qquad 0 \to \mathcal{R}(t) \to \left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(t)\right)^{\oplus s} \to \mathcal{E}\left(t + r_1(\mathcal{E})\right) \to 0 \ .$$

Now choose $r_2(\mathcal{E}) = \max\{r_0 + r_1(\mathcal{E}), r_2(\mathcal{R})\}$, where r_0 is as in 4.1.2. Then we can conclude from the long exact cohomology sequence associated to 4.1.8 that for $r \geqslant r_2(\mathcal{E})$

$$H^1(\mathbb{X}_n, \mathcal{E}(r)) = 0 \ .$$

$\square$

Proposition 4.1.9. *Fix $r \in \mathbb{Z}$.*

(i) There is $c_1 = c_1(r) \in \mathbb{Z}_{\geqslant 0}$ such that for all $i > 0$ the sheaf $R^i \mathrm{pr}_* \left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(r) \right)$ on $\mathbb{X}$ is annihilated by p^{c_1}. Furthermore, it is a skyscraper sheaf, and it is a coherent module over $\widetilde{\mathcal{D}}_{0,k}^{(m)}$.

(ii) There is $c_2 = c_2(r) \in \mathbb{Z}_{\geqslant 0}$ such that for all $i > 0$ the cohomology group $H^i(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}(r))$ is annihilated by p^{c_2}. Furthermore, it is a finitely generated module over $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$.

(iii) Let $\mathcal{E}$ be a coherent $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module. There is $c_3 = c_3(\mathcal{E}) \in \mathbb{Z}_{\geqslant 0}$ such that for all $i > 0$ the cohomology group $H^i(\mathbb{X}_n, \mathcal{E})$ is annihilated by p^{c_3}. Furthermore, it is a finitely generated module over $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$.

Proof. (i) The direct image functor commutes with inductive limits, and we will thus consider the sheaves

$$\widetilde{\mathcal{D}}_{n,k;d}^{(m)}(r) = \widetilde{\mathcal{D}}_{n,k;d}^{(m)} \otimes_{\mathcal{O}_{\mathbb{X}_n}} \mathcal{L}_n^{\otimes r} \ ,$$

where, as before, we consider $\widetilde{\mathcal{D}}_{n,k;d}^{(m)}$ as a *right* $\mathcal{O}_{\mathbb{X}_n}$-module. Because the fibers of pr are of dimension zero or one, we have

$$R^i \mathrm{pr}_* \left(\widetilde{\mathcal{D}}_{n,k;d}^{(m)}(r) \right) = 0$$

for $i > 1$, by [21, ch. III, 11.2]. Now let $i = 1$. Using the projection formula and 3.2.1 we deduce that

$$
\begin{aligned}
R^1 \mathrm{pr}_* \left(\widetilde{\mathcal{D}}_{n,k;d}^{(m)}(r) \right) &= \widetilde{\mathcal{D}}_{0,k;d}^{(m)} \otimes R^1 \mathrm{pr}_* \left(\mathcal{L}_n^{\otimes r} \right) \\
&= \widetilde{\mathcal{D}}_{0,k;d}^{(m)} \otimes \mathcal{O}_{\mathbb{X}}(r \cdot a_0) \otimes R^1 \mathrm{pr}_* \left(\mathcal{O}_{\mathbb{X}_n/\mathbb{X}}(1)^{\otimes r} \right) \ .
\end{aligned}
$$

(4.1.10)

By 2.3.2 and 2.3.4 we can conclude that $R^1 \mathrm{pr}_* \left(\mathcal{O}_{\mathbb{X}_n/\mathbb{X}}(1)^{\otimes r} \right)$ is a torsion sheaf, and because it is coherent, it is thus annihilated by a p^{c_1} for some $c_1 = c_1(r) \geqslant 0$. Furthermore, because pr is an isomorphism from $\mathrm{pr}^{-1}(\mathbb{X} \backslash \mathbb{X}(\mathbb{F}_q))$ to $\mathbb{X} \backslash \mathbb{X}(\mathbb{F}_q)$, the sheaf $R^1 \mathrm{pr}_* \left(\widetilde{\mathcal{D}}_{n,k;d}^{(m)}(r) \right)$ is supported on $\mathbb{X}(\mathbb{F}_q)$. It follows from 4.1.10 that

$$R^1 \mathrm{pr}_* \left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(r) \right) = \widetilde{\mathcal{D}}_{0,k}^{(m)} \otimes \mathcal{O}_{\mathbb{X}}(r \cdot a_0) \otimes R^1 \mathrm{pr}_* \left(\mathcal{O}_{\mathbb{X}_n/\mathbb{X}}(1)^{\otimes r} \right) \ ,$$

which shows that it is a coherent $\widetilde{\mathcal{D}}_{0,k}^{(m)}$-module.

(ii) By 4.1.1 there is nothing to show for $i > 1$. In order to compute H^1 we consider the exact sequence of low-degree terms for the Leray spectral sequence for the morphism pr and the sheaf $\widetilde{\mathcal{D}}^{(m)}_{n,k;d}(r)$ on $\mathbb{X}_n$:

$$
\begin{aligned}
0 \;\to\; & H^1\left(\mathbb{X}, \widetilde{\mathcal{D}}^{(m)}_{0,k;d} \otimes \mathrm{pr}_*\left(\mathcal{L}^{\otimes r}_n\right)\right) \;\to\; H^1\left(\mathbb{X}_n, \widetilde{\mathcal{D}}^{(m)}_{n,k;d}(r)\right) \\
& \to\; H^0\left(\mathbb{X}, \widetilde{\mathcal{D}}^{(m)}_{0,k;d} \otimes R^1\mathrm{pr}_*\left(\mathcal{L}^{\otimes r}_n\right)\right) \to 0 \,.
\end{aligned}
$$
(4.1.11)

In order to show the first assertion in (ii) we use (i) and are thus reduced to showing that

$$
H^1\left(\mathbb{X}, \widetilde{\mathcal{D}}^{(m)}_{0,k} \otimes \mathrm{pr}_*\left(\mathcal{L}^{\otimes r}_n\right)\right)
$$

is annihilated by a power of p. We will first show that it is a finitely generated $\mathfrak{o}$-module, and then use the fact that coherent $\widetilde{\mathcal{D}}^{(m)}_{0,k,\mathbb{Q}}$-modules have vanishing higher cohomology, by [3]. We put $\mathcal{F}_d = \widetilde{\mathcal{D}}^{(m)}_{0,k;d}$ and

$$
\mathcal{F}_{d,r} = \widetilde{\mathcal{D}}^{(m)}_{0,k;d} \otimes_{\mathcal{O}_{\mathbb{X}}} \mathrm{pr}_*\left(\mathcal{L}^{\otimes r}_n\right) \,,
$$

where, as before, we consider $\widetilde{\mathcal{D}}^{(m)}_{0,k;d}$ as a *right* $\mathcal{O}_{\mathbb{X}}$-module.[3] More generally, when $\mathcal{G}$ is a $\mathcal{O}_{\mathbb{X}}$-module, we put $\mathcal{G}_r = \mathcal{G} \otimes \mathrm{pr}_*\left(\mathcal{L}^{\otimes r}_n\right)$. We let $d, e \geqslant 0$ and consider the exact sequence

$$
0 \longrightarrow \mathcal{F}_d \longrightarrow \mathcal{F}_{d+e} \longrightarrow \mathcal{F}_{d+e}/\mathcal{F}_d \longrightarrow 0 \,.
$$
(4.1.12)

Because the sheaves in this sequence are locally free, tensoring with $\mathrm{pr}_*\left(\mathcal{L}^{\otimes r}_n\right)$ gives an exact sequence

$$
0 \longrightarrow \mathcal{F}_{d,r} \longrightarrow \mathcal{F}_{d+e,r} \longrightarrow \left(\mathcal{F}_{d+e}/\mathcal{F}_d\right)_r \longrightarrow 0 \,.
$$
(4.1.13)

and thus an exact sequence

$$
0 \longrightarrow \left(\mathcal{F}_{d+e}/\mathcal{F}_d\right)_r \longrightarrow \left(\mathcal{F}_{d+e+1}/\mathcal{F}_d\right)_r \longrightarrow \left(\mathcal{F}_{d+e+1}/\mathcal{F}_{d+e}\right)_r \longrightarrow 0 \,.
$$
(4.1.14)

It follows from 4.1.12, together with 3.2.1, that the sheaf on the right of 4.1.14 is isomorphic to

---

[3]We note that, with the convention introduced above, the sheaf $\mathcal{F}_d(r)$ is not the same as $\mathcal{F}_{d,r}$.

$$\frac{q_d^{(m)}!}{d!}\widetilde{\mathcal{T}}_{0,k}^{\otimes(d+e+1)}\otimes\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right) \;=\; \frac{q_d^{(m)}!}{d!}\varpi^{k(d+e+1)}\mathcal{T}_{\mathbb{X}}^{\otimes(d+e+1)}\otimes\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)$$

$$\simeq\; \mathcal{O}_{\mathbb{X}}(2(d+e+1))\otimes\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right).$$

Let d_0 be such that for all $d \geqslant d_0$ the sheaf $\mathcal{O}_{\mathbb{X}}(2(d+1))\otimes\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)$ is acyclic. Using this fact and 4.1.14, we see by induction on e that all sheaves $\mathcal{F}_{d_0+e,r}/\mathcal{F}_{d_0,r}$, where $e \geqslant 0$, are acyclic as well. The long exact cohomology sequence associated to 4.1.13 shows that, for all $e \geqslant 0$, the canonical map

(4.1.15) $$H^i(\mathbb{X},\mathcal{F}_{d_0,r}) \longrightarrow H^i(\mathbb{X},\mathcal{F}_{d_0+e,r})$$

is an isomorphism for $i \geqslant 2$ and surjective for $i = 1$. This proves that

$$H^i\left(\mathbb{X},\widetilde{\mathcal{D}}_{0,k}^{(m)}\otimes\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)\right)$$

is a finitely generated $\mathfrak{o}$-module, as cohomology commutes with inductive limits on a noetherian scheme, and $\mathcal{F}_{d_0}$ is coherent over $\mathcal{O}_{\mathbb{X}}$.

By 3.2.1 the sheaf $\widetilde{\mathcal{D}}_{0,k}^{(m)}\otimes\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)$ is a coherent $\widetilde{\mathcal{D}}_{0,k}^{(m)}$-module which is quasi-coherent as $\mathcal{O}_{\mathbb{X}}$-module. By [3], coherent $\widetilde{\mathcal{D}}_{0,k,\mathbb{Q}}^{(m)}$-modules which are quasi-coherent as $\mathcal{O}_{\mathbb{X},\mathbb{Q}}$-modules have vanishing higher cohomology. This implies that

$$H^1\left(\mathbb{X},\widetilde{\mathcal{D}}_{0,k}^{(m)}\otimes\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)\right)$$

is annihilated by a power of p. Analogous to 4.1.11 we have an exact sequence

$$0 \;\rightarrow\; H^1\left(\mathbb{X},\widetilde{\mathcal{D}}_{0,k}^{(m)}\otimes\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)\right) \;\rightarrow\; H^1\left(\mathbb{X}_n,\widetilde{\mathcal{D}}_{n,k}^{(m)}(r)\right)$$

$$\rightarrow\; H^0\left(\mathbb{X},\widetilde{\mathcal{D}}_{0,k}^{(m)}\otimes R^1\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)\right) \rightarrow 0.$$

As we have just seen, the cohomology group on the left is a finite group of p-power order. Now we consider the cohomology group on the right. Because $R^1\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)$ is a skyscraper sheaf, the canonical homomorphism

$$H^0\left(\mathbb{X},\widetilde{\mathcal{D}}_{0,k}^{(m)}\right)\otimes_{\mathfrak{o}} H^0\left(\mathbb{X},R^1\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)\right) \longrightarrow H^0\left(\mathbb{X},\widetilde{\mathcal{D}}_{0,k}^{(m)}\otimes R^1\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)\right),$$

is easily seen to be surjective. This shows that $H^0\left(\mathbb{X}, \widetilde{\mathcal{D}}_{0,k}^{(m)} \otimes R^1\mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right)\right)$ is a finitely generated module over $H^0(\mathbb{X}, \widetilde{\mathcal{D}}_{0,k}^{(m)})$, but the latter is equal to $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$, by 3.2.1. Therefore, $H^1\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}(r)\right)$ is also a finitely generated $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$-module.

(iii) The proof proceeds by descending induction on i. For $i \geqslant 2$ there is nothing to show by 4.1.1. Using 4.1.7 we obtain a surjection of $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-modules $\beta : \left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(-r)\right)^{\oplus s} \twoheadrightarrow \mathcal{E}$ for suitable s and r. The $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module $\mathcal{R} = \ker(\beta)$ is coherent too, by 3.2.1. Because the statement is true for $i = 2$ (for all coherent $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-modules, hence for $\mathcal{R}$), we get from the exact sequence

$$H^1\left(\mathbb{X}_n, \left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(-r)\right)^{\oplus s}\right) \to H^1(\mathbb{X}_n, \mathcal{E}) \to H^2(\mathbb{X}_n, \mathcal{R}) = 0 \,,$$

together with part (ii), that the assertion is true for $i = 1$. $\square$

Proposition 4.1.16. *Fix $r \in \mathbb{Z}$.*

(i) The canonical injective homomorphism

$$H^0\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k;d}^{(m)}(r)\right) \Big/ H^0\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k;d-1}^{(m)}(r)\right) \longrightarrow H^0\left(\mathbb{X}_n, \mathrm{gr}_d\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(r)\right)\right)$$

is an isomorphism for $d \gg 0$.

(ii) The canonical map of graded abelian groups

$$\mathrm{gr}\left(H^0\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}(r)\right)\right) \longrightarrow H^0\left(\mathbb{X}_n, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(r)\right)\right)$$

is injective and its cokernel is finitely generated over $\mathfrak{o}$.

Proof. The arguments are similar to those used in the proof of 4.1.9 (ii). As there, put

$$\mathcal{F}_{d,r} = \widetilde{\mathcal{D}}_{0,k;d}^{(m)} \otimes_{\mathcal{O}_\mathbb{X}} \mathrm{pr}_*\left(\mathcal{L}_n^{\otimes r}\right) \,.$$

Note that $H^0\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k;d}^{(m)}(r)\right) = H^0(\mathbb{X}, \mathcal{F}_{d,r})$, by 3.2.1. From 4.1.13 we see that there is a map of exact sequences

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{F}_{d-1,r} & \to & \mathcal{F}_{d,r} & \to & \left(\mathcal{F}_d/\mathcal{F}_{d-1}\right)_r & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F}_{d,r} & \to & \mathcal{F}_{d+1,r} & \to & \left(\mathcal{F}_{d+1}/\mathcal{F}_d\right)_r & \to & 0
\end{array}
$$

where the vertical arrow on the right is the zero map. From this we get a map of long exact sequences

$$(4.1.17) \quad 0 \longrightarrow H^0(\mathcal{F}_{d-1,r}) \longrightarrow H^0(\mathcal{F}_{d,r}) \longrightarrow H^0\Big((\mathcal{F}_d/\mathcal{F}_{d-1})_r\Big) \longrightarrow H^1(\mathcal{F}_{d-1,r})$$

$$\Big\downarrow \qquad\qquad \Big\downarrow \qquad\qquad \Big\downarrow \sigma_d \qquad\qquad \Big\downarrow \tau_d$$

$$0 \longrightarrow H^0(\mathcal{F}_{d,r}) \longrightarrow H^0(\mathcal{F}_{d+1,r}) \longrightarrow H^0\Big((\mathcal{F}_{d+1}/\mathcal{F}_d)_r\Big) \longrightarrow H^1(\mathcal{F}_{d,r})$$

(We write $H^i(-)$ for $H^i(\mathbb{X}, -)$ here.) There is d_0 such that for all $d \geqslant d_0$ one has

$$H^1\Big(\mathbb{X}, (\mathcal{F}_d/\mathcal{F}_{d-1})_r\Big) = 0 \ .$$

and the map $H^1(\mathcal{F}_{d-1,r}) \to H^1(\mathcal{F}_{d,r})$ is thus surjective for $d \geqslant d_0$. Hence $H^1(\mathcal{F}_{d_0-1,r}) \to H^1(\mathcal{F}_{d,r})$ is surjective for $d \geqslant d_0$. For $d \geqslant d_0$ consider the submodules

$$N'_d = \ker\Big(H^1(\mathcal{F}_{d_0-1,r}) \to H^1(\mathcal{F}_{d,r}) \Big)$$

and

$$N_d = \ker\Big(H^1(\mathcal{F}_{d,r}) \to H^1(\mathcal{F}_{d+1,r}) \Big) \ .$$

The submodules N'_d form an increasing sequence, and since $H^1\Big(\mathcal{F}_{d_0-1,r}\Big)$ is noetherian, this sequence must be stationary, i.e., there exists $d_1 \geqslant d_0$ such that $N'_d = N'_{d+1}$ for all $d \geqslant d_1$. But then $N_d = 0$ for all $d \geqslant d_1$, and the maps $H^1(\mathcal{F}_{d,r}) \to H^1(\mathcal{F}_{d+1,r})$ are isomorphisms for $d \geqslant d_1$. Now we consider the map τ_d in 4.1.17. For $d \geqslant d_1 + 1$ the map τ_d is an isomorphism, by what we have just seen. Since the map σ_d in 4.1.17 is the zero map, we conclude that the map $H^0\Big((\mathcal{F}_d/\mathcal{F}_{d-1})_r\Big) \to H^1(\mathcal{F}_{d,r})$ is the zero map for $d \geqslant d_1 + 1$, and the sequence

$$0 \longrightarrow H^0(\mathcal{F}_{d-1,r}) \longrightarrow H^0(\mathcal{F}_{d,r}) \longrightarrow H^0\Big((\mathcal{F}_d/\mathcal{F}_{d-1})_r\Big) \longrightarrow 0$$

is therefore exact. This proves (i), and (ii) follows immediately from (i). $\square$

Corollary 4.1.18. *Let $\mathcal{E}$ be a coherent $\widetilde{\mathcal{D}}^{(m)}_{n,k}$-module. Then $H^0(\mathbb{X}_n, \mathcal{E})$ is a finitely generated module over $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}^{(m)}_{n,k})$.*

Proof. Using 4.1.7 we get an exact sequence of coherent $\widetilde{\mathcal{D}}^{(m)}_{n,k}$-modules

$$0 \longrightarrow \mathcal{R} \longrightarrow \Big(\widetilde{\mathcal{D}}^{(m)}_{n,k}(-r)\Big)^{\oplus s} \longrightarrow \mathcal{E} \longrightarrow 0 \ .$$

passing to the long exact cohomology sequence gives

$$0 \to H^0(\mathbb{X}_n, \mathcal{R}) \to H^0\left(\mathbb{X}_n, \left(\widetilde{\mathcal{D}}_{n,k}^{(m)}(-r)\right)^{\oplus s}\right) \to H^0(\mathbb{X}_n, \mathcal{E}) \xrightarrow{\delta} H^1(\mathbb{X}_n, \mathcal{R}) .$$

By 4.1.9 the group on the right is a finitely generated $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$-module. Because $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$ is noetherian, cf. 3.2.1, it follows that $\mathrm{im}(\delta)$ is also finitely generated over this ring. We are therefore reduced to showing that $H^0\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}(-r)\right)$ is a finitely generated $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k})$-module. This is the case if

$$\mathrm{gr}\left(H^0\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}(-r)\right)\right)$$

is a finitely generated $\mathrm{gr}\left(H^0\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}\right)\right)$-module. We know that

$$H^0\left(\mathbb{X}_n, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right)(-r)\right)$$

is a finitely generated $H^0\left(\mathbb{X}_n, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right)\right)$-module, by 3.2.5 and 3.2.1. By 3.2.1 we have $H^0\left(\mathbb{X}_n, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right)\right) = \mathrm{gr}\left(H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})\right)$, and the injective canonical map

$$\mathrm{gr}\left(H^0\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}(-r)\right)\right) \longrightarrow H^0\left(\mathbb{X}_n, \mathrm{gr}\left(\widetilde{\mathcal{D}}_{n,k}^{(m)}\right)(-r)\right)$$

is a homomorphism of $\mathrm{gr}\left(H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})\right)$-modules. Because the latter ring is noetherian, we conclude that $\mathrm{gr}\left(H^0\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}(-r)\right)\right)$ is a finitely generated $\mathrm{gr}\left(H^0\left(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}\right)\right)$-module. $\square$

4.2. Cohomology of coherent $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m)}$-modules.

Proposition 4.2.1.[4] *Let $\mathcal{E}$ be a coherent $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module on $\mathbb{X}_n$ and $\widehat{\mathcal{E}} = \varprojlim_j \mathcal{E}/p^{j+1}\mathcal{E}$ its p-adic completion, which we consider as a sheaf on $\mathfrak{X}_n$.*

(i) For all $i \geqslant 0$ one has $H^i(\mathfrak{X}_n, \widehat{\mathcal{E}}) = \varprojlim_j H^i(\mathbb{X}_n, \mathcal{E}/p^{j+1}\mathcal{E})$.

(ii) For all $i > 0$ one has $H^i(\mathfrak{X}_n, \widehat{\mathcal{E}}) = H^i(\mathbb{X}_n, \mathcal{E})$.

(iii) $H^0(\mathfrak{X}_n, \widehat{\mathcal{E}}) = \varprojlim_j H^0(\mathbb{X}_n, \mathcal{E})/p^{j+1}H^0(\mathbb{X}_n, \mathcal{E})$.

Proof. (i) Put $\mathcal{E}_j = \mathcal{E}/p^{j+1}\mathcal{E}$. Let $\mathcal{E}_t$ be the subsheaf defined by $\mathcal{E}_t(U) = \mathcal{E}(U)_{\mathrm{tor}}$, where the right hand side denotes the group of torsion elements in $\mathcal{E}(U)$. This is indeed a sheaf

---

[4]Statement and proof are as in [23, 3.2].

(and not only a presheaf) because $\mathbb{X}_n$ is a noetherian space. Furthermore, $\mathcal{E}_t$ is a $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-submodule of $\mathcal{E}$. Because the sheaf $\widetilde{\mathcal{D}}_{n,k}^{(m)}$ has noetherian rings of sections over open affine subsets of $\mathbb{X}_n$, cf. 3.2.1, the submodule $\mathcal{E}_t$ is a coherent $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module. $\mathcal{E}_t$ is thus generated by a coherent $\mathcal{O}_{\mathbb{X}_n}$-submodule $\mathcal{F}$ of $\mathcal{E}_t$. The submodule $\mathcal{F}$ is annihilated by a fixed power p^c of p, and so is $\mathcal{E}_t$. Put $\mathcal{G} = \mathcal{E}/\mathcal{E}_t$, which is again a coherent $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module. Using 4.1.9, we can then assume, after possibly replacing c by a larger number, that

$$
\begin{aligned}
&(a) \quad p^c \mathcal{E}_t = 0 \ , \\
&(b) \quad \text{for all } i > 0 : p^c H^i(\mathbb{X}_n, \mathcal{E}) = 0 \ , \\
&(c) \quad \text{for all } i > 0 : p^c H^i(\mathbb{X}_n, \mathcal{G}) = 0 \ .
\end{aligned}
$$

Let j, ℓ be integers which are greater or equal to c. Denote by $\nu_\ell : \mathcal{G} \to \mathcal{E}$ the map induced by multiplication by $p^{\ell+1}$, so that we have an exact sequence

$$(4.2.2) \qquad\qquad 0 \longrightarrow \mathcal{G} \xrightarrow{\ \nu_\ell\ } \mathcal{E} \longrightarrow \mathcal{E}_\ell \longrightarrow 0 \ ,$$

where $\mathcal{E}_j = \mathcal{E}/p^{j+1}\mathcal{E}$. Consider the morphism of two such sequences

$$(4.2.3)$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G} & \xrightarrow{\ \nu_{\ell+j}\ } & \mathcal{E} & \longrightarrow & \mathcal{E}_{\ell+j} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle p^j} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G} & \xrightarrow{\ \nu_\ell\ } & \mathcal{E} & \longrightarrow & \mathcal{E}_\ell & \longrightarrow & 0
\end{array}
$$

We get hence a morphism of long exact sequences

$$(4.2.4)$$

$$
\begin{array}{ccccccc}
H^i(\mathbb{X}_n, \mathcal{G}) & \xrightarrow{H^i(\nu_{\ell+j})} & H^i(\mathbb{X}_n, \mathcal{E}) & \xrightarrow{\alpha_{\ell+j}} & H^i(\mathbb{X}_n, \mathcal{E}_{\ell+j}) & \longrightarrow & H^{i+1}(\mathbb{X}_n, \mathcal{G}) \\
\downarrow{\scriptstyle p^j} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \lambda_{\ell+j,\ell}} & & \downarrow{\scriptstyle p^j} \\
H^i(\mathbb{X}_n, \mathcal{G}) & \xrightarrow{H^i(\nu_\ell)} & H^i(\mathbb{X}_n, \mathcal{E}) & \xrightarrow{\alpha_\ell} & H^i(\mathbb{X}_n, \mathcal{E}_\ell) & \xrightarrow{\beta_\ell} & H^{i+1}(\mathbb{X}_n, \mathcal{G})
\end{array}
$$

Because we assume $j \geqslant c$ the vertical map on the right is the zero map, and hence $\beta_\ell \circ \lambda_{\ell+j,\ell} = 0$, which means that $\mathrm{im}(\lambda_{\ell+j,\ell})$ is contained in $\mathrm{im}(\alpha_\ell)$. Since $\lambda_{\ell+j,\ell} \circ \alpha_{\ell+j} = \alpha_\ell$, we find that $\mathrm{im}(\lambda_{\ell+j,\ell}) = \mathrm{im}(\alpha_\ell)$ for all $j \geqslant c$. Therefore, the projective system $(H^i(\mathbb{X}_n, \mathcal{E}_j))_j$, with the transition maps given by the $\lambda_{j',j}$ with $j' \geqslant j$, satisfies the Mittag-Leffler condition for any $i \geqslant 0$.

We now want to apply the proposition [20, ch. 0, 13.3.1]. The transition maps of the system $(\mathcal{E}_j)_j$ are obviously surjective, which gives the third condition of that proposition. Moreover, if $U \subset \mathbb{X}_n$ is an open affine subset, then we have $H^i(U, \mathcal{E}_j) = 0$ for $i > 0$, because

$\mathcal{E}_j$ is quasi-coherent over $\mathcal{O}_{\mathbb{X}_n}$. This implies the second condition of that proposition. It follows that the exact sequence

$$0 \longrightarrow \mathcal{E}_\ell \xrightarrow{p^j} \mathcal{E}_{\ell+j} \longrightarrow \mathcal{E}_j \longrightarrow 0$$

stays exact after taking sections over U. Hence the system $(H^0(U, \mathcal{E}_j))_j$ satisfies the Mittag-Leffler condition, which, together with the vanishing of higher cohomology on U observed before, means that the first condition of [20, ch. 0, 13.3.1] is fulfilled. Because we have proven above that the system $(H^i(\mathbb{X}_n, \mathcal{E}_j))_j$ satisfies the Mittag-Leffler condition for any $i \geqslant 0$, it now follows from [20, ch. 0, 13.3.1] that for all $i \geqslant 0$

$$H^i(\mathfrak{X}_n, \widehat{\mathcal{E}}) = \varprojlim_j H^i\left(\mathbb{X}_n, \mathcal{E}/p^{j+1}\mathcal{E}\right) .$$

Hence we have proved the first assertion.

(ii) For the second assertion we consider again the diagram 4.2.4 and notice that the maps $H^i(\nu_\ell)$ are zero maps for $i > 0$ and $\ell \geqslant c$, and thus α_ℓ is an isomorphism onto its image, for these i and ℓ. Therefore, the projective limit of the $(H^i(\mathbb{X}_n, \mathcal{E}_j))_j$ is equal to $H^i(\mathbb{X}_n, \mathcal{E})$ when $i > 0$.

(iii) As above, we let $\ell, j \geqslant c$ be integers, and we consider the exact sequence

$$0 \to \mathcal{E}_t \to \mathcal{E} \xrightarrow{p^{\ell+1}} \mathcal{E}_\ell \to 0 .$$

This sequence splits into two exact sequences

$$0 \to \mathcal{E}_t \to \mathcal{E} \xrightarrow{u} \mathcal{G} \to 0 \quad \text{and} \quad 0 \to \mathcal{G} \xrightarrow{\nu_\ell} \mathcal{E} \to \mathcal{E}_\ell \to 0 .$$

The long exact cohomology sequences to these sequences begin as follows:

$$0 \to H^0(\mathbb{X}_n, \mathcal{E}_t) \to H^0(\mathbb{X}_n, \mathcal{E}) \xrightarrow{u} H^0(\mathbb{X}_n, \mathcal{G}) \to H^1(\mathbb{X}_n, \mathcal{E}_t)$$

$$0 \to H^0(\mathbb{X}_n, \mathcal{G}) \xrightarrow{\nu_\ell} H^0(\mathbb{X}_n, \mathcal{E}) \to H^0(\mathbb{X}_n, \mathcal{E}_\ell) \to H^1(\mathbb{X}_n, \mathcal{G}) .$$

From the second of these exact sequences and the morphism of exact sequences 4.2.3 we obtain the following morphism of exact sequences

$$(4.2.5) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbb{X}_n, \mathcal{G}) & \xrightarrow{\nu_{\ell+j}} & H^0(\mathbb{X}_n, \mathcal{E}) & \xrightarrow{\alpha_{\ell+j}} & H^0(\mathbb{X}_n, \mathcal{E}_{\ell+j}) & \longrightarrow & H^1(\mathbb{X}_n, \mathcal{G}) \\ & & \downarrow{p^j} & & \downarrow{\mathrm{id}} & & \downarrow{\lambda_{\ell+j,\ell}} & & \downarrow{p^j} \\ 0 & \longrightarrow & H^0(\mathbb{X}_n, \mathcal{G}) & \xrightarrow{\nu_\ell} & H^0(\mathbb{X}_n, \mathcal{E}) & \xrightarrow{\alpha_\ell} & H^0(\mathbb{X}_n, \mathcal{E}_\ell) & \xrightarrow{\beta_\ell} & H^1(\mathbb{X}_n, \mathcal{G}) \end{array}$$

The composition $\nu_\ell \circ u$ is equal to the multiplication by $p^{\ell+1}$, and we get therefore a canonical surjection

$$\sigma_\ell : H^0(\mathbb{X}_n, \mathcal{E})/p^{\ell+1}H^0(\mathbb{X}_n, \mathcal{E}) \twoheadrightarrow H^0(\mathbb{X}_n, \mathcal{E})\Big/\nu_\ell\Big(H^0(\mathbb{X}_n, \mathcal{G})\Big) \ .$$

The homomorphisms σ_ℓ form a morphism of projective systems. Because ν_ℓ is injective, we have a canonical isomorphism

$$\begin{aligned}
\ker(\sigma_\ell) \ &= \ \nu_\ell\Big(H^0(\mathbb{X}_n, \mathcal{G})\Big)\Big/p^{\ell+1}H^0(\mathbb{X}_n, \mathcal{E}) \\[2mm]
&= \ \nu_\ell\Big(H^0(\mathbb{X}_n, \mathcal{G})\Big)\Big/\nu_\ell\Big(u\Big(H^0(\mathbb{X}_n, \mathcal{E})\Big)\Big) \\[2mm]
&\simeq \ H^0(\mathbb{X}_n, \mathcal{G})\Big/u\Big(H^0(\mathbb{X}_n, \mathcal{E})\Big) = \mathrm{coker}(H^0(u)) \ ,
\end{aligned}$$

and $\mathrm{coker}(H^0(u))$ embeds into $H^1(\mathbb{X}_n, \mathcal{G})$ which is annihilated by p^c. Furthermore, the morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_t & \longrightarrow & \mathcal{E} & \overset{p^{\ell+j+1}}{\longrightarrow} & \mathcal{E} & \longrightarrow & \mathcal{E}_{\ell+j} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p^j} & & \downarrow{\scriptstyle p^j} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}_t & \longrightarrow & \mathcal{E} & \overset{p^{\ell+1}}{\longrightarrow} & \mathcal{E} & \longrightarrow & \mathcal{E}_\ell & \longrightarrow & 0
\end{array}$$

induces a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 \to \mathrm{coker}(H^0(u)) & \rightarrowtail & H^0(\mathbb{X}_n, \mathcal{E})/p^{\ell+j+1}H^0(\mathbb{X}_n, \mathcal{E}) & \twoheadrightarrow & H^0(\mathbb{X}_n, \mathcal{E})\Big/\nu_{\ell+j}\Big(H^0(\mathbb{X}_n, \mathcal{G})\Big) & \to 0 \\
\downarrow{\scriptstyle p^j} & & \downarrow & & \downarrow & \\
0 \to \mathrm{coker}(H^0(u)) & \longrightarrow & H^0(\mathbb{X}_n, \mathcal{E})/p^{\ell+1}H^0(\mathbb{X}_n, \mathcal{E}) & \longrightarrow & H^0(\mathbb{X}_n, \mathcal{E})\Big/\nu_\ell\Big(H^0(\mathbb{X}_n, \mathcal{G})\Big) & \longrightarrow 0
\end{array}$$

The projective limit $\varprojlim_\ell \ker(\sigma_\ell)$ vanishes thus, and the system $(\sigma_\ell)_\ell$ induces an isomorphism

$$\varprojlim_\ell H^0(\mathbb{X}_n, \mathcal{E})/p^{\ell+1}H^0(\mathbb{X}_n, \mathcal{E}) \overset{\simeq}{\longrightarrow} \varprojlim_\ell H^0(\mathbb{X}_n, \mathcal{E})\Big/\nu_\ell\Big(H^0(\mathbb{X}_n, \mathcal{G})\Big)$$

and, by looking at 4.2.5, we see that the right hand side is in turn canonically isomorphic to $\varprojlim_\ell H^0(\mathbb{X}_n, \mathcal{E}_\ell) = H^0(\mathfrak{X}_n, \widehat{\mathcal{E}})$, by assertion (i). $\qquad\square$

Corollary 4.2.6. *The ring* $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{(m)}_{n,k})$ *is noetherian.*

Proof. This follows from (iii) of the preceding proposition together with 3.3.3.   □

Proposition 4.2.7. *Let $\mathscr{E}$ be a coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-module.*

(i) There is $r_1(\mathscr{E}) \in \mathbb{Z}$ such that for all $r \geqslant r_1(\mathscr{E})$ there is $s \in \mathbb{Z}_{\geqslant 0}$ and an epimorphism of $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-modules

$$\left(\widetilde{\mathscr{D}}_{n,k}^{(m)}(-r) \right)^{\oplus s} \twoheadrightarrow \mathscr{E} \ .$$

(ii) There is $r_2(\mathscr{E}) \in \mathbb{Z}$ such that for all $r \geqslant r_2(\mathscr{E})$ and all $i > 0$

$$H^i \left(\mathfrak{X}_n, \mathscr{E}(r) \right) = 0 \ .$$

Proof. (i) Because $\mathscr{E}$ is a coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-module, and because $H^0(U, \widetilde{\mathscr{D}}_{n,k}^{(m)})$ is a noetherian ring for all open affine subsets $U \subset \mathfrak{X}_n$, cf. 3.3.7, the torsion submodule $\mathscr{E}_t \subset \mathscr{E}$ is again a coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-module. As $\mathfrak{X}_n$ is quasi-compact, there is $c \in \mathbb{Z}_{\geqslant 0}$ such that $p^c \mathscr{E}_t = 0$. Put $\mathscr{G} = \mathscr{E}/\mathscr{E}_t$ and $\mathscr{G}_0 = \mathscr{G}/p\mathscr{G}$. For $j \geqslant c$ one has an exact sequence

$$0 \to \mathscr{G}_0 \xrightarrow{p^{j+1}} \mathscr{E}_{j+1} \to \mathscr{E}_j \to 0 \ .$$

We note that the sheaf $\mathscr{G}_0$ is a coherent module over $\widetilde{\mathscr{D}}_{n,k}^{(m)}/p\widetilde{\mathscr{D}}_{n,k}^{(m)}$. Because the canonical map of sheaves of rings on $\mathfrak{X}_n$

$$(4.2.8) \qquad\qquad \widetilde{\mathcal{D}}_{n,k}^{(m)}/p\widetilde{\mathcal{D}}_{n,k}^{(m)} \longrightarrow \widetilde{\mathscr{D}}_{n,k}^{(m)}/p\widetilde{\mathscr{D}}_{n,k}^{(m)}$$

(note that the source is a sheaf on $\mathbb{X}_n$ which is supported on the special fibre and thus considered a sheaf on $\mathfrak{X}_n$) is an isomorphism, $\mathscr{G}_0$ can be considered as a module over the sheaf on the left hand side of 4.2.8, and thus as a $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module. It is then automatically a coherent $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module. Hence we can apply 4.1.7 and deduce that there is $r_2(\mathscr{G}_0)$ such that for all $r \geqslant r_2(\mathscr{G}_0)$ one has $H^1(\mathfrak{X}_n, \mathscr{G}_0(r)) = 0$. The canonical maps

$$(4.2.9) \qquad\qquad H^0(\mathfrak{X}_n, \mathscr{E}_{j+1}(r)) \longrightarrow H^0(\mathfrak{X}_n, \mathscr{E}_j(r))$$

are thus surjective for $r \geqslant r_2(\mathscr{G}_0)$ and $j \geqslant c$. Similarly, $\mathscr{E}_c$ is a coherent module over $\widetilde{\mathcal{D}}_{n,k}^{(m)}/p^c\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module, in particular a coherent $\widetilde{\mathcal{D}}_{n,k}^{(m)}$-module. By 4.1.7 there is $r_1(\mathscr{E}_c)$ such that for every $r \geqslant r_1(\mathscr{E}_c)$ there is $s \in \mathbb{Z}_{\geqslant 0}$ and a surjection

$$\lambda : \left(\widetilde{\mathcal{D}}_{n,k}^{(m)}/p^c\widetilde{\mathcal{D}}_{n,k}^{(m)} \right)^{\oplus s} \twoheadrightarrow \mathscr{E}_c(r) \ .$$

Let $r_1(\mathscr{E}) = \max\{r_2(\mathscr{G}_0), r_1(\mathscr{E}_c)\}$, and assume from now on that $r \geqslant r_1(\mathscr{E})$. Let $e_1, \ldots, e_s$ be the standard basis of the domain of λ, and use 4.2.9 to lift each $\lambda(e_t)$, $1 \leqslant t \leqslant s$, to an element of

$$\varprojlim_j H^0(\mathfrak{X}_n, \mathscr{E}_j(r)) \simeq H^0(\mathfrak{X}_n, \widehat{\mathscr{E}(r)}) \;,$$

by 4.2.1 (i). But $\widehat{\mathscr{E}(r)} = \widehat{\mathscr{E}}(r)$, and $\widehat{\mathscr{e}} = \mathscr{E}$, as follows from [5, 3.2.3 (v)]. This defines a morphism $\left(\widetilde{\mathscr{D}}_{n,k}^{(m)}\right)^{\oplus s} \longrightarrow \mathscr{E}(r)$ which is surjective because, modulo p^c, it is a surjective morphism of sheaves coming from coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-modules by reduction modulo p^c, cf. [5, 3.2.2 (ii)].

(ii) We deduce from 4.1.2 and 4.2.1 that for all $i > 0$

$$H^i\left(\mathfrak{X}_n, \widetilde{\mathscr{D}}_{n,k}^{(m)}(r)\right) = 0 \;,$$

whenever $r \geqslant r_0$, where r_0 is as in 2.3.5. We prove the statement in (ii) by descending induction on i. When $i \geqslant 2$ there is nothing to show because $\mathfrak{X}_n$ is a one-dimensional noetherian space. We will now deduce it for $i = 1$. Using (i) we find an epimorphism of $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-modules

$$\beta : \left(\widetilde{\mathscr{D}}_{n,k}^{(m)}\right)^{\oplus s} \twoheadrightarrow \mathscr{E}\left(r_1(\mathscr{E})\right) \;.$$

The kernel $\mathscr{R} = \ker(\beta)$ is a coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-module and we have an exact sequence

$$0 \to \mathscr{R} \to \left(\widetilde{\mathscr{D}}_{n,k}^{(m)}\right)^{\oplus s} \to \mathscr{E}\left(r_1(\mathscr{E})\right) \to 0 \;,$$

which gives for any $t \in \mathbb{Z}$ the exact sequence

$$(4.2.10) \qquad 0 \to \mathscr{R}(t) \to \left(\widetilde{\mathscr{D}}_{n,k}^{(m)}(t)\right)^{\oplus s} \to \mathscr{E}\left(t + r_1(\mathscr{E})\right) \to 0 \;.$$

Now choose $r_2(\mathscr{E}) = \max\{r_0 + r_1(\mathscr{E}), r_2(\mathscr{R})\}$, where r_0 is as before. Then we can conclude from the long exact cohomology sequence associated to 4.2.10 that for $r \geqslant r_2(\mathscr{E})$

$$H^1(\mathfrak{X}_n, \mathscr{E}(r)) = 0 \;.$$

$\square$

Proposition 4.2.11. *Let $\mathscr{E}$ be a coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-module.*

(i) There is $c = c(\mathscr{E}) \in \mathbb{Z}_{\geqslant 0}$ such that for all $i > 0$ the cohomology group $H^i(\mathfrak{X}_n, \mathscr{E})$ is annihilated by p^c. Furthermore, it is a finitely generated module over $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}_{n,k}^{(m)})$.

(ii) $H^0(\mathfrak{X}_n, \mathscr{E}) = \varprojlim_j H^0(\mathfrak{X}_n, \mathscr{E})/p^j H^0(\mathfrak{X}_n, \mathscr{E})$.

Proof. (i) There is nothing to show for $i \geqslant 2$. We now deduce the case $i = 1$ from the case $i = 2$. Use 4.2.7 (i) to find a surjection of $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-modules

$$\beta : \left(\widetilde{\mathscr{D}}_{n,k}^{(m)}(-r) \right)^{\oplus s} \twoheadrightarrow \mathscr{E} \ .$$

Let $\mathscr{R} = \ker(\beta)$ which is again a coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-module. Applying the long exact cohomology sequence to the exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \left(\widetilde{\mathscr{D}}_{n,k}^{(m)}(-r) \right)^{\oplus s} \longrightarrow \mathscr{E} \longrightarrow 0$$

we find that $H^1\left(\mathfrak{X}_n, \widetilde{\mathscr{D}}_{n,k}^{(m)}(-r) \right)^{\oplus s}$ surjects onto $H^1(\mathfrak{X}_n, \mathscr{E})$. By 4.2.1 we have

$$H^1(\mathfrak{X}_n, \widetilde{\mathscr{D}}_{n,k}^{(m)}(-r)) = H^1(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}(-r)) \ ,$$

and this is annihilated by a finite power of p, by 4.1.9. Furthermore, $H^1(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)}(-r))$ is a finitely generated $H^0(\mathbb{X}_n, \widetilde{\mathcal{D}}_{n,k}^{(m)})$-module, by 4.1.9, and hence a finitely generated $H^0(\mathbb{X}_n, \widetilde{\mathscr{D}}_{n,k}^{(m)})$-module.

(ii) Let $\mathscr{E}_t \subset \mathscr{E}$ be the subsheaf of torsion elements and $\mathscr{G} = \mathscr{E}/\mathscr{E}_t$. Then the discussion in the beginning of the proof of 4.2.1 shows that there is $c \in \mathbb{Z}_{\geqslant 0}$ such that $p^c \mathscr{E}_t = 0$. Part (i) gives that $p^c H^1(\mathfrak{X}_n, \mathscr{E}) = p^c H^1(\mathfrak{X}_n, \mathscr{G}) = 0$, after possibly increasing c. Now we can apply the same reasoning as in the proof of 4.2.1 (iii) to conclude that assertion (ii) is true. $\qquad\square$

Proposition 4.2.12. *Let $\mathscr{E}$ be a coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-module. Then $H^0(\mathfrak{X}_n, \mathscr{E})$ is a finitely generated $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}_{n,k}^{(m)})$-module.*

Proof. By 4.2.7 there is an exact sequence of coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-modules

$$0 \longrightarrow \mathscr{R} \longrightarrow \left(\widetilde{\mathscr{D}}_{n,k}^{(m)}(-r) \right)^{\oplus s} \longrightarrow \mathscr{E} \longrightarrow 0 \ .$$

The long exact cohomology sequence begins as follows

$$0 \to H^0(\mathbb{X}_n, \mathscr{R}) \to H^0\left(\mathbb{X}_n, \left(\widetilde{\mathscr{D}}_{n,k}^{(m)}(-r)\right)^{\oplus s}\right) \to H^0(\mathbb{X}_n, \mathscr{E}) \to H^1(\mathbb{X}_n, \mathscr{R}) \ .$$

By 4.2.11 the group on the right is finitely generated over $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}_{n,k}^{(m)})$. Because $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}_{n,k}^{(m)})$ is noetherian, cf. 4.2.6,

$$\mathrm{im}\left(H^0(\mathbb{X}_n, \mathscr{E}) \to H^1(\mathbb{X}_n, \mathscr{R})\right)$$

is finitely generated over $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}_{n,k}^{(m)})$ as well. It follows from 4.1.18 and 4.2.1 that the group

$$H^0\left(\mathbb{X}_n, \left(\widetilde{\mathscr{D}}_{n,k}^{(m)}(-r)\right)^{\oplus s}\right)$$

is finitely generated over $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}_{n,k}^{(m)})$, and so is then $H^0(\mathbb{X}_n, \mathscr{E})$. $\square$

4.2.13. Let $\mathrm{Coh}(\widetilde{\mathscr{D}}_{n,k}^{(m)})$ (resp. $\mathrm{Coh}(\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m)})$) be the category of coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-modules (resp. $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m)}$-modules). Let $\mathrm{Coh}(\widetilde{\mathscr{D}}_{n,k}^{(m)})_{\mathbb{Q}}$ be the category of coherent $\widetilde{\mathscr{D}}_{n,k}^{(m)}$-modules up to isogeny. We recall that this means that $\mathrm{Coh}(\widetilde{\mathscr{D}}_{n,k}^{(m)})_{\mathbb{Q}}$ has the same class of objects as $\mathrm{Coh}(\widetilde{\mathscr{D}}_{n,k}^{(m)})$, and for any two objects $\mathcal{M}$ and $\mathcal{N}$ one has

$$\mathrm{Hom}_{\mathrm{Coh}(\widetilde{\mathscr{D}}_{n,k}^{(m)})_{\mathbb{Q}}}(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathrm{Coh}(\widetilde{\mathscr{D}}_{n,k}^{(m)})}(\mathcal{M}, \mathcal{N}) \otimes_{\mathbb{Z}} \mathbb{Q} \ .$$

Proposition 4.2.14. *(i) The functor $\mathcal{M} \mapsto \mathcal{M}_{\mathbb{Q}} = \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}$ induces an equivalence between $\mathrm{Coh}(\widetilde{\mathscr{D}}_{n,k}^{(m)})_{\mathbb{Q}}$ and $\mathrm{Coh}(\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m)})$.*

(ii) For every coherent $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{\dagger}$-module $\mathscr{M}$ there is $m \geqslant 0$ and a coherent $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m)}$-module $\mathscr{M}_m$ and an isomorphism of $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{\dagger}$-modules

$$\varepsilon : \widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{\dagger} \otimes_{\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m)}} \mathscr{M}_m \xrightarrow{\simeq} \mathscr{M} \ .$$

If $(m', \mathscr{M}_{m'}, \varepsilon')$ is another such triple, then there is $\ell \geqslant \max\{m, m'\}$ and an isomorphism of $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(\ell)}$-modules

$$\varepsilon_\ell : \widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(\ell)} \otimes_{\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m)}} \mathscr{M}_m \xrightarrow{\simeq} \widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(\ell)} \otimes_{\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m')}} \mathscr{M}_{m'}$$

such that $\varepsilon' \circ \left(\mathrm{id}_{\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{\dagger}} \otimes \varepsilon_\ell\right) = \varepsilon$.

Proof. (i) This is [5, 3.4.5]. Note that the sheaf $\widetilde{\mathscr{D}}^{(m)}_{n,k}$ satisfies the conditions in [5, 3.4.1], by 3.3.7. We point out that the formal scheme $\mathcal{X}$ in [5, sec. 3.4] is not supposed to be smooth over a discrete valuation ring, but only locally noetherian, cf. [5, sec. 3.3].

(ii) This is [5, 3.6.2]. In this reference the formal scheme is supposed to be noetherian and quasi-separated, but not necessarily smooth over a discrete valuation ring. $\square$

Corollary 4.2.15. *Let $\mathscr{E}$ be a coherent $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$-module. Then $H^0(\mathfrak{X}_n, \mathscr{E})$ is a finitely generated $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}})$-module.*

Proof. By 4.2.14 there is a coherent $\widetilde{\mathscr{D}}^{(m)}_{n,k}$-module $\mathscr{F}$ such that $\mathscr{F} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathscr{E}$. Now we can apply 4.2.12 to $\mathcal{F}$. $\square$

Theorem 4.2.16. *Let $\mathscr{E}$ be a coherent $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$-module (resp. $\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}$-module).*

(i) There is $r(\mathscr{E}) \in \mathbb{Z}$ such that for all $r \geqslant r(\mathscr{E})$ there is $s \in \mathbb{Z}_{\geqslant 0}$ and an epimorphism of $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$-modules (resp. $\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}$-modules)

$$\left(\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}(-r) \right)^{\oplus s} \twoheadrightarrow \mathscr{E} \qquad \left(\text{resp. } \left(\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}(-r) \right)^{\oplus s} \twoheadrightarrow \mathscr{E} \right) .$$

(ii) For all $i > 0$ one has $H^i(\mathfrak{X}_n, \mathscr{E}) = 0$.

Proof. (a) We first show both assertions (i) and (ii) for a coherent $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$-module $\mathscr{E}$. By 4.2.14 (i) there is a coherent $\widetilde{\mathscr{D}}^{(m)}_{n,k}$-module $\mathscr{F}$ such that $\mathscr{F} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathscr{E}$. We use 4.2.7 to find for every $r \geqslant r_1(\mathscr{F})$ a surjection

$$\left(\widetilde{\mathscr{D}}^{(m)}_{n,k}(-r) \right)^{\oplus s} \twoheadrightarrow \mathscr{F} ,$$

for some s (depending on r). Tensoring with $\mathbb{Q}$ gives then the desired surjection onto $\mathscr{E}$. Hence assertion (i). Furthermore, for $i > 0$

$$H^i(\mathfrak{X}_n, \mathscr{E}) = H^i(\mathfrak{X}_n, \mathscr{F}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0 ,$$

by 4.2.11, and this proves (ii).

(b) Now suppose $\mathscr{E}$ is a coherent $\widetilde{\mathcal{D}}^{\dagger}_{n,k,\mathbb{Q}}$-module. By 4.2.14 (ii) there is $m \geqslant 0$ and a coherent module $\mathscr{E}_m$ over $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$ and an isomorphism of $\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}$-modules

$$\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}} \otimes_{\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}} \mathscr{E}_m \xrightarrow{\simeq} \mathscr{E} .$$

Now use what we have just shown for $\mathscr{E}_m$ in (a) and get the sought for surjection after tensoring with $\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}$. This proves the first assertion. We have

$$\mathscr{E} = \widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}} \otimes_{\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}} \mathscr{E}_m = \varinjlim_{\ell \geqslant m} \widetilde{\mathscr{D}}^{(\ell)}_{n,k,\mathbb{Q}} \otimes_{\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}} \mathscr{E}_m = \varinjlim_{\ell \geqslant m} \mathscr{E}_\ell$$

where $\mathscr{E}_\ell = \widetilde{\mathscr{D}}^{(\ell)}_{n,k,\mathbb{Q}} \otimes_{\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}} \mathscr{E}_m$ is a coherent $\widetilde{\mathscr{D}}^{(\ell)}_{n,k,\mathbb{Q}}$-module. Then we have for $i > 0$

$$H^i(\mathfrak{X}_n, \mathscr{E}) = \varinjlim_{\ell \geqslant m} H^i(\mathfrak{X}_n, \mathscr{E}_\ell) = 0 \ ,$$

by part (a). And this proves assertion (ii). $\qquad\square$

4.3. $\mathfrak{X}_n$ is $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$-affine and $\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}$-affine.

Proposition 4.3.1. *(i) Let $\mathscr{E}$ be a coherent $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$-module. Then $\mathscr{E}$ is generated by its global sections as $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$-module. Furthermore, $\mathscr{E}$ has a resolution by finite free $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$-modules.*

(ii) Let $\mathscr{E}$ be a coherent $\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}$-module. Then $\mathscr{E}$ is generated by its global sections as $\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}$-module. $H^0(\mathfrak{X}_n, \mathscr{E})$ is a $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}})$-module of finite presentation. Furthermore, $\mathscr{E}$ has a resolution by finite free $\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}$-modules.

Proof. Using 4.2.16 we can argue exactly as in [23]. $\qquad\square$

4.3.2. *The functors $\mathscr{L}oc^{(m)}_{n,k}$ and $\mathscr{L}oc^{\dagger}_{n,k}$.* Let E be a finitely generated $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}})$-module (resp. a finitely presented $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}})$-module). Then we let $\mathscr{L}oc^{(m)}_{n,k}(E)$ (resp. $\mathscr{L}oc^{\dagger}_{n,k}(E)$) be the sheaf on $\mathbb{X}_n$ associated to the presheaf

$$U \mapsto \widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}(U) \otimes_{H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}})} E \quad \left(\text{resp. } U \mapsto \widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}(U) \otimes_{H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}})} E \right) .$$

It is obvious that $\mathscr{L}oc^{(m)}_{n,k}$ (resp. $\mathscr{L}oc^{\dagger}_{n,k}$) is a functor from the category of finitely generated $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}})$-modules (resp. finitely presented $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}})$-modules) to the category of sheaves of modules over $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$ (resp. $\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}$).

Theorem 4.3.3. *(i) The functors $\mathscr{L}oc^{(m)}_{n,k}$ and H^0 (resp. $\mathscr{L}oc^{\dagger}_{n,k}$ and H^0) are quasi-inverse equivalences between the categories of finitely generated $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}})$-modules and coherent $\widetilde{\mathscr{D}}^{(m)}_{n,k,\mathbb{Q}}$-modules (resp. finitely presented $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}})$-modules and coherent $\widetilde{\mathscr{D}}^{\dagger}_{n,k,\mathbb{Q}}$-modules).*

(ii) The functor $\mathscr{L}oc^{(m)}_{n,k}$ (resp. $\mathscr{L}oc^{\dagger}_{n,k}$) is an exact functor.

Proof. The proofs of [23, 5.2.1, 5.2.3] for the first and the second assertion, respectively, carry over word for word. $\qquad\square$

5. Localization of representations of $\mathrm{GL}_2(L)$

5.1. **Finitely generated modules over compact type algebras.** A space of *compact type* is an L-vector space $V = \varinjlim_{m \in \mathbb{N}} V_m$ equal to a locally convex inductive limit over a countable system of Banach spaces V_m where the transition maps $V_m \to V_{m+1}$ are injective compact linear maps [14, Def. 1.1.16]. Any space of compact type is Hausdorff. A *compact type topological algebra* (for short, a compact type algebra) is a topological L-algebra $A = \varinjlim_m A_m$ equal to a locally convex inductive limit of Banach algebras A_m with injective, continuous, compact linear and multiplicative transition maps $A_m \to A_{m+1}$ [15, Def. 2.3.1]. We have the following generalization of [38, Prop. 2.1]. All modules are *left* modules.

Proposition 5.1.1. *Let $A = \varinjlim_m A_m$ be a compact type algebra with noetherian Banach algebras A_m.*

(i) Each finitely generated A-module M carries a unique compact type topology (called its canonical topology) $M = \varinjlim_m M_m$ such that the A-module structure map $A \times M \to M$ becomes continuous; each M_m can be chosen to be a finitely generated A_m-submodule of M.

(ii) every finitely generated A-submodule of M is closed in the canonical topology;

(iii) any linear map between finitely generated A-modules is continuous and strict with closed image for the canonical topologies;

(iv) if M admits a finite presentation $A^{\oplus s} \to A^{\oplus r} \xrightarrow{f} M \to 0$, then the quotient topology on $M \simeq A^{\oplus r}/\ker(f)$ coincides with the canonical topology.

Proof. (i) Let $x_1, ..., x_r$ be generators for M and put $M_m := \sum_i A_m x_i \subset M$. According to [38, Prop. 2.1] the A_m-module M_m has a unique structure as Banach module and the natural injection $M_m \to M_{m+1}$ is continuous. Moreover, the latter injection is a compact linear map as follows from the compactness of $A_m \to A_{m+1}$ along the lines of the proof of [38, Lem. 6.1]. Hence, the inductive limit topology on $M = \varinjlim_m M_m$ is as claimed and it remains to show uniqueness. However, if M has a second such topology, say τ, then each inclusion $M_m \hookrightarrow M^\tau$ is continuous. The identity map $M^{can} \to M^\tau$ is therefore a continuous bijection. By the open mapping theorem, in the version of [14, Thm. 1.1.17], it is therefore strict. Thus, it is a topological isomorphism. This shows (i).

(ii) If $N \subseteq M$ is any submodule of M, then $N_m := M_m \cap N$ is a submodule of the A_m-module M_m and hence closed by [38, Prop.2.1]. The inductive limit topology on $N' = \varinjlim_m N_m$ makes the natural map $N' \to M$ a closed embedding [14, Prop. 1.1.41]. However, if N is finitely generated, then $N = N'$ which proves (ii).

(iii) Given a linear map $f : M \to N$ between finitely generated modules, we have $f(M_m) \subseteq N_m$ for all large enough m. For such m, the induced map $f_m : M_n \to N_m$ is continuous,

cf. [38, Prop. 2.1], and hence, so is $f = \varinjlim_m f_m$. The image $\mathrm{im}(f) \subseteq N$ is closed by (ii). Moreover, $f : M \to \mathrm{im}(f)$ is a continuous surjection and therefore strict by [14, Thm. 1.1.7]. This proves (iii).

(iv) According to (ii) the submodule $\ker(f) \subseteq A^{\oplus r}$ is closed and the quotient topology on $M \simeq A^{\oplus r}/\ker(f)$ is therefore of compact type, cf. [37, Prop. 1.2]. Since it makes the A-module structure map $A \times M \to M$ continuous, it must coincide with the canonical topology by the uniqueness part in (i). $\qquad\square$

Remark 5.1.2. Suppose there is a central ideal $I_m \subset A_m$ for each m with $I_{m+1} = A_{m+1}I_m$ and $I_{m+1} \cap A_m = I_m$. This yields a central ideal $I \subset A$ with $A/I = \varinjlim_m A_m/I_m$. The algebra A/I is then a compact type algebra.

We will need the following lemma which justifies the replacement of completed tensor products which occur in certain situations when working with weak Fréchet-Stein structures, cf. [14, Def. 1.2.6], by ordinary tensor products. The key point here is that we consider finitely presented modules over compact type algebras $A = \varinjlim_m A_m$ with noetherian defining Banach algebras A_m. Recall that, if A is a locally convex topological algebra and M is a locally convex topological module, then M is said to be a *finitely generated topological module*, if there is an A-module map $A^{\oplus r} \to M$, which is a continuous and strict surjection [14, Def. 1.2.1]. In this situation, the module is called *a finitely generated free topological module* if the map $A^{\oplus r} \to M$ is additionally injective, i.e. a topological isomorphism. Finally, given a continuous homomorphism between locally convex topological L-algebras $A \to B$ and a locally convex topological A-module M, the completed tensor product $B \hat{\otimes}_A M$ will always be understood as in [14, Lem. 1.2.3], i.e. by regarding $B \otimes_A M$ as a quotient of $B \otimes_{L,\pi} M$[5], giving it the quotient topology and then forming the Hausdorff completion.

Lemma 5.1.3. *Let A' and B' be compact type algebras with noetherian defining Banach algebras. Let further $A' \subseteq A$ (resp. $B' \subseteq B$) be a topological algebra which is a finitely generated free topological module over A' (resp. B'). Let $A \to B$ be a continuous homomorphism. Suppose M is a finitely presented (abstract) A-module. Giving M its canonical topology as finitely generated A'-module, the natural map $B \otimes_A M \xrightarrow{\simeq} B \hat{\otimes}_A M$ is bijective.*

Proof. We choose a finite presentation of the A-module M

$$(\mathcal{R}) \qquad A^{\oplus s} \to A^{\oplus r} \xrightarrow{f} M \to 0 \ .$$

This is a finite presentation of M as A'-module, so that the map f is in fact strict by 5.1.1. This shows that M is a finitely generated topological A-module. According to [14, Prop. 1.2.5], the natural map $B \otimes_A M \to B \hat{\otimes}_A M$ is surjective and the map $B \otimes f$ occurring in

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[5]The symbol $\otimes_{L,\pi}$ refers to the projective tensor product, cf. [14, (0.9)].

$$(B \otimes \mathcal{R}) \qquad B^{\oplus s} \to B^{\oplus r} \overset{B \otimes f}{\to} B \otimes_A M \to 0$$

is strict. It remains to see that $B \otimes_A M$ is Hausdorff. But $B \otimes_A M$ is a finitely presented B-module and therefore has its canonical topology as finitely generated B'-module. Since $B \otimes \mathcal{R}$ is a finite presentation of $B \otimes_A M$ as B'-module, the map $B \otimes f$ is a strict surjection for the canonical topology, again by 5.1.1. Hence the original topology on $B \otimes_A M$ coincides with the canonical topology and hence, is Hausdorff. $\square$

5.2. **Modules over** $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)_{\theta_0}$**.** We will apply these considerations to the algebra of global sections $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^\dagger_{n,n,\mathbb{Q}})$. Recall that it is canonically isomorphic to the coherent L-algebra $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)_{\theta_0}$, cf. Prop. 3.3.7.

Lemma 5.2.1. *The algebras $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$ and $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)_{\theta_0}$ are compact type algebras with noetherian defining Banach algebras.*

Proof. The discussion in [27, 5.3] applied to the smooth and affine $\mathfrak{o}$-group scheme $\mathbb{G}(n)$ shows the property for $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$. According to the remark after Prop. 5.1.1 the property then passes to the central reduction $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)_{\theta_0}$. $\square$

We consider the locally L-analytic compact group $G_0 = \mathrm{GL}_2(\mathfrak{o})$ with its series of congruence subgroups $G_{n+1} = \mathbb{G}(n)^\circ(L)$. The group G_0 acts by translations on the space $C^{\mathrm{cts}}(G_0, L)$ of continuous L-valued functions. Following [14, (5.3)] let $D(\mathbb{G}(n)^\circ, G_0)$ be the strong dual of the space of $\mathbb{G}(n)^\circ$-analytic vectors

$$D(\mathbb{G}(n)^\circ, G_0) := (C^{\mathrm{cts}}(G_0, L)_{\mathbb{G}(n)^\circ - \mathrm{an}})'_b \, .$$

It is a locally convex topological L-algebra naturally isomorphic to the crossed product of the ring $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$ with the finite group G_0/G_{n+1}. In particular,

$$(5.2.2) \qquad D(\mathbb{G}(n)^\circ, G_0) = \oplus_{g \in G_0/G_{n+1}} \mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ) * \delta_g$$

is a finitely generated free topological module over $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$. Denoting by $C^{\mathrm{la}}(G_0, L)$ the space of L-valued locally analytic functions and dualizing the isomorphism

$$\varinjlim_n C^{\mathrm{cts}}(G_0, L)_{\mathbb{G}(n)^\circ - \mathrm{an}} \overset{\simeq}{\longrightarrow} C^{\mathrm{la}}(G_0, L)$$

yields an isomorphism of topological algebras

$$D(G_0) \overset{\simeq}{\longrightarrow} \varprojlim_n D(\mathbb{G}(n)^\circ, G_0) \, .$$

This is the weak Fréchet-Stein structure on the locally analytic distribution algebra $D(G_0) = D(G_0, L)$ as introduced by Emerton in [14, Prop. 5.3.1]. In an obviously similar manner, we may construct the ring $D(\mathbb{G}(n)^\circ, G_0)_{\theta_0}$ and obtain an isomorphism $D(G_0)_{\theta_0} \xrightarrow{\cong} \varprojlim_n D(\mathbb{G}(n)^\circ, G_0)_{\theta_0}$.

We consider an admissible locally analytic G_0-representation V, its coadmissible module $M := V_b'$ and its subspace of $\mathbb{G}(n)^\circ$-analytic vectors $V_{\mathbb{G}(n)^\circ - \mathrm{an}} \subseteq V$. The latter subspace is naturally a nuclear Fréchet space [14, Lem. 6.1.6] and we let $(V_{\mathbb{G}(n)^\circ - \mathrm{an}})_b'$ be its strong dual. It is a space of compact type and a topological $D(\mathbb{G}(n)^\circ, G_0)$-module which is finitely generated [14, Lem. 6.1.13]. According to [14, Thm. 6.1.20] the modules $M_n := (V_{\mathbb{G}(n)^\circ - \mathrm{an}})'$ form a $(D(\mathbb{G}(n)^\circ, G_0))_{n \in \mathbb{N}}$-sequence, in the sense of [14, Def. 1.3.8], for the coadmissible module M relative to the weak Fréchet-Stein structure on $D(G_0)$. This implies that one has

$$(5.2.3) \qquad\qquad M_n = D(\mathbb{G}(n)^\circ, G_0) \hat{\otimes}_{D(G_0)} M$$

as $D(\mathbb{G}(n)^\circ, G_0)$-modules for any n. Furthermore, there are natural isomorphisms

$$D(\mathbb{G}(n-1)^\circ, G_0) \hat{\otimes}_{D(\mathbb{G}(n)^\circ, G_0)} M_n \xrightarrow{\cong} M_{n-1} \ ,$$

of $D(\mathbb{G}(n-1)^\circ, G_0)$-modules for any n. Here, the completed tensor product are understood in the sense of [14, Lem. 1.2.3] as we have explained above. We have the following slight refinements of these results.

Lemma 5.2.4. *(i) The $D(\mathbb{G}(n)^\circ, G_0)$-module M_n is finitely presented.*

(ii) The natural map

$$D(\mathbb{G}(n-1)^\circ, G_0) \otimes_{D(\mathbb{G}(n)^\circ, G_0)} M_n \xrightarrow{\cong} D(\mathbb{G}(n-1)^\circ, G_0) \hat{\otimes}_{D(\mathbb{G}(n)^\circ, G_0)} M_n$$

is bijective.

Proof. (i) Let $(M_n')_{n \in \mathbb{N}}$ be any $(D(\mathbb{G}(n)^\circ, G_0))_{n \in \mathbb{N}}$-sequence for the Fréchet-Stein structure of $D(G_0)$ exhibited in [14, Cor. 5.3.19]. Fix a number n. The space M_n' is a finitely generated module over the noetherian Banach algebra A equal to the crossed product of $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)^{(m_n)}$, the level-$m_n$ part of $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$, with the finite group G_0/G_{n+1}. Here, m_n is a sequence of natural numbers with $m_n \to \infty$ for $n \to \infty$. In particular,

$$(5.2.5) \qquad\qquad M_n' = A \otimes_{D(G_0)} M = A \hat{\otimes}_{D(G_0)} M.$$

The algebra A is naturally contained in $B := D(\mathbb{G}(n)^\circ, G_0)$ as shown in equation (5.3.20) of loc.cit. By 5.2.3 and 5.2.5 we have $M_n = B \hat{\otimes}_A M_n'$. However, M_n' is a finitely presented A-module and the topological algebras A and B satisfy the assumptions of Lem. 5.1.3:

take $A' = A$ and $B' = \mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$, cf. 5.2.2. Hence, the natural map $B \otimes_A M'_n \xrightarrow{\simeq} B \hat{\otimes}_A M'_n$ is bijective. This proves (i).

(ii) The $D(\mathbb{G}(n)^\circ, G_0)$-module M_n is finitely presented according to (i). As we have indicated above, M_n is a space of compact type and a topological $D(\mathbb{G}(n)^\circ, G_0)$-module. By the uniqueness part in 5.1.1 its topology therefore coincides with the canonical topology as finitely generated $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$-module. The assertion follows therefore from Lem. 5.1.3. $\qquad\square$

Remark: These results have obvious analogues when the character θ_0 is involved.

5.3. **The functor** $\mathscr{L}oc^\dagger$. For every $n \geqslant 0$ we put $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}} := \widetilde{\mathscr{D}}^\dagger_{n,n,\mathbb{Q}}$, and we denote the functor $\mathscr{L}oc^\dagger_{n,n}$ by $\mathscr{L}oc^\dagger_n$ in the following. The group $G_0 = \mathrm{GL}_2(\mathfrak{o})$ acts on the formal scheme $\mathfrak{X}_n$, and there is an induced (left) action of G_0 on the sheaf $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}$. Given $g \in G_0$ and a local section s of $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}$, there is thus a local section $g.s$ of $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}$. We can then consider the abelian category $\mathrm{Coh}(\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}, G_0)$ of (left) G_0-equivariant coherent $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}$-modules. Furthermore, the group G_{n+1} is contained in $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$ as a set of delta distributions, and for $h \in G_{n+1}$ we write δ_h for its image in $H^0(\mathfrak{X}_n, \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}) = \mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)_{\theta_0}$. For $g \in G_0$, $h \in G_{n+1}$, we have $g.\delta_h = \delta_{ghg^{-1}}$, and for a local section s of $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}$ we have then the relation

$$(5.3.1) \qquad\qquad g.(s\delta_h) = (g.s)(g.\delta_h) = (g.s)\delta_{ghg^{-1}} \ .$$

We denote by $\mathrm{pr}_n := \mathrm{pr}_{n,n-1}$ the blow-up morphism $\mathfrak{X}_n \to \mathfrak{X}_{n-1}$. There is a natural morphism of sheaves of rings

$$(5.3.2) \qquad\qquad (\mathrm{pr}_n)_* \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}} \to \widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}}$$

which is G_0-equivariant. Given a coherent $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}$-module $\mathscr{M}_n$ the $\widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}}$-module

$$\widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}} \otimes_{(\mathrm{pr}_n)_* \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}} (\mathrm{pr}_n)_* \mathscr{M}_n$$

is G_0-equivariant via $g.(s\otimes m) = (g.s)\otimes(g.m)$ for local sections s, m and $g \in G_0$. Consider its submodule $\mathcal{R}_{n-1}$ locally generated by all elements $s\delta_h \otimes m - s \otimes (h.m)$ for $h \in G_n$. Because of 5.3.1 the submodule $\mathcal{R}_{n-1}$ is G_0-stable. We put

$$\widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}} \otimes_{(\mathrm{pr}_n)_* \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}, G_n} (\mathrm{pr}_n)_* \mathscr{M}_n := \left(\widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}} \otimes_{(\mathrm{pr}_n)_* \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}} (\mathrm{pr}_n)_* \mathscr{M}_n \right) / \mathcal{R}_{n-1} \ .$$

Definition 5.3.3. A G_0-*equivariant coadmissible module* on $\mathfrak{X}_\infty := \varprojlim_n \mathfrak{X}_n$ consists of a family $\mathscr{M} := (\mathscr{M}_n)_{n\in\mathbb{N}}$ of objects $\mathscr{M}_n \in \mathrm{Coh}(\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}, G_0)$ together with isomorphisms

$$(5.3.4) \qquad \widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}} \otimes_{(\mathrm{pr}_n)_* \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}, G_n} (\mathrm{pr}_n)_* \mathscr{M}_n \xrightarrow{\simeq} \mathscr{M}_{n-1}$$

of G_0-equivariant $\widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}}$-modules for all $n \geqslant 1$.

A morphism $\mathscr{M} \to \mathscr{N}$ between two such modules consists of morphisms $\mathscr{M}_n \to \mathscr{N}_n$ in $\mathrm{Coh}(\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}, G_0)$ compatible with the isomorphisms above.

Let $\mathscr{M}$ be a G_0-equivariant coadmissible module on $\mathfrak{X}_\infty$. The isomorphisms 5.3.4 induce morphisms $(\mathrm{pr}_n)_* \mathscr{M}_n \to \mathscr{M}_{n-1}$ having global sections $H^0(\mathfrak{X}_n, \mathscr{M}_n) \to H^0(\mathfrak{X}_{n-1}, \mathscr{M}_{n-1})$. We let

$$H^0(\mathfrak{X}_\infty, \mathscr{M}) := \varprojlim_n H^0(\mathfrak{X}_n, \mathscr{M}_n) \ .$$

On the other hand, we consider the category of coadmissible $D(G_0)_{\theta_0}$-modules. Given such a module M we have its associated admissible locally analytic G_0-representation $V = M'_b$ together with its subspace of $\mathbb{G}(n)^\circ$-analytic vectors $V_{\mathbb{G}(n)^\circ-\mathrm{an}}$. The latter is stable under the G_0-action and its dual $M_n := (V_{\mathbb{G}(n)^\circ-\mathrm{an}})'_b$ is a finitely presented $D(\mathbb{G}(n)^\circ, G_0)_{\theta_0}$-module, cf. 5.2.4.

According to Thm. 4.3.3 we have the coherent $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}$-module

$$\mathscr{L}oc_n^\dagger(M_n) = \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}} \otimes_{\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)_{\theta_0}} M_n$$

on $\mathfrak{X}_n$. Using the contragredient G_0-action on the dual space M_n, we put

$$g.(s \otimes m) := (g.s) \otimes (g.m)$$

for $g \in G_0, m \in M_n$ and a local section s. In this way, $\mathscr{L}oc_n^\dagger(M_n)$ becomes an object of $\mathrm{Coh}(\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}, G_0)$.

Proposition 5.3.5. *(i) The family $\mathscr{L}oc_n^\dagger(M_n), n \in \mathbb{N}$, forms a G_0-equivariant coadmissible module on $\mathfrak{X}_\infty$. Call it $\mathscr{L}oc^\dagger(M)$. The formation of $\mathscr{L}oc^\dagger(M)$ is functorial in M.*

(ii) The functors $\mathscr{L}oc^\dagger$ and $H^0(\mathfrak{X}_\infty, \cdot)$ are quasi-inverse equivalences between the categories of coadmissible $D(G_0)_{\theta_0}$-modules and G_0-equivariant coadmissible modules on $\mathfrak{X}_\infty$.

Proof. We let $H_n := G_n/G_{n+1}$ and we denote a system of representatives in G_n for the cosets in H_n by the same symbol. For simplicity, we abbreviate in this proof

$$D(n) := \mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^{\circ})_{\theta_0} \quad \text{and} \quad D(n, G_0) := D(\mathbb{G}(n)^{\circ}, G_0)_{\theta_0} \,.$$

We have the natural inclusion $D(n) \hookrightarrow D(n, G_0)$ from 5.2.2 which is compatible with variation in n. Now suppose M is a $D(n, G_0)$-module. We then have the free $D(n-1)$-module

$$D(n-1)^{\oplus M \times H_n}$$

on a basis $e_{m,h}$ indexed by the elements (m, h) of the set $M \times H_n$. Its formation is functorial in M: if M' is another module and $f : M \to M'$ a linear map, then $e_{m,h} \to e_{f(m),h}$ induces a linear map between the corresponding free modules. The free module comes with a linear map

$$f_M : D(n-1)^{\oplus M \times H_n} \to D(n-1) \otimes_{D(n)} M$$

given by

$$\oplus_{(m,h)} \lambda_{m,h} e_{m,h} \mapsto (\lambda_{m,h} \delta_h) \otimes m - \lambda_{m,h} \otimes (h \cdot m)$$

for $\lambda_{m,h} \in D(n-1)$ where we consider M a $D(n)$-module via the inclusion $D(n) \hookrightarrow D(n, G_0)$. The map is visibly functorial in M and gives rise to the sequence of linear maps

$$D(n-1)^{\oplus M \times H_n} \xrightarrow{f_M} D(n-1) \otimes_{D(n)} M \xrightarrow{can_M} D(n-1, G_0) \otimes_{D(n,G_0)} M \longrightarrow 0$$

where the second map is induced from the inclusion $D(n) \hookrightarrow D(n, G_0)$. The sequence is functorial in M, since so are both occuring maps.

Claim 1: If M is a finitely presented $D(n, G_0)$-module, then the above sequence is exact.

Proof. We show $\ker(can_M) \subseteq \mathrm{im}(f_M)$, the rest is clear. Choose a finite presentation of the $D(n, G_0)$-module M. Since the sequence is functorial in M, we obtain a commutative diagram with exact columns. The 3×3-lemma reduces us to the case of a finite free module $M = D(n, G_0)^{\oplus r}$. Since it suffices to see that $\ker(can_M)$ lies in the submodule generated by the images of the elements $e_{m_i,h}$, for generators $m_1, \ldots, m_r$ of M and $h \in H_n$, we are further reduced to the case $r = 1$. In this case the statement follows directly from 5.2.2. $\qquad\square$

Claim 2: Suppose M is a finitely presented $D(n)$-module and let $\mathscr{M} := \mathscr{L}oc_n^{\dagger}(M)$. The natural morphism

$$\mathscr{L}oc_{n-1}^{\dagger}(D(n-1) \otimes_{D(n)} M) \xrightarrow{\simeq} \widetilde{\mathscr{D}}_{n-1,\mathbb{Q}}^{\dagger} \otimes_{(\mathrm{pr}_n)_* \widetilde{\mathscr{D}}_{n,\mathbb{Q}}^{\dagger}} (\mathrm{pr}_n)_* \mathscr{M}$$

is bijective.

Proof. We have the equality of sheaves on $\mathbb{X}_{n-1}$

$$(5.3.6) \qquad R^1(\mathrm{pr}_n)_* \widetilde{\mathscr{D}}_{n,n}^{(m)} = R^1(\mathrm{pr}_n)_* \widetilde{\mathcal{D}}_{n,n}^{(m)} = 0$$

for any m. Indeed, it suffices to verify this over affine open neighborhoods U of points $P \in \mathbb{X}_{n-1}^{\mathrm{sm}}(\mathbb{F}_q)$. A Čech cohomology argument as in the proof of Lem.2.3.4 shows that

$$H^1(\mathrm{pr}_n^{-1}(U), \widetilde{\mathcal{T}}_{n,n}^{\otimes d}) = 0$$

for any $d \geqslant 0$. The cohomology sequence associated to the exact sequence of sheaves over $\mathrm{pr}_n^{-1}(U)$

$$0 \to \widetilde{\mathcal{D}}_{n,n;d-1}^{(m)} \to \widetilde{\mathcal{D}}_{n,n;d}^{(m)} \to \frac{q_d^{(m)}!}{d!} \widetilde{\mathcal{T}}_{n,n}^{\otimes d} \to 0 \ ,$$

cf. Prop. 3.2.1, shows by induction on d that

$$H^1(\mathrm{pr}_n^{-1}(U), \widetilde{\mathcal{D}}_{n,n;d}^{(m)}) = 0$$

for any $d \geqslant 0$. Passing to the inductive limit over d gives

$$(5.3.7) \qquad H^1(\mathrm{pr}_n^{-1}(U), \widetilde{\mathcal{D}}_{n,n}^{(m)}) = 0 \ .$$

In particular, $R^1(\mathrm{pr}_n)_* \widetilde{\mathcal{D}}_{n,n}^{(m)} = 0$. Let $V = \mathrm{pr}_n^{-1}(U)$ and let V_j be the reduction of U modulo p^{j+1}. We may now argue as in the proof of (i) and (ii) of Prop. 4.2.1: using 5.3.7 together with a commutative diagram similar to 4.2.4 shows that the projective limit of the system

$$(H^1(V_j, \widetilde{\mathcal{D}}_{n,n}^{(m)}/p^{j+1}\widetilde{\mathcal{D}}_{n,n}^{(m)}))_{j \geqslant 0}$$

is equal to $H^1(V, \widetilde{\mathcal{D}}_{n,n}^{(m)})$ and that the system

$$(H^0(V_j, \widetilde{\mathcal{D}}_{n,n}^{(m)}/p^{j+1}\widetilde{\mathcal{D}}_{n,n}^{(m)}))_{j \geqslant 0}$$

satisfies the Mittag-Leffler condition. The latter condition allows us to apply [20, ch. 0, 13.3.1] where we use a base for the topology of V consisting of open affines. This yields a bijection between the projective limit of the $H^1(V_j, \widetilde{\mathcal{D}}_{n,n}^{(m)}/p^{j+1}\widetilde{\mathcal{D}}_{n,n}^{(m)})$ and $H^1(V, \widetilde{\mathscr{D}}_{n,n}^{(m)})$. This completes the proof of 5.3.6. We conclude by inverting p and passing to the inductive limit over m that $R^1(\mathrm{pr}_n)_* \widetilde{\mathscr{D}}_{n,\mathbb{Q}}^{\dagger} = 0$. Moreover, since $\mathfrak{X}_n$ is a noetherian space of dimension

one, we have $R^2(\mathrm{pr}_n)_* \mathscr{E} = 0$ for any abelian sheaf $\mathscr{E}$ on $\mathfrak{X}_n$ by Grothendieck's vanishing theorem. According to 4.3.3, we therefore have $R^1(\mathrm{pr}_n)_* \mathscr{E} = 0$ for any coherent $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}$-module $\mathscr{E}$, in other words, the functor $(\mathrm{pr}_n)_*$ is right-exact on such modules. Choosing a finite presentation of M reduces us now to the case $M = D(n)$ which is obvious. $\qquad\square$

Now let M be a finitely presented $D(n, G_0)$-module. Let $m_1, \ldots, m_r$ be generators for M as a $D(n)$-module. We have a sequence of $D(n-1)$-modules

$$\bigoplus_{i,h} D(n-1)e_{m_i,h} \xrightarrow{f'_M} D(n-1) \otimes_{D(n)} M \xrightarrow{can_M} D(n-1, G_0) \otimes_{D(n,G_0)} M \longrightarrow 0$$

where f'_M denotes the restriction of the map f_M to the free submodule of $D(n-1)^{\oplus M \times H_n}$ generated by the finitely many vectors $e_{m_i,h}, i = 1, \ldots, r, h \in H_n$. Since $\mathrm{im}(f'_M) = \mathrm{im}(f_M)$ the sequence is exact by the first claim. Since it consists of finitely presented $D(n-1)$-modules, we may apply the exact functor $\mathscr{L}oc^\dagger_{n-1}$ to it. By the second claim, we get an exact sequence

$$(\widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}})^{\oplus r|H_n|} \to \widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}} \otimes_{(\mathrm{pr}_n)_* \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}} (\mathrm{pr}_n)_* \mathscr{M} \to \mathscr{L}oc^\dagger_{n-1}(D(n-1, G_0) \otimes_{D(n,G_0)} M) \to 0$$

where $\mathscr{M} = \mathscr{L}oc^\dagger_n(M)$. The cokernel of the first map in this sequence equals by definition

$$\widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}} \otimes_{(\mathrm{pr}_n)_* \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}, G_n} (\mathrm{pr}_n)_* \mathscr{M} \; ,$$

whence an isomorphism

$$\widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}} \otimes_{(\mathrm{pr}_n)_* \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}, G_n} (\mathrm{pr}_n)_* \mathscr{M} \xrightarrow{\simeq} \mathscr{L}oc^\dagger_{n-1}(D(n-1, G_0) \otimes_{D(n,G_0)} M) \; .$$

This implies both parts of the proposition. $\qquad\square$

5.4. **The functor $\mathscr{L}oc^\dagger_\infty$ and $\mathrm{GL}_2(L)$-equivariance.** Let $G := \mathrm{GL}_2(L)$. Denote the canonical projection map $\mathfrak{X}_\infty \to \mathfrak{X}_n$ by sp_n for each n. We define the following sheaf of rings on $\mathfrak{X}_\infty$. Assume $V \subseteq \mathfrak{X}_\infty$ is an open subset of the form $\mathrm{sp}_N^{-1}(U)$ with an open subset $U \subseteq \mathfrak{X}_N$ for some N. We have that

$$\mathrm{sp}_n(V) = \mathrm{pr}_{n,N}^{-1}(U)$$

for any $n \geqslant N$ and so, in particular, $\mathrm{sp}_n(V) \subseteq \mathfrak{X}_n$ is an open subset for these n. Moreover,

$$\mathrm{pr}_{n,n-1}^{-1}(\mathrm{sp}_{n-1}(V)) = \mathrm{sp}_n(V)$$

for all $n > N$. The morphism 5.3.2 induces the ring homomorphism

$$(5.4.1) \qquad \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}(\mathrm{sp}_n(V)) = (\mathrm{pr}_{n,n-1})_* \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}((\mathrm{sp}_{n-1}(V))) \to \widetilde{\mathscr{D}}^\dagger_{n-1,\mathbb{Q}}(\mathrm{sp}_{n-1}(V))$$

for all $n > N$ and we form the projective limit

$$\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}(V) := \varprojlim_{n>N} \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}(\mathrm{sp}_n(V))$$

over all these maps. The open subsets of the form V form a basis for the topology on $\mathfrak{X}_\infty$ and $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$ is a presheaf on this basis. We denote the associated sheaf on $\mathfrak{X}_\infty$ by the symbol $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$ as well. It is a G_0-equivariant sheaf of rings on $\mathfrak{X}_\infty$.

Suppose $\mathscr{M} := (\mathscr{M}_n)_n$ is a G_0-equivariant coadmissible module on $\mathfrak{X}_\infty$ as defined in the previous subsection. The isomorphisms 5.3.4 induce G_0-equivariant maps $(\mathrm{pr}_n)_* \mathscr{M}_n \to \mathscr{M}_{n-1}$ which are linear relative to the morphism 5.3.2. In a completely analogous manner as above, we obtain a sheaf $\mathscr{M}_\infty$ on $\mathfrak{X}_\infty$. It is a G_0-equivariant (left) $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$-module on $\mathfrak{X}_\infty$ whose formation is functorial in $\mathscr{M}$.

Proposition 5.4.2. *The functor $\mathscr{M} \to \mathscr{M}_\infty$ from G_0-equivariant coadmissible modules on $\mathfrak{X}_\infty$ to G_0-equivariant $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$-modules is a fully faithful embedding.*

Proof. We have $\mathrm{sp}_n(\mathfrak{X}_\infty) = \mathfrak{X}_n$ for all n. The global sections of M_∞ are therefore equal to

$$\Gamma(\mathfrak{X}_\infty, \mathscr{M}_\infty) = \varprojlim_n \Gamma(\mathfrak{X}_n, \mathscr{M}_n) = H^0(\mathfrak{X}, \mathscr{M})$$

in the notation of the previous section. Thus, the functor $\mathscr{L}oc^\dagger \circ \Gamma(\mathfrak{X}_\infty, -)$ is a left quasi-inverse according to Prop. 5.3.5. $\qquad\square$

We tentatively call the abelian subcategory of all G_0-equivariant (left) $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$-modules equal to the essential image of the functor $(\cdot)_\infty$ the *coadmissible G_0-equivariant* $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$-modules. We denote by $\mathscr{L}oc^\dagger_\infty$ the composite of the functor $\mathscr{L}oc^\dagger$ with $(\cdot)_\infty$.

5.4.3. The G_0-action on the space $\mathfrak{X}_\infty$ admits a natural extension to an action of the full group G. Indeed, let $\mathcal{T}$ be the Bruhat-Tits tree for the group $\mathrm{PGL}_2(L)$ with its natural G-action. Its vertices correspond to the homothety classes of lattices $\Lambda \subset L^{\oplus 2}$. Given a vertex $v \in \mathcal{T}$ we denote by $|\mathcal{T}(v)|_{\leqslant n} \subset \mathcal{T}$ the set of points whose distance to v is less or equal to n. There is a unique maximal compact subgroup $G_v \subseteq G$ fixing the vertex v. We denote by v_0 the vertex corresponding to the class of the standard lattice $\Lambda_0 = \mathfrak{o} \oplus \mathfrak{o}$. In this case $G_{v_0} = G_0$. As mentioned in 2.1, all constructions in sections 2-4 are associated to the standard lattice Λ_0 but can be done canonically for (the homothety class of) an arbitrary lattice $\Lambda \subset L^{\oplus 2}$. We distinguish the various constructions from each other by

adding the corresponding vertex $v = [\Lambda]$ to them, i.e. we write $\mathbb{X}_n(v), \mathfrak{X}_n(v), \widetilde{\mathscr{D}}_{n,k}^{(m)}(v)$ and so on.

An element $g \in G$ induces a morphism

$$\mathbb{X}_0(v) = \mathrm{Proj}(\Lambda) \xrightarrow{g\cdot} \mathrm{Proj}(g\Lambda) = \mathbb{X}_0(gv)$$

which satisfies $(gh). = (g.) \circ (h.)$ and $1. = \mathrm{id}$ for $g, h \in G$. By the universal property of blow-up, this induces a morphism $\mathbb{X}_n(v) \xrightarrow{g\cdot} \mathbb{X}_n(gv)$ compatible with variation in n. These morphisms pass to the formal completions $\mathfrak{X}_n(v)$. Given an inclusion

$$|\mathcal{T}(v)|_{\leqslant n} \subseteq |\mathcal{T}(v')|_{\leqslant n'}$$

the space $\mathbb{X}_{n'}(v')$ equals the blow-up of $\mathbb{X}_n(v)$ in the $\mathbb{F}_q$-valued points of $\mathbb{X}_n(v)^{\mathrm{sm}}$ which correspond to the vertices in the complement of $|\mathcal{T}(v)|_{\leqslant n}$ in $|\mathcal{T}(v')|_{\leqslant n'}$. The action of G is compatible with the blow-up morphism. The same holds for the formal completions. We obtain thus a G-action on the projective limit over all $\mathfrak{X}_n(v)$. Since a given set $|\mathcal{T}(v)|_{\leqslant n}$ is contained in $|\mathcal{T}(v_0)|_{\leqslant n'}$ for large enough n', this projective limit equals $\mathfrak{X}_\infty$ by cofinality. We thus have indeed a natural G-action on $\mathfrak{X}_\infty$.

We note that, by definition of the action, there is an equality

$$(5.4.4) \qquad\qquad \mathrm{sp}_{n,gv}(g.V) = g.\mathrm{sp}_{n,v}(V)$$

in $\mathfrak{X}_n(g.v)$ for any subset $V \subseteq \mathfrak{X}_\infty$ and the specialization map $\mathrm{sp}_{n,v} : \mathfrak{X}_\infty \to \mathfrak{X}_n(v)$.

Proposition 5.4.5. *The G_0-equivariant structure on the sheaf $\widetilde{\mathscr{D}}_{\infty,\mathbb{Q}}^\dagger$ extends to a G-equivariant structure.*

Proof. Let $g \in G$. The isomorphism $\mathfrak{X}_n(v) \xrightarrow{g\cdot} \mathfrak{X}_n(gv)$ induces a ring isomorphism

$$(5.4.6) \qquad\qquad \widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger(v)(U) \xrightarrow{g\cdot} \widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger(gv)(g.U)$$

for any open subset $U \subseteq \mathfrak{X}_n(v)$. In particular, for an open subset $V \subseteq \mathfrak{X}_\infty$ of the form $V = \mathrm{sp}_{N,v}^{-1}(U)$ with $U \subseteq \mathfrak{X}_N(v)$ open, this gives a ring homomorphism for $n > N$

$$(5.4.7) \qquad \widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger(v)(\mathrm{sp}_{n,v}(V)) \xrightarrow{g\cdot} \widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger(gv)(g.\mathrm{sp}_{n,v}(V)) = \widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger(gv)(\mathrm{sp}_{n,gv}(gV)) \ .$$

where we have used 5.4.4. Given an inclusion $|\mathcal{T}(v)|_{\leqslant n} \subseteq |\mathcal{T}(v')|_{\leqslant n'}$ the blow-up morphism

$$\mathrm{pr}_{(n',v'),(n,v)} : \mathfrak{X}_{n'}(v') \longrightarrow \mathfrak{X}_n(v)$$

induces a morphism of sheaves of rings

$$(5.4.8) \qquad (\mathrm{pr}_{(n',v'),(n,v)})_* \widetilde{\mathscr{D}}^\dagger_{n',\mathbb{Q}}(v') \longrightarrow \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}(v)$$

in a manner analogous to 5.3.2. Indeed, let $\mathbb{G}_v$ denote the smooth affine Bruhat-Tits group scheme over $\mathfrak{o}$ associated with the vertex v and let $\mathfrak{g}_v$ be its Lie-algebra. From the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Lie}(\mathbb{G}_{v'}(n')) & = & \varpi^{n'}\mathfrak{g}_{v'} & \longrightarrow & H^0(\mathbb{X}_{n'}(v'), \mathcal{T}_{\mathbb{X}_{n'}(v')^{\log}}) \\
& & \downarrow & & \downarrow \\
\mathrm{Lie}(\mathbb{G}_v(n)) & = & \varpi^n\mathfrak{g}_v & \longrightarrow & H^0(\mathbb{X}_n(v), \mathcal{T}_{\mathbb{X}_n(v)^{\log}})
\end{array}$$

we deduce a morphism

$$(\mathrm{pr}_{(n',v'),(n,v)})_* \widetilde{\mathcal{T}}^{(m)}_{n',n'}(v') \to \widetilde{\mathcal{T}}^{(m)}_{n,n}(v)$$

for any m. Passing to differential operators yields the claimed morphism.

Given $V \subset \mathfrak{X}_\infty$ of the form $V = \mathrm{sp}^{-1}_{\tilde{n},\tilde{v}}(U)$ with an open set $U \subseteq \mathfrak{X}_{\tilde{n}}(\tilde{v})$, the morphism 5.4.8 induces a ring homomorphism

$$(5.4.9) \quad \widetilde{\mathscr{D}}^\dagger_{n',\mathbb{Q}}(v')(\mathrm{sp}_{n',v'}(V)) = (\mathrm{pr}_{(n',v'),(n,v)})_* \widetilde{\mathscr{D}}^\dagger_{n',\mathbb{Q}}(v')(\mathrm{sp}_{n,v}(V)) \to \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}(v)(\mathrm{sp}_{n,v}(V))$$

whenever there are inclusions

$$|\mathcal{T}(\tilde{v})|_{\leqslant \tilde{n}} \subseteq |\mathcal{T}(v)|_{\leqslant n} \subseteq |\mathcal{T}(v')|_{\leqslant n'}$$

and we may form the projective limit

$$\varprojlim_{(n,v)} \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}(v)(\mathrm{sp}_{n,v}(V)) \ .$$

However, again by cofinality, this projective limit equals $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}(V)$. Since the homomorphism 5.4.7 is compatible with variation in n we deduce for a given $g \in G$ a ring homomorphism

$$\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}(V) = \varprojlim_{(n,v)} \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}(v)(\mathrm{sp}_{n,v}(V)) \xrightarrow{g.} \varprojlim_{(n,gv)} \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}(gv)(\mathrm{sp}_{n,gv}(gV)) = \widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}(gV) \ .$$

It implies that the sheaf $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$ is G-equivariant. It is clear from the construction that the G-equivariant structure extends the G_0-structure. $\qquad\square$

A pair consisting of a coadmissible G_0-equivariant $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$-module M together with a G-action on M which extends its G_0-action will be called a coadmissible G-equivariant $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$-module. In the following we will simply write M for such a pair, the G-action being understood.

Theorem 5.4.10. *The functors $\mathscr{L}oc^\dagger_\infty$ and $\Gamma(\mathfrak{X}_\infty, \cdot)$ are quasi-inverse equivalences between the categories of coadmissible $D(G_0)_{\theta_0}$-modules and coadmissible G_0-equivariant $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$-modules. They lift to quasi-inverse equivalences between the categories of coadmissible $D(G)_{\theta_0}$-modules and coadmissible G-equivariant $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$-modules (also denoted by $\mathscr{L}oc^\dagger_\infty$ and $\Gamma(\mathfrak{X}_\infty, \cdot)$). We thus have a commutative diagramm of functors*

$$
\left\{ \begin{array}{c} coadmissible \\ D(G)_{\theta_0} - modules \end{array} \right\} \xrightarrow[\simeq]{\mathscr{L}oc^\dagger_\infty} \left\{ \begin{array}{c} coadmissible\ G - equiv. \\ \widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}} - modules \end{array} \right\}
$$

$$
\left\{ \begin{array}{c} coadmissible \\ D(G_0)_{\theta_0} - modules \end{array} \right\} \xrightarrow[\simeq]{\mathscr{L}oc^\dagger_\infty} \left\{ \begin{array}{c} coadmissible\ G_0 - equiv. \\ \widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}} - modules \end{array} \right\}
$$

where the left-hand vertical arrow is the restriction functor coming from the homomorphism $D(G_0)_{\theta_0} \to D(G)_{\theta_0}$ and the right-hand vertical arrow is the forgetful functor.

Proof. We only need to show the second statement. It is clear that a coadmissible $D(G_0)_{\theta_0}$-module which comes from a coadmissible G-equivariant $\widetilde{\mathscr{D}}^\dagger_{\infty,\mathbb{Q}}$-module is a $D(G)_{\theta_0}$-module. For the converse, we consider a vertex $v \in \mathcal{T}$ and the corresponding localisation functor $\mathscr{L}oc^\dagger_{n,\mathbb{Q}}(v)$ which is an equivalence between finitely presented $\mathcal{D}^{an}(\mathbb{G}_v(n)^\circ)_{\theta_0}$-modules and coherent $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}(v)$-modules on $\mathfrak{X}_n(v)$. Here, $\mathbb{G}_v$ denotes the smooth affine Bruhat-Tits group scheme over $\mathfrak{o}$ associated with the vertex v. The adjoint action of G on its Lie algebra induces a ring isomorphism

$$
(5.4.11) \qquad\qquad \mathcal{D}^{an}(\mathbb{G}_v(n)^\circ) \xrightarrow{g \cdot} \mathcal{D}^{an}(\mathbb{G}_{gv}(n)^\circ)
$$

for any $g \in G$. Now consider a coadmissible $D(G)_{\theta_0}$-module M with dual space $V = M'$. We have the family $(\mathscr{M}_{n,v})_{(n,v)}$ where

$$
\mathscr{M}_{n,v} = \mathscr{L}oc^\dagger_{n,\mathbb{Q}}(v)(M_{n,v}) = \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}(v) \otimes_{\mathcal{D}^{an}(\mathbb{G}_v(n)^\circ)_{\theta_0}} M_{n,v}
$$

and $M_{n,v} = (V_{\mathbb{G}_v(n)^\circ - an})'$. Let $g \in G$. The map $m \mapsto gm$ on M induces a map $M_{n,v} \to M_{n,gv}$ which is linear relative to 5.4.11. We therefore have for any open subset $U \subseteq \mathfrak{X}_n(v)$ a homomorphism

$$
\mathscr{M}_{n,v}(U) \xrightarrow{g \cdot} \mathscr{M}_{n,gv}(g.U)
$$

which is induced by the map

$$s \otimes m \mapsto (g.s) \otimes gm \ .$$

for $s \in \widetilde{\mathscr{D}}^{\dagger}_{n,\mathbb{Q}}(v)(U), m \in M_{n,v}$ and where $g.$ is the ring isomorphism 5.4.6. In particular, for an open subset $V \subseteq \mathfrak{X}_{\infty}$ of the form $V = \mathrm{sp}^{-1}_{N,v}(U)$ with $U \subseteq \mathfrak{X}_N(v)$ open, this gives a homomorphism for $n > N$

$$(5.4.12) \qquad \mathscr{M}_{n,v}(\mathrm{sp}_{n,v}(V)) \xrightarrow{g.} \mathscr{M}_{n,gv}(g.\mathrm{sp}_{n,v}(V)) = \mathscr{M}_{n,gv}(\mathrm{sp}_{n,gv}(gV))$$

which is linear relative to the ring homomorphism 5.4.7.

Now for every inclusion

$$|\mathcal{T}(v)|_{\leqslant n} \subseteq |\mathcal{T}(v')|_{\leqslant n'} \ ,$$

we have a morphism

$$(5.4.13) \qquad (\mathrm{pr}_{(n',v'),(n,v)})_* \mathscr{M}_{n',v'} \longrightarrow \mathscr{M}_{n,v}$$

compatible with the morphism of rings 5.4.8 as follows. First of all, one has

$$(\mathrm{pr}_{(n',v'),(n,v)})_* \left(\mathscr{L}oc^{\dagger}_{n',\mathbb{Q}}(v')(M_{n',v'}) \right) = \left((\mathrm{pr}_{(n',v'),(n,v)})_* (\widetilde{\mathscr{D}}^{\dagger}_{n',\mathbb{Q}}(v')) \right) \otimes_{\mathcal{D}^{an}(\mathbb{G}_{v'}(n')^{\circ})_{\theta_0}} M_{n',v'}$$

which can be proved along the lines of claim 2 in the proof of Prop. 5.3.5. Moreover, we have inclusions $\mathbb{G}_{v'}(n') \subseteq \mathbb{G}_v(n)$ and thus

$$V_{\mathbb{G}_v(n)^{\circ}-\mathrm{an}} \subseteq V_{\mathbb{G}_{v'}(n')^{\circ}-\mathrm{an}} \ .$$

The dual map $M_{n',v'} \to M_{n,v}$ is linear relative to the natural inclusion

$$\mathcal{D}^{an}(\mathbb{G}_{v'}(n')^{\circ}) \to \mathcal{D}^{an}(\mathbb{G}_v(n)^{\circ}) \ .$$

The latter inclusion is compatible with the morphism of rings 5.4.8 via taking global sections. Hence, we have a morphism 5.4.13 as claimed. We now have everything at hand to follow the arguments in the proof of the preceding proposition word for word and to conclude that the projective limit $\mathscr{M}_{\infty}$ has a G-action which extends its G_0-action and which makes it a G-equivariant $\widetilde{\mathscr{D}}^{\dagger}_{\infty,\mathbb{Q}}$-module. This completes the proof of the theorem. □

Remark: Let $L \subset K$ be a complete and discretely valued extension field. If we consider the K-algebras $\mathcal{D}^{an}(\mathbb{G}(k)^\circ)\hat{\otimes}_L K, D(G_0)\hat{\otimes}_L K$ etc. as well as the sheaves of K-algebras $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{(m)}\hat{\otimes}_L K, \widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{\dagger}\hat{\otimes}_L K$, then there are obvious versions 'over K' of the theorems 4.3.3 and 5.4.10.

6. Examples of localizations

6.1. **Smooth representations.** If V is a smooth G-representation (i.e. the stabilizer of each vector $v \in V$ is an open subgroup of G), then $V_{\mathbb{G}(n)^\circ - an}$ equals the space of fixed vectors $V^{G_{n+1}}$ in V under the action of the compact subgroup G_{n+1}. Suppose V is admissible (i.e. the vector space $V^{G_{n+1}}$ is finite-dimensional). Since $\mathfrak{g}V = 0$, one finds

$$(6.1.1) \qquad \mathscr{L}oc_n^{\dagger}((V^{G_{n+1}})') = \mathcal{O}_{\mathfrak{X}_n,\mathbb{Q}} \otimes_L (V^{G_{n+1}})',$$

where G_0 acts diagonally and $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^{\dagger}$ acts through its natural action on $\mathcal{O}_{\mathfrak{X}_n,\mathbb{Q}}$.

6.2. **Representations attached to $U(\mathfrak{g})$-modules.** In this section, we will compute the G_0-equivariant coadmissible module $\mathscr{L}oc^{\dagger}(\mathbf{M})$ on $\mathfrak{X}_\infty$ for a class of coadmissible $D(G)$-modules $\mathbf{M}$ related to the pair $(\mathfrak{g}, B)$. Here $\mathbb{B} \subset \mathrm{GL}_{2,L}$ denotes the Borel subgroup of upper triangular matrices and B denotes its group of L-rational points. Let $\mathfrak{b}$ be its Lie-algebra. Let $T \subset B$ be the subgroup of diagonal matrices with Lie algebra $\mathfrak{t}$.

Let $L \subset K$ be a complete and discretely valued extension field. To simplify notation, we make the convention that, when dealing with universal enveloping algebras, distribution algebras, differential operators etc. we write $U(\mathfrak{g})$, $D(G_0)$, $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{\dagger}$ etc. to denote the corresponding objects *after base change to K*, i.e., what is precisely $U(\mathfrak{g}_K)$, $D(G_0)\hat{\otimes}_L K$, $\widetilde{\mathscr{D}}_{n,k,\mathbb{Q}}^{\dagger}\hat{\otimes}_L K$ and so on (compare also final remark in sect. 5).

The group G and its subgroup B act via the adjoint representation on $U(\mathfrak{g})$ and we denote by

$$(6.2.1) \qquad D(\mathfrak{g}, B) := D(B) \otimes_{U(\mathfrak{b})} U(\mathfrak{g})$$

the corresponding skew-product ring. The skew-multiplication here is induced by

$$(\delta_{b'} \otimes x') \cdot (\delta_b \otimes x) = \delta_{b'b} \otimes \delta_{b^{-1}}(x')x$$

for $b, b' \in B$ and $x, x' \in U(\mathfrak{g})$. If λ denotes a locally analytic character of B, then we have a canonical algebra isomorphism

$$(6.2.2) \qquad D(\mathfrak{g}, B)/D(\mathfrak{g}, B)I(\lambda) \simeq U(\mathfrak{g})/U(\mathfrak{g})I(d\lambda)$$

where $I(\lambda)$ resp. $I(d\lambda)$ denote the ideals equal to the kernel of $D(B) \xrightarrow{\lambda} K$ resp. $\mathfrak{b} \xrightarrow{d\lambda} K$. Replacing B by $B_0 = B \cap G_0$, we obtain a skew-product ring $D(\mathfrak{g}, B_0)$ with similar properties. Given a $D(\mathfrak{g}, B)$-module M one has

$$(6.2.3) \qquad\qquad D(G) \otimes_{D(\mathfrak{g}, B)} M = D(G_0) \otimes_{D(\mathfrak{g}, B_0)} M$$

as $D(G_0)$-modules [35, 4.2]. We consider the functor

$$M \mapsto \mathbf{M} := D(G) \otimes_{D(\mathfrak{g}, B)} M$$

from $D(\mathfrak{g}, B)$-modules to $D(G)$-modules [32]. If M is finitely generated as $U(\mathfrak{g})$-module, then $\mathbf{M}$ is coadmissible by [35, 4.3]. From now on we assume that M is a finitely generated $U(\mathfrak{g})$-module. We let $V := \mathbf{M}'_b$ be the locally analytic G-representation corresponding to $\mathbf{M}$ and denote by

$$\mathbf{M}_n := (V_{\mathbb{G}(n)^\circ - \mathrm{an}})'_b$$

the dual of the subspace of its $\mathbb{G}(n)^\circ$-analytic vectors. According to 5.2.4 the $D(\mathbb{G}(n)^\circ, G_0)$-module $\mathbf{M}_n$ is finitely presented and has its canonical topology.

Lemma 6.2.4. *The canonical map*

$$D(\mathbb{G}(n)^\circ, G_0) \otimes_{D(G_0)} \mathbf{M} \xrightarrow{\simeq} \mathbf{M}_n$$

is an isomorphism.

Proof. It suffices to show that the left-hand side is complete. We have

$$D(\mathbb{G}(n)^\circ, G_0) \otimes_{D(G_0)} \mathbf{M} = D(\mathbb{G}(n)^\circ, G_0) \otimes_{D(\mathfrak{g}, B_0)} M \ ,$$

according to 6.2.3. The group B_0 is topologically finitely generated. As in the proof of [35, 4.3] we may write the module $D(\mathbb{G}(n)^\circ, G_0) \otimes_{D(G_0)} \mathbf{M}$ as the quotient of the finitely presented module $D(\mathbb{G}(n)^\circ, G_0) \otimes_{U(\mathfrak{g})} M$ by a finitely generated submodule. By (ii) of 5.1.1, this quotient is complete. $\qquad\square$

Recall the congruence subgroup $G_{n+1} = \mathbb{G}(n)^\circ(L)$ of G_0. Put $B_{n+1} := G_{n+1} \cap B_0$. The corresponding skew-product ring $D(\mathfrak{g}, B_{n+1})$ is contained in $\mathcal{D}^{an}(\mathbb{G}(n)^\circ)$ according to 5.2.2. Let $C(n)$ be a system of representatives in G_0 containing 1 for the residue classes in G_0/G_{n+1} modulo the subgroup B_0/B_{n+1}. Note that for an element $g \in G_0$ and a $\mathcal{D}^{an}(\mathbb{G}(n)^\circ)$-submodule N of $D(G_0)$, the abelian group $\delta_g N$ is again a $\mathcal{D}^{an}(\mathbb{G}(n)^\circ)$-submodule because of the formula $x\delta_g = \delta_g \mathrm{Ad}(g^{-1})(x)$ for any $x \in \mathcal{D}^{an}(\mathbb{G}(n)^\circ)$.

Lemma 6.2.5. *The natural map of* $(\mathcal{D}^{an}(\mathbb{G}(n)^\circ), D(\mathfrak{g}, B_0))$*-bimodules*

$$\sum : \bigoplus_{g \in C(n)} \delta_g \left(\mathcal{D}^{an}(\mathbb{G}(n)^\circ) \otimes_{D(\mathfrak{g}, B_{n+1})} D(\mathfrak{g}, B_0) \right) \xrightarrow{\simeq} D(\mathbb{G}(n)^\circ, G_0)$$

is an isomorphism.

Proof. Since $Lie(B_{n+1}) = Lie(B_0) = \mathfrak{b}$, the definition 6.2.1 implies that $D(\mathfrak{g}, B_0)$ equals the direct sum over B_0/B_{n+1} copies of $D(\mathfrak{g}, B_{n+1})$. Hence, 5.2.2 implies the claim.   □

The two lemmas allow us to write

$$\mathbf{M}_n = \oplus_{g \in C(n)} \delta_g \left(\mathcal{D}^{an}(\mathbb{G}(n)^\circ) \otimes_{D(\mathfrak{g}, B_{n+1})} M \right) = \oplus_{g \in C(n)} \delta_g M_n^{an}$$

as modules over $\mathcal{D}^{an}(\mathbb{G}(n)^\circ)$. Here

$$M_n^{an} := \mathcal{D}^{an}(\mathbb{G}(n)^\circ) \otimes_{D(\mathfrak{g}, B_{n+1})} M,$$

a finitely presented $\mathcal{D}^{an}(\mathbb{G}(n)^\circ)$-module. If M has character θ_0, so has M_n^{an}. As explained above, the 'twisted' module $\delta_g M_n^{an}$ can and will be viewed as having the same underlying group as M_n^{an} but with an action of $\mathcal{D}^{an}(\mathbb{G}(n)^\circ)$ pulled-back by the automorphism $\mathrm{Ad}(g^{-1})$. Since the adjoint action of G_0 fixes the center in $U(\mathfrak{g})$, the character of the module $\delta_g M_n^{an}$ (if existing) does not depend on g.

If M has character θ_0, then the $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$-module $\mathscr{L}oc_n^\dagger(\delta_g M_n^{an})$ on $\mathfrak{X}_n$ can be described as follows. For any $g \in G_0$ let g_* denote the direct image functor coming from the automorphism g of $\mathfrak{X}_n$. If N denotes a (coherent) $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$-module, then $g_* N$ is a (coherent) $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$-module via the isomorphism $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger \xrightarrow{\simeq} g_* \widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$.

Lemma 6.2.6. *One has*

$$\mathscr{L}oc_n^\dagger(\delta_g M_n^{an}) = g_* \mathscr{L}oc_n^\dagger(M_n^{an}) = g_* \left(\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger \otimes_{D(\mathfrak{g}, B_{n+1})} M \right).$$

Proof. The second identity follows from the definition of $\mathscr{L}oc_n^\dagger$ by contracting tensor products. The automorphism $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger \xrightarrow{\simeq} g_* \widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$ equals $\mathrm{Ad}(g^{-1})$ on global sections. The identity map is a $\mathcal{D}^{an}(\mathbb{G}(n)^\circ)$-linear isomorphism between the global sections of $\mathscr{L}oc_n^\dagger(\delta_g M_n^{an})$ and $g_* \mathscr{L}oc_n^\dagger(M_n^{an})$. Since both sheaves are coherent $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$-modules, the isomorphism extends to the sheaves by 4.3.3.   □

Since $\mathscr{L}oc_n^\dagger$ commutes with direct sums the lemma implies

$$\mathscr{L}oc_n^\dagger(\mathbf{M}_n) = \oplus_{g \in C(n)} g_* \left(\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger \otimes_{D(\mathfrak{g}, B_{n+1})} M \right)$$

as $\widetilde{\mathscr{D}}^{\dagger}_{n,\mathbb{Q}}$-modules. Of course, $\mathscr{L}oc^{\dagger}_n(\mathbf{M}_n)$ is G_0-equivariant and the collection of the $\mathscr{L}oc^{\dagger}_n(\mathbf{M}_n)$ for all n, is equal to the G_0-equivariant coadmissible module on $\mathfrak{X}_\infty$ corresponding to $\mathbf{M}$ by the equivalence 5.3.5.

6.3. Principal series representations.

6.3.1. *A general formula.* Let λ be a locally analytic K-valued character of T, viewed as a character of B. Let

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} K_{d\lambda}$$

be the induced module. Then $M(\lambda)$ is a $D(\mathfrak{g}, B)$-module [35]. Let $\mathbf{M}(\lambda)$ be the coadmissible $D(G)$-module associated with $M(\lambda)$. Its dual equals the locally analytic principal series representation

$$V := \mathrm{Ind}^G_B(\lambda^{-1}) = \{f \in C^{\mathrm{la}}(G, K) : f(gb) = \lambda(b)f(g) \text{ for all } g \in G, b \in B\}$$

with G acting by left translations. Here, $C^{\mathrm{la}}(\cdot, K)$ denotes K-valued locally analytic functions. We now choose n large enough such that the restriction of λ to $T \cap G_{n+1}$ is $\mathbb{T}(n)^\circ$-analytic. Denote by $\mathbb{U}^-$ the unipotent radical of the Borel subgroup scheme opposite to $\mathbb{B}$. Restriction of functions induces an isomorphism

$$V_{\mathbb{G}(n)^\circ - \mathrm{an}} \simeq \oplus_{g \in G_0/G_{n+1}B_0} C^{an}(\mathbb{U}^-(n)^{\circ, g}, K)$$

where $\mathbb{U}^-(n)^{\circ, g} = g\mathbb{U}^-(n)^\circ g^{-1}$ and where $C^{\mathrm{an}}(\cdot, K)$ denotes K-valued rigid analytic functions. Indeed, it is not difficult to see that, as G_{n+1}-representations, one has

$$V|_{G_{n+1}} = \oplus_{g \in G_0/G_{n+1}B_0} \mathrm{Ind}^{G_{n+1}}_{G_{n+1} \cap B^g_0}(\lambda^{-1, g})$$

where $B^g_0 = gB_0g^{-1}$ and where λ^g denotes the character of B^g_0 defined by $\lambda^g(h) = \lambda(g^{-1}hg)$. Now the $\mathbb{G}(n)^\circ$-analytic vectors in $V_g := \mathrm{Ind}^{G_{n+1}}_{G_{n+1} \cap B^g_0}(\lambda^{-1, g})$ are the preimage of V_g under the (injective) restriction map

$$C^{\mathrm{an}}(\mathbb{G}(n)^\circ, K) \to C^{\mathrm{la}}(G_{n+1}, K),$$

so that

$$(V_g)_{\mathbb{G}(n)^\circ - \mathrm{an}} = \{f \in C^{\mathrm{an}}(\mathbb{G}(n)^\circ, K) : f(hb) = \lambda^g(b)f(h) \text{ for all } h \in G_{n+1}, b \in G_{n+1} \cap B^g_0\}.$$

Our assertion follows now from the rigid-analytic (Iwahori) decomposition

$$\mathbb{G}(n)^\circ = \mathbb{U}^-(n)^{\circ, g} \times \mathbb{T}(n)^{\circ, g} \times \mathbb{U}(n)^{\circ, g}$$

together with the fact that λ^g is $\mathbb{T}(n)^{\circ, g}$-analytic for all g.

We have

$$M(\lambda) = D(\mathfrak{g}, B_{n+1})/D(\mathfrak{g}, B_{n+1})I_{n+1}(\lambda)$$

as a $D(\mathfrak{g}, B_{n+1})$-module where $I_{n+1}(\lambda)$ denotes the kernel of $D(B_{n+1}) \xrightarrow{\lambda} K$, cf. 6.2.2. Let $\mathcal{D}^{an}(\mathbb{B}(n)^\circ)$ be the analytic distribution algebra of $\mathbb{B}(n)^\circ$. By our assumption on n, the character $d\lambda$ extends to a character of $\mathcal{D}^{an}(\mathbb{B}(n)^\circ)$ whose kernel is generated by $I(d\lambda) \subset U(\mathfrak{b})$. It follows that

$$M(\lambda)_n^{an} = \mathcal{D}^{an}(\mathbb{G}(n)^\circ)/\mathcal{D}^{an}(\mathbb{G}(n)^\circ)I_{n+1}(\lambda) = \mathcal{D}^{an}(\mathbb{G}(n)^\circ) \otimes_{U(\mathfrak{g})} M(\lambda) \,.$$

Let $\mathcal{D}_{\mathbb{P}^1_L}$ be the sheaf of usual algebraic differential operators on the algebraic variety $\mathbb{P}^1_L$. The Beilinson-Bernstein localization [3] of a finitely generated $U(\mathfrak{g})$-module M with character θ_0 is a coherent $\mathcal{D}_{\mathbb{P}^1_L}$-module $\mathrm{Loc}(M)$. Let $\mathbb{P}^{1,\mathrm{rig}}_L$ be the rigid-analytic space associated to $\mathbb{P}^1_L$ with its canonical morphism $\rho : \mathbb{P}^{1,\mathrm{rig}}_L \to \mathbb{P}^1_L$. Let

$$\mathrm{sp}_{\mathfrak{X}_n} : \mathbb{P}^{1,\mathrm{rig}}_L \longrightarrow \mathfrak{X}_n$$

denote the specialization morphism[6]. Then $(\mathrm{sp}_{\mathfrak{X}_n})_* \rho^* \mathrm{Loc}(M)$ is an $\mathcal{O}_{\mathfrak{X}_n,\mathbb{Q}}$-module with an action of the sheaf $(\mathrm{sp}_{\mathfrak{X}_n})_* \rho^* \mathcal{D}_{\mathbb{P}^1_L}$. We denote its base change along the natural morphism

$$(\mathrm{sp}_{\mathfrak{X}_n})_* \rho^* \mathcal{D}_{\mathbb{P}^1_L} \longrightarrow \widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}$$

by $\mathrm{Loc}(M)_n^\dagger$, a coherent $\widetilde{\mathscr{D}}^\dagger_{n,\mathbb{Q}}$-module. Suppose now that $d\lambda$ is associated by the Harish-Chandra isomorphism to the central character θ_0. We then have

$$\mathrm{Loc}(M(\lambda))_n^\dagger = \mathscr{L}oc_n^\dagger(M(\lambda)_n^{an})$$

by the above. We can thus conclude

$$(6.3.2) \qquad \mathscr{L}oc_n^\dagger(\mathbf{M}(\lambda)_n) = \oplus_{g \in C(n)} \, g_* \mathrm{Loc}(M(\lambda))_n^\dagger \,.$$

6.3.3. *The cases when $\lambda = \mathbf{1}$ or $\lambda = \rho^{-2}$.* Here we consider the character $\rho^{-2} : T = \mathbf{T}(L) \to L^*$ defined by

$$\rho^{-2}\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right) = \frac{t_2}{t_1} \,,$$

---

[6] note that the specialization morphism factors into the three canonical maps $\mathbb{P}^{1,\mathrm{rig}}_L \to \mathbb{P}^{1,\mathrm{ad}}_L$, $\mathbb{P}^{1,\mathrm{ad}}_L \to \mathfrak{X}_\infty$ and $\mathrm{sp}_n : \mathfrak{X}_\infty \longrightarrow \mathfrak{X}_n$, where the adic space $\mathbb{P}^{1,\mathrm{ad}}_L$ is homeomorphic to the projective limit over *all* formal models of $\mathbb{P}^{1,\mathrm{rig}}_L$.

i.e., the negative root. Then $d\rho^{-2}$ and $0 = d\mathbf{1}$ are associated to θ_0 by the Harish-Chandra homomorphism. Next we consider the well-known exact sequence of $U(\mathfrak{g})$-modules

$$(6.3.4) \qquad\qquad 0 \longrightarrow M(\rho^{-2}) \longrightarrow M(\mathbf{1}) \longrightarrow L_0 \longrightarrow 0 ,$$

where L_0 denotes the one-dimensional trivial $U(\mathfrak{g})$-module. The functor $\mathcal{F}_B^G$ of [32] associates to the sequence 6.3.4 the exact sequence of locally analytic representations

$$(6.3.5) \qquad\qquad 0 \longrightarrow \mathrm{ind}_B^G(\mathbf{1}) \longrightarrow \mathrm{Ind}_B^G(\mathbf{1}) \longrightarrow \mathrm{Ind}_B^G(\rho^2) \longrightarrow 0 ,$$

where $\mathrm{ind}_B^G(\mathbf{1})$ denotes the smooth induction of the trivial character $\mathbf{1}$ of B. Passing in 6.3.5 to $\mathbb{G}(n)^\circ$-analytic vectors, and then to the continuous duals of the spaces of $\mathbb{G}(n)^\circ$-analytic vectors, furnishes an exact sequence of $D(\mathbb{G}(n)^\circ, G_0)$-modules

$$(6.3.6) \qquad\qquad 0 \longrightarrow \mathbf{M}(\rho^{-2})_n \longrightarrow \mathbf{M}(\mathbf{1})_n \longrightarrow \bigoplus_{g \in C(n)} g_* L_0 \longrightarrow 0 ,$$

where we denote the extension of L_0 to the $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)$-module corresponding to the trivial one-dimensional representation of $\mathbb{G}(n)^\circ$ by L_0, too. The Beilinson-Bernstein localization of $M(\rho^{-2})$ is a skyscraper sheaf supported at the fixed point of B on $\mathbb{P}_L^1$, cf. [22, sec. 12.3], whereas the Beilinson-Bernstein localization of the trivial one-dimensional representation is the structure sheaf of $\mathbb{P}_L^1$. Applying $\mathscr{L}oc_n^\dagger$ and 6.3.2 to 6.3.6 gives then the following exact sequence of sheaves of $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$-modules on $\mathfrak{X}_n$:

$$(6.3.7) \quad 0 \to \oplus_{g \in C(n)} g_* \mathrm{Loc}(M(\rho^{-2}))_n^\dagger \to \oplus_{g \in C(n)} g_* \mathrm{Loc}(M(\mathbf{1}))_n^\dagger \to \oplus_{g \in C(n)} g_* \mathcal{O}_{\mathfrak{X}_n,\mathbb{Q}} \to 0 .$$

The sheaf on the left is a skyscraper sheaf whose support is the set $\mathfrak{X}_n^{\mathrm{sm}}(\mathbb{F}_q)$ of smooth $\mathbb{F}_q$-rational points of $\mathfrak{X}_n$, a set which is naturally in bijection with $\mathbb{P}^1(\mathfrak{o}_L/(\varpi^{n+1}))$.

6.4. **Two representations furnished by the p-adic upper half plane.** In this section we consider the $\mathrm{GL}_2(L)$-equivariant line bundles $\mathcal{O}$, the structure sheaf, and Ω, the sheaf of rigid-analytic differential forms, on $\mathbb{P}_L^{1,\mathrm{rig}}$ and their restrictions to the p-adic upper half plane $\Sigma_{0,L}$. Their global sections over $\Sigma_{0,L}$ furnish coadmissible $D(\mathrm{GL}_2(L), L)$-modules, which are closely related to the coadmissible module $\mathbf{M}(\rho^{-2})$ considered in the previous section. As this material is not used later on, we only give indications of the proofs. We also note that one can treat all line bundles $\mathcal{O}(k)$ in a similar way, but only when $k \in \{-2, 0\}$ do the corresponding coadmissible modules have central character θ_0, which is why we restrict to these cases.

6.4.1. *The p-adic upper half plane.* Let $\mathbb{P}_L^{1,\mathrm{rig}}$ be the rigid-analytic space attached to the scheme $\mathbb{P}_L^1$ over L. It is well-known and not difficult to prove that the subset

$$\Sigma_{0,L} = \mathbb{P}_L^{1,\mathrm{rig}} \setminus \mathbb{P}_L^1(L)$$

is an admissible open subset of $\mathbb{P}_L^{1,\mathrm{rig}}$, and is hence a rigid-analytic space in its own right, called the *p-adic upper half plane over L*. The inductive limit of the formal schemes $\mathfrak{X}_n^\circ$ furnishes a formal model $\widehat{\Sigma}_{0,L}$ for $\Sigma_{0,L}$:

$$\widehat{\Sigma}_{0,L} = \varinjlim_n \mathfrak{X}_n^\circ \ ,$$

cf. [7].

Proposition 6.4.2. *(i) There is a canonical exact sequence of locally analytic G-representations*

$$0 \longrightarrow \mathrm{Ind}_B^G(\rho^2) \longrightarrow \mathcal{O}(\Sigma_{0,L})' \longrightarrow \mathbf{1} \longrightarrow 0 \ ,$$

and hence an exact sequence of coadmissible $D(G,L)$-modules

$$0 \longrightarrow L_0 \longrightarrow \mathcal{O}(\Sigma_{0,L}) \longrightarrow \mathbf{M}(\rho^{-2}) \longrightarrow 0 \ .$$

(ii) There is a canonical exact sequence of locally analytic G-representations

$$0 \longrightarrow \mathrm{St} \longrightarrow \Omega(\Sigma_{0,L})' \longrightarrow \mathrm{Ind}_B^G(\rho^2) \longrightarrow 0 \ ,$$

where St *denotes the smooth Steinberg representation (realized on an L-vector space). Dually, there is an exact sequence of coadmissible $D(G,L)$-modules*

$$0 \longrightarrow \mathbf{M}(\rho^{-2}) \longrightarrow \Omega(\Sigma_{0,L}) \longrightarrow \mathrm{St}' \longrightarrow 0 \ .$$

Proof. We will deduce the exact sequences of locally analytic representations from the description given in [31] and [32, sec. 7] (see also p. 4 of the introduction of [32]). The exact sequences of $D(G,L)$-modules follow immediately from these, by duality. Let $\mathcal{L} = \mathcal{O}$ or $\mathcal{L} = \Omega$. Put $\mathcal{L}(\Sigma_{0,L})^0 = H^0(\Sigma_{0,L}, \mathcal{L})$ and $\mathcal{L}(\Sigma_{0,L})^1 = H^0(\mathbb{P}_L^1, \mathcal{L})$. Then there is a canonical exact sequence of locally analytic G-representations

$$(6.4.3) \qquad 0 \longrightarrow \mathrm{St} \otimes_L H^1(\mathbb{P}_L^1, \mathcal{L})' \longrightarrow \left(\mathcal{L}(\Sigma_{0,L})^0 \Big/ \mathcal{L}(\Sigma_{0,L})^1 \right)' \longrightarrow \mathrm{Ind}_B^G(\rho^2) \longrightarrow 0 \ .$$

For $\mathcal{L} = \mathcal{O}$ (resp. $\mathcal{L} = \Omega$) this is equivalent to the exact sequence in (i) (resp. (ii)). $\qquad \square$

For $n \in \mathbb{Z}_{\geqslant 0}$ put

$$\mathcal{O}(\Sigma_{0,L})_n = \left([\mathcal{O}(\Sigma_{0,L})']_{\mathbb{G}(n)^\circ - \mathrm{an}} \right)', \quad \Omega(\Sigma_{0,L})_n = \left([\Omega(\Sigma_{0,L})']_{\mathbb{G}(n)^\circ - \mathrm{an}} \right)',$$

and denote by St_n the space of G_{n+1}-invariant vectors in St.

Corollary 6.4.4. *Let n be a non-negative integer.*

(i) There is a canonical exact sequence of G_0-equivariant sheaves of $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$-modules on $\mathfrak{X}_n$

$$0 \longrightarrow \mathcal{O}_{\mathfrak{X}_n,\mathbb{Q}} \longrightarrow \mathscr{L}oc_n^\dagger\left(\mathcal{O}(\Sigma_{0,L})_n \right) \longrightarrow \oplus_{g \in C(n)} \, g_* \mathrm{Loc}(M(\rho^{-2}))_n^\dagger \longrightarrow 0 \, .$$

(ii) There is a canonical exact sequence of G_0-equivariant sheaves of $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$-modules on $\mathfrak{X}_n$

$$0 \longrightarrow \oplus_{g \in C(n)} \, g_* \mathrm{Loc}(M(\rho^{-2}))_n^\dagger \longrightarrow \mathscr{L}oc_n^\dagger\left(\Omega(\Sigma_{0,L})_n \right) \longrightarrow \mathrm{St}_n' \otimes_L \mathcal{O}_{\mathfrak{X}_n,\mathbb{Q}} \longrightarrow 0 \, .$$

Proof. This follows from 6.4.2 together with 6.3.2. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.4.5. As was pointed out in the previous section, the sheaf on the right (resp. left) of the first (resp. second) exact sequence in 6.4.4 is a skyscraper sheaf whose support is the set $Z_{0,n} := \mathfrak{X}_n^{\mathrm{sm}}(\mathbb{F}_q)$ of smooth $\mathbb{F}_q$-rational points of $\mathfrak{X}_n$. Using [9], one can show that

$$\mathcal{O}(\Sigma_{0,L})_n = \mathcal{O}(\mathfrak{X}_n^{\circ,\mathrm{rig}})^\dagger$$

is the space of overconvergent rigid-analytic functions, i.e., the space of rigid-analytic functions on

$$\mathfrak{X}_n^{\circ,\mathrm{rig}} = \mathrm{sp}_{\mathfrak{X}_n}^{-1}\left(\mathfrak{X}_n^\circ \right)$$

which converge (to an unspecified extent) into the residue discs of the points in Z_n. It follows from this that the sheaf $\mathscr{L}oc_n^\dagger\left(\mathcal{O}(\Sigma_{0,L})_n \right)$ is equal to the sheaf $\mathcal{O}(^\dagger Z_{0,n})$ of functions on $\mathfrak{X}_n$ which are regular on $\mathfrak{X}_n^\circ$ and have overconvergent singularities along the closed subset $Z_{0,n}$. This is an overconvergent isocrystal on $\mathfrak{X}_n$. Similarly, one can show that

$$\Omega(\Sigma_{0,L})_n = \Omega(\mathfrak{X}_n^{\circ,\mathrm{rig}})^\dagger$$

is the space of overconvergent rigid-analytic differential forms on $\mathfrak{X}_n^{\circ,\mathrm{rig}}$. The sheaf of differential forms $\Omega(^\dagger Z_{0,n})$ on $\mathfrak{X}_n$ which are regular on $\mathfrak{X}_n^\circ$ and have overconvergent singularities along $Z_{0,n}$ is not an isocrystal, however. Otherwise the localization of the space of global sections (on $\mathfrak{X}_n$) of this sheaf, namely $\mathscr{L}oc_n^\dagger(\Omega(\mathfrak{X}_n^{\mathrm{rig}})^\dagger)$, would have to be equal

to $\Omega(^\dagger Z_{0,n})$. But it follows from the second exact sequence in 6.4.4 that the restriction of $\mathscr{L}oc_n^\dagger(\Omega(\mathfrak{X}_n^{\mathrm{rig}})^\dagger)$ to $\mathfrak{X}_n^\circ$ is locally free of rank $(q+1)q^n - 1$, and not locally free of rank one.

7. Representations furnished by an étale covering of the p-adic upper half plane

The theory in section 4 might be useful when one tries to decide whether certain modules M over $D(\mathrm{GL}_2(\mathfrak{o}))$ are coadmissible, especially if M 'comes from geometry' in a way that makes it accessible to the techniques used here. In this section we will show how the techniques from the theory of arithmetic differential operators can be used to prove that some particular locally analytic representations are admissible. The representations considered here are those furnished by the structure sheaf of the *first Drinfeld covering* of the p-adic upper half plane.

7.1. The first Drinfeld covering.

7.1.1. Denote by $\hat{L}^{\mathrm{nr}}$ the completion of the maximal unramified extension of L, and consider the base change $\Sigma_{0,L} \times_{\mathrm{Sp}(L)} \mathrm{Sp}(\hat{L}^{\mathrm{nr}})$ which we will from now on denote by Σ_0. This rigid space has a natural semistable formal model

$$\widehat{\Sigma}_0 = \varinjlim_n \mathfrak{X}_n^\circ \times_{\mathrm{Spf}(\mathfrak{o})} \mathrm{Spf}(\hat{\mathfrak{o}}^{\mathrm{nr}}) \, ,$$

where $\hat{\mathfrak{o}}^{\mathrm{nr}}$ is the ring of integers of $\hat{L}^{\mathrm{nr}}$, cf. section 6.4.1. By [13] (cf. also [7]) the formal scheme $\widehat{\Sigma}_0$ represents a deformation functor of a particular two-dimensional formal group F over $\hat{\mathfrak{o}}^{\mathrm{nr}}/(\varpi)$ which is equipped with an action of the maximal compact subring O_D of a quaternion division algebra D over L. Let $\varpi_D \subset O_D$ be a uniformizer. Denote by $\mathfrak{F} \to \widehat{\Sigma}_0$ the universal deformation of F, and let $\mathfrak{F}[\varpi_D^j] \subset \mathfrak{F}$ be the finite flat group scheme of ϖ_D^j-torsion points. Denote by $\mathfrak{F}[\varpi_D^j]^{\mathrm{rig}}$ the associated rigid analytic space. Note that $\mathfrak{F}[1]$ is just the zero section of $\mathfrak{F}$, which is isomorphic to $\widehat{\Sigma}_0$. Then

$$\Sigma_1 = \mathfrak{F}[\varpi_D]^{\mathrm{rig}} \setminus \mathfrak{F}[1]^{\mathrm{rig}} = \mathfrak{F}[\varpi_D]^{\mathrm{rig}} \setminus \Sigma_0 \, .$$

Multiplication by ϖ_D is a map $\mathfrak{F}[\varpi_D] \to \mathfrak{F}[1]$ and hence induces a morphism $f : \Sigma_1 \to \Sigma_0$ which is an étale covering space with Galois group $(\mathcal{O}_D/(\varpi_D))^* \simeq \mathbb{F}_{q^2}^*$. We will henceforth write $\mathbb{F}_{q^2}^*$ for this covering group.

Remark 7.1.2. The group $\mathrm{GL}_2(L)$ acts on $\Sigma_{0,L}$ by fractional linear transformations, and therefore also on $\Sigma_0 = \Sigma_{0,L} \times_{\mathrm{Sp}(L)} \mathrm{Sp}(\hat{L}^{\mathrm{nr}})$, and on the formal model $\widehat{\Sigma}_0$. However, this action, which is $\hat{\mathfrak{o}}^{\mathrm{nr}}$-linear, is *not* the action which is induced by the modular interpretation of $\widehat{\Sigma}_0$ as a deformation space, cf. [7, I.6]. However, the group

$$\mathrm{GL}_2(L)^{(0)} = \left\{ g \in \mathrm{GL}_2(L) \mid \det(g) \in \mathfrak{o}_L^* \right\}$$

does act $\hat{\mathfrak{o}}^{\mathrm{nr}}$-linearly on $\widehat{\Sigma}_0$, and $\hat{L}^{\mathrm{nr}}$-linearly on Σ_0, and the action of this group lifts to Σ_1. But in this section we will anyway only consider the action of $G_0 = \mathrm{GL}_2(\mathfrak{o}_L)$.

The subspaces $\Sigma_0(n)$ and $\Sigma_1(n)$. There is a map

$$r : \Sigma_0 \to |\mathcal{T}|$$

from Σ_0 to the geometric realization $|\mathcal{T}|$ of the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(L)$. We refer to [7, I.2.2] for the definition of this map (it is denoted by λ in this paper). Let $v_0 \in \mathcal{T}$ be the vertex which is fixed by $\mathrm{GL}_2(\mathfrak{o})$, and we denote by $|\mathcal{T}|_{\leqslant n} \subset |\mathcal{T}|$ the set of points whose distance to v_0 is less or equal to n. For instance, $|\mathcal{T}|_{\leqslant 0} = \{v_0\}$. Then

$$\Sigma_0(n) = r^{-1}\Big(|\mathcal{T}|_{\leqslant n}\Big) ,$$

is an affinoid subdomain of Σ_0, and is equal to the generic fiber of $\widehat{\Sigma}_0(n) := \mathfrak{X}_n^\circ \times_{\mathrm{Spf}(\mathfrak{o})} \mathrm{Spf}(\hat{\mathfrak{o}}^{\mathrm{nr}})$. We also put

$$\Sigma_1(n) = f^{-1}\Big(\Sigma_0(n)\Big) ,$$

and we let $f_n : \Sigma_1(n) \to \Sigma_0(n)$ denote the restriction of f.

7.1.3. *A semistable model for Σ_1.* In [42] J. Teitelbaum has shown that there is a finite (tamely ramified) extension K of $\hat{L}^{\mathrm{nr}}$ such that $\Sigma_{1,K} = \Sigma_1 \times_{\mathrm{Sp}(L)} \mathrm{Sp}(K)$ has a semistable formal model $\widehat{\Sigma}_1$ over $\mathrm{Spf}(\mathfrak{o}_K)$. It follows from [42, p. 72] that $\Sigma_{1,K}$ has $q - 1$ connected components, which are transitively permuted by the action of the covering group, and each of these is an étale covering of $\Sigma_{0,K}$ of degree $q + 1$. Consequently, the formal scheme $\widehat{\Sigma}_1$ has $q - 1$ connected components, each of which is finite of degree $q + 1$ over $\widehat{\Sigma}_0 \times_{\mathrm{Spf}(\mathfrak{o}_L)} \mathrm{Spf}(\mathfrak{o}_K)$.

As a general convention, we consider from now on the rigid spaces Σ_0, $\Sigma_0(n)$, Σ_1, $\Sigma_1(n)$, etc., as rigid spaces over K (i.e., we perform a base change to $\mathrm{Sp}(K)$), and we consider the formal schemes $\widehat{\Sigma}_0$, $\widehat{\Sigma}_0(n)$, etc., as formal schemes over $\mathrm{Spf}(\mathfrak{o}_K)$ (i.e., we perform a base change to $\mathrm{Spf}(\mathfrak{o}_K)$). The morphism $f : \Sigma_1 \to \Sigma_0$ extends to a finite flat morphism $\widehat{f} : \widehat{\Sigma}_1 \to \widehat{\Sigma}_0$ which is étale over the complement of the singular points of the special fiber of $\widehat{\Sigma}_0$. The irreducible components of the special fiber of $\widehat{\Sigma}_1$ are isomorphic to

$$(7.1.4) \qquad C = \left\{ [x : y : z] \in \mathbb{P}^2 \mid xy^q - x^q y = z^{q+1} \right\} .$$

According to [12, sec. 2], it was V. Drinfeld who proved that the discrete series representations of $\mathrm{SL}_2(\mathbb{F}_q)$ are furnished by the ℓ-adic étale cohomology of the affine curve $xy^q - x^q y = 1$. Each of these projective curves C lies over a unique irreducible component

of the special fiber of $\widehat{\Sigma}_0$, isomorphic to $\mathbb{P}^1$, and there are $q - 1$ such curves C over each irreducible component of the special fiber of $\widehat{\Sigma}_0$. The induced morphism from C to $\mathbb{P}^1$, considered as an irreducible component, is given by the formula $[x : y : z] \mapsto [x : y]$. The formal scheme

$$\widehat{\Sigma}_1(n) \overset{\text{def}}{=} \widehat{\Sigma}_1 \times_{\widehat{\Sigma}_0} \widehat{\Sigma}_0(n)$$

is then a formal model for $\Sigma_1(n)$, and it is an open formal subscheme of $\widehat{\Sigma}_1$. The morphism $\widehat{f}$ restricts to a morphism

$$\widehat{f}_n : \widehat{\Sigma}_1(n) \longrightarrow \widehat{\Sigma}_0(n)$$

which is simply the projection from $\widehat{\Sigma}_1(n) = \widehat{\Sigma}_1 \times_{\widehat{\Sigma}_0} \widehat{\Sigma}_0(n)$ to the second factor. Denote by $\widehat{\Sigma}_1(n)^c$ the formal completion of $\widehat{\Sigma}_1$ along the *closure* of $\widehat{\Sigma}_1(n)$. Similarly, let $\widehat{\Sigma}_0(n)^c$ be the completion of $\widehat{\Sigma}_0$ along the *closure* of $\widehat{\Sigma}_0(n)$. The morphism $\widehat{f}_n$ extends uniquely to a morphism $\widehat{f}_n^c : \widehat{\Sigma}_1(n)^c \to \widehat{\Sigma}_0(n)^c$ such that one has a commutative diagram of morphisms of formal schemes

$$(7.1.5) \qquad \begin{array}{ccccc} \widehat{\Sigma}_1(n) & \longrightarrow & \widehat{\Sigma}_1(n)^c & \longrightarrow & \widehat{\Sigma}_1 \\ \downarrow{\scriptstyle \widehat{f}_n} & & \downarrow{\scriptstyle \widehat{f}_n^c} & & \downarrow{\scriptstyle \widehat{f}} \\ \widehat{\Sigma}_0(n) & \longrightarrow & \widehat{\Sigma}_0(n)^c & \longrightarrow & \widehat{\Sigma}_0 \end{array}$$

The special fiber of a formal scheme will always be denoted by the subscript "s", e.g., $\widehat{\Sigma}_1(n)_s$ denotes the special fiber of $\widehat{\Sigma}_1(n)$. When it is clear from the context that it is only the special fiber of a given formal scheme $\mathcal{Y}$ which matters, we may drop the index "s" and write $\mathcal{Y}$ instead of $\mathcal{Y}_s$. Put $Z_{1,n} = \widehat{\Sigma}_1(n)_s^c \setminus \widehat{\Sigma}_1(n)_s$. This is a finite set of points of $\widehat{\Sigma}_{1,s}$.

7.1.6. *Sheaves of overconvergent functions.* Let

$$\mathrm{sp}_{\widehat{\Sigma}_1} : \Sigma_1 \longrightarrow \widehat{\Sigma}_1$$

be the specialization map, and let

$$v_n : \widehat{\Sigma}_1(n)_s \hookrightarrow \widehat{\Sigma}_{1,s}$$

be the open immersion. Associated to this morphism is the sheaf $v_n^\dagger \mathcal{O}_{\Sigma_1}$ on Σ_1, cf. [6, 4.0.1], and we consider the sheaf

$$(7.1.7) \qquad\qquad \mathscr{F}_n \overset{\text{def}}{=} (\mathrm{sp}_{\widehat{\Sigma}_1})_* v_n^\dagger \mathcal{O}_{\Sigma_1}$$

on $\widehat{\Sigma}_{1,s}$ which is supported on $\widehat{\Sigma}_1(n)_s^c$. Another way to describe $\mathscr{F}_n$ is as the sheaf of rigid-analytic functions on $\widehat{\Sigma}_1(n)^c$ which are regular on (the tubular neighborhood of) $\widehat{\Sigma}_1(n)$ and have overconvergent singularities along $Z_{1,n}$:

$$\mathscr{F}_n = \mathcal{O}_{\widehat{\Sigma}_1(n)^c, \mathbb{Q}}\left(^\dagger Z_{1,n}\right) .$$

By its very definition, the space of global sections of $\mathscr{F}_n$ on $\widehat{\Sigma}_1$ is equal to the space of overconvergent functions on $\Sigma_1(n)$, which we denote by $H^0(\Sigma_1(n), \mathcal{O})^\dagger$, i.e.,

$$H^0\left(\widehat{\Sigma}_1, \mathscr{F}_n\right) = H^0\left(\widehat{\Sigma}_1(n)^c, \mathscr{F}_n\right) = H^0(\Sigma_1(n), \mathcal{O})^\dagger .$$

7.2. **Main result: coadmissibility of the space of global functions.** Recall that G_0 denotes the group $\mathrm{GL}_2(\mathfrak{o}_L)$. Let $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ)^{\mathrm{nr}}$ be the completed tensor product over L of $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^\circ)$ with the finite extension K of $\hat{L}^{\mathrm{nr}}$. Similarly, denote by $D(\mathbb{G}(n)^\circ, G_0)^{\mathrm{nr}}$ resp. $D(G_0, K)$ the completed tensor product of $D(\mathbb{G}(n)^\circ, G_0)$ resp. $D(G_0)$ with K over L.

Theorem 7.2.1. *(i) For every $n \in \mathbb{Z}_{\geqslant 0}$ the space $H^0(\Sigma_1(n), \mathcal{O})^\dagger$ is a $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)^{\mathrm{nr}}$-module of finite presentation.*

(ii) For every $n \in \mathbb{Z}_{\geqslant 0}$ the space $H^0(\Sigma_1(n), \mathcal{O})^\dagger$ is a finitely generated $D(\mathbb{G}(n)^\circ, G_0)^{\mathrm{nr}}$-module.

(iii) For every $n \in \mathbb{Z}_{\geqslant 1}$ the restriction map

$$H^0(\Sigma_1(n), \mathcal{O})^\dagger \longrightarrow H^0(\Sigma_1(n-1), \mathcal{O})^\dagger$$

induces an isomorphism of $D(\mathbb{G}(n)^\circ, G_0)^{\mathrm{nr}}$-modules

$$(7.2.2) \qquad D(\mathbb{G}(n-1)^\circ, G_0)^{\mathrm{nr}} \otimes_{D(\mathbb{G}(n)^\circ, G_0)^{\mathrm{nr}}} H^0(\Sigma_1(n), \mathcal{O})^\dagger \xrightarrow{\simeq} H^0(\Sigma_1(n-1), \mathcal{O})^\dagger .$$

(iv) The space $H^0(\Sigma_1, \mathcal{O})$ is a coadmissible $D(G_0, K)$-module.

Proof. (i) This will be shown in the following section 7.3, where we treat first the case when $n = 0$, and in section 7.4, which deals with the case of general n.

(ii) This follows from (i) because $D(\mathbb{G}(n)^\circ, G_0)^{\mathrm{nr}}$ contains $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ)^{\mathrm{nr}}$.

(iii) This will be shown in section 7.5.

(iv) This is a consequence of (ii) and (iii), by [14, 6.1.20]. $\qquad\qquad\square$

Consider a character $\chi : \mathbb{F}_{q^2}^* \to K^*$ of the covering group $\mathbb{F}_{q^2}^*$, and denote by $H^0(\Sigma_1, \mathcal{O})^\chi$ the χ-isotypic component of $H^0(\Sigma_1, \mathcal{O})$. As G_0 commutes with the action of $\mathbb{F}_{q^2}^*$, this is a $D(G_0, K)$-submodule of $H^0(\Sigma_1, \mathcal{O})$, and we have the decomposition of $D(G_0, K)$-modules

$$H^0(\Sigma_1, \mathcal{O}) = \bigoplus_\chi H^0(\Sigma_1, \mathcal{O})^\chi \, ,$$

where χ runs over all characters of $\mathbb{F}_{q^2}^*$.

Corollary 7.2.3. *For every character $\chi : \mathbb{F}_{q^2}^* \to K^*$ the space $H^0(\Sigma_1, \mathcal{O})^\chi$ is a coadmissible $D(G_0, K)$-module.*

Proof. The projection map

$$H^0(\Sigma_1, \mathcal{O}) \twoheadrightarrow H^0(\Sigma_1, \mathcal{O})^\chi \, , \quad f \mapsto \frac{1}{q^2 - 1} \sum_{\zeta \in \mathbb{F}_{q^2}^*} \chi(\zeta) \cdot (\zeta^{-1}.f) \, ,$$

is continuous and has as section the natural inclusion. Hence each isotypic component $H^0(\Sigma_1, \mathcal{O})^\chi$ is the quotient of a coadmissible module by a closed submodule, and is thus coadmissible too. $\qquad\qquad\square$

Remark 7.2.4. Consider the de Rham sequence

$$0 \longrightarrow K \longrightarrow H^0(\Sigma_1, \mathcal{O}) \longrightarrow H^0(\Sigma_1, \Omega) \longrightarrow H^1_{\mathrm{dR}}(\Sigma_1/K) \longrightarrow 0 \, .$$

It is exact because Σ_1 is a Stein space (as is Σ_0). One can show that the de Rham cohomology on the right is the (algebraic) dual space of a finite number of irreducible smooth representations of $\mathrm{GL}_2(L)^{(0)}$ (cf. 7.1.2 for the notation)[7]. It follows from [38, 3.2] that the category of coadmissible modules is closed under extensions. Therefore, 7.2.1 implies that $H^0(\Sigma_1, \Omega)$ is a coadmissible $D(G_0, K)$-module, and the same is true for the isotypic components $H^0(\Sigma_1, \Omega)^\chi$.

Remark 7.2.5. Denote by $\mathfrak{X}_n^{\mathrm{nr}}$ and $\mathfrak{X}_0^{\mathrm{nr},\circ}$ the fibre products $\mathfrak{X}_n \times_{\mathrm{Spf}(\mathfrak{o})} \mathrm{Spf}(\mathfrak{o}_K)$ and $\mathfrak{X}_n^\circ \times_{\mathrm{Spf}(\mathfrak{o})} \mathrm{Spf}(\mathfrak{o}_K)$, respectively. Making this base change is necessary later on to relate $\mathfrak{X}_n^{\mathrm{nr}}$ to $\widehat{\Sigma}_0$. Indeed, in the following we will often consider certain sheaves on $\widehat{\Sigma}_0(n)_s^c$ as sheaves on the special fiber $\mathfrak{X}_{n,s}^{\mathrm{nr}}$ of $\mathfrak{X}_n^{\mathrm{nr}}$. This will in fact be one of the key features of the arguments which will be employed in 7.3.1 and 7.4.11 below. We will here elaborate on this relation.

According to our conventions set up in 7.1.3, we have, by definiton, $\widehat{\Sigma}_0(n) = \mathfrak{X}_n^{\mathrm{nr},\circ}$. It therefore suffices to consider only points in $\widehat{\Sigma}_0(n)_s^c \backslash \widehat{\Sigma}_0(n)_s$.

---

[7]As far as we know, a proof of this has not been written up so far. It should be possible to do so by using the work of E. Grosse-Klönne on Hyodo-Kato cohomology for such spaces as Σ_1.

Let us consider the blow-up morphism $\mathrm{pr}_{n+1,n} : \mathfrak{X}^{\mathrm{nr}}_{n+1} \to \mathfrak{X}^{\mathrm{nr}}_n$. This morphism induces an isomorphism $\mathrm{pr}^{-1}_{n+1,n}\left(\mathfrak{X}^{\mathrm{nr},\circ}_n\right) \xrightarrow{\sim} \mathfrak{X}^{\mathrm{nr},\circ}_n$. The closure of $\widehat{\Sigma}_0(n)_s = \mathfrak{X}^{\mathrm{nr},\circ}_{n,s}$ in $\widehat{\Sigma}_{0,s}$ is the same as the closure of $\mathrm{pr}^{-1}_{n+1,n}\left(\mathfrak{X}^{\mathrm{nr},\circ}_{n,s}\right)$ in $\mathfrak{X}^{\mathrm{nr}}_{n+1,s}$, i.e., it is the *strict transform* of $\mathfrak{X}^{\mathrm{nr},\circ}_{n,s}$.

In order to describe the situation locally, we let $\mathfrak{A} = \mathrm{Spf}(\mathfrak{o}_K\langle x\rangle) \subset \mathfrak{X}^{\mathrm{nr}}_n$ be the smooth locus of an outermost component of $\mathfrak{X}^{\mathrm{nr}}_n$. We consider the point in the special fiber where x vanishes. Blowing up $\mathfrak{A}$ in this point leads to a formal scheme $\mathfrak{A}'$ which is glued together from $\mathrm{Spf}\left(\mathfrak{o}_K\langle x, y\rangle/(xy - \varpi)\right)$ and $\mathrm{Spf}\left(\mathfrak{o}_K\langle x, z\rangle/(x - \varpi z)\right)$. These two formal schemes are glued together along the open subsets where y and z, respectively, are invertible, according to the relation $yz = 1$. Denote by k the residue field of K (it is an algebraic closure of the residue field $\mathbb{F}_q$ of L). It is easy to see that the strict transform of the special fiber $\mathfrak{A}_s = \mathrm{Spec}(k[x])$ in $\mathfrak{A}'$ is the closed subscheme of $\mathfrak{A}'$ defined by the ideal (ϖ, y), which is $\mathrm{Spec}\left(\mathfrak{o}_K\langle x, y\rangle/(xy - \varpi)\big/(y, \varpi)\right) = \mathrm{Spec}(k[x])$. Completing $\mathfrak{A}'$ along this closed subscheme gives the formal scheme $\mathfrak{A}^c := \mathrm{Spf}\left(\mathfrak{o}_K\langle x\rangle[[y]]/(xy - \varpi)\right)$ (which is not a p-adic formal scheme). The upshot is then that the open immersion $\mathfrak{A}^\circ := \mathrm{Spf}\left(\mathfrak{o}_K\langle x, \frac{1}{x}\rangle\right) \hookrightarrow \mathfrak{A}$ factors as follows: $\mathfrak{A}^\circ \hookrightarrow \mathfrak{A}^c \to \mathfrak{A}$, and the morphism $\mathfrak{A}^c \to \mathfrak{A}$ corresponds on affine coordinate rings to the ring homomorphism $\mathfrak{o}_K\langle x\rangle \to \mathfrak{o}_K\langle x\rangle[[y]]/(xy - \varpi)$. The induced morphism on the special fibers of these formal schemes (modulo their largest ideals of definition) is an isomorphism: $\mathfrak{A}^c_s \xrightarrow{\sim} \mathfrak{A}_s$.

The induced morphism of rigid analytic spaces $\mathfrak{A}^{c,\mathrm{rig}} \to \mathfrak{A}^{\mathrm{rig}}$ corresponds to the inclusion of the semi-open annulus $\{|\varpi| < |x| \leqslant 1\}$ into the disc $\{|x| \leqslant 1\}$. Moreover, the family of strict neighborhoods of $\mathfrak{A}^{\circ,\mathrm{rig}} = \mathrm{Spf}\left(\mathfrak{o}_K\langle x, \frac{1}{x}\rangle\right)^{\mathrm{rig}} = \{|x| = 1\}$ in the annulus $\mathfrak{A}^{c,\mathrm{rig}} \simeq \{|\varpi| < |x| \leqslant 1\}$ is cofinal in the family of strict neighborhoods of $\mathfrak{A}^{\circ,\mathrm{rig}}$ in $\mathfrak{A}^{\mathrm{rig}} \simeq \{|x| \leqslant 1\}$, because $\mathfrak{A}^{c,\mathrm{rig}}$ is itself a strict neighborhood of $\mathfrak{A}^{\circ,\mathrm{rig}}$. We summarize the local situation by the following diagram in which the upper horizontal arrows are open immersions of rigid analytic spaces, and where the vertical arrows are specialization morphisms

$$
(7.2.6) \qquad
\begin{array}{ccccc}
\mathfrak{A}^{\circ,\mathrm{rig}} & \longrightarrow & \mathfrak{A}^{c,\mathrm{rig}} & \longrightarrow & \mathfrak{A}^{\mathrm{rig}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{A}^\circ_s = \mathrm{Spec}(k[x, x^{-1}]) & \longrightarrow & \mathfrak{A}^c_s & \xrightarrow{\simeq} & \mathfrak{A}_s
\end{array}
$$

From this local description we deduce the following. Firstly, the open immersion $\widehat{\Sigma}_0(n) = \mathfrak{X}^{\mathrm{nr},\circ}_n \hookrightarrow \mathfrak{X}^{\mathrm{nr}}_n$ factors as follows: $\widehat{\Sigma}_0(n) \hookrightarrow \widehat{\Sigma}_0(n)^c \to \mathfrak{X}^{\mathrm{nr}}_n$. Moreover, the morphism induced by $\widehat{\Sigma}_0(n)^c \to \mathfrak{X}^{\mathrm{nr}}_n$ on the special fibers of these formal schemes is an isomorphism: $\widehat{\Sigma}_0(n)^c_s \xrightarrow{\sim} \mathfrak{X}^{\mathrm{nr}}_{n,s}$.

Let $]\widehat{\Sigma}_0(n)_s^c[$ be the tubular neighborhood of $\widehat{\Sigma}_0(n)_s^c$ in Σ_0, i.e., $]\widehat{\Sigma}_0(n)_s^c[= \mathrm{sp}_{\widehat{\Sigma}_0}^{-1}\left(\widehat{\Sigma}_0(n)_s^c\right)$. This rigid analytic space is canonically isomorphic to the rigid-analytic space associated to the formal scheme $\widehat{\Sigma}_0(n)^c$. And the morphism of formal schemes $\widehat{\Sigma}_0(n)^c \to \mathfrak{X}_n^{\mathrm{nr}}$ induces an open embedding of rigid analytic spaces $]\widehat{\Sigma}_0(n)_s^c[\hookrightarrow (\mathfrak{X}_n^{\mathrm{nr}})^{\mathrm{rig}} \simeq \mathbb{P}_K^{1,\mathrm{rig}}$. Finally the system of strict neighborhood of $\Sigma_0(n)$ in $]\widehat{\Sigma}_0(n)_s^c[$ (or Σ_0) is cofinal in the system of all strict neighborhoods of $\Sigma_0(n)$ in $(\mathfrak{X}_n^{\mathrm{nr}})^{\mathrm{rig}}$, because $]\widehat{\Sigma}_0(n)_s^c[$ is itself a strict neighborhood of $\Sigma_0(n)$. This is why an overconvergent sheaf on $(\widehat{\Sigma}_0(n)_s, \widehat{\Sigma}_0(n)_s^c)$ is the same as an overconvergent sheaf on $(\mathfrak{X}_{n,s}^{\mathrm{nr},\circ}, \mathfrak{X}_{n,s}^{\mathrm{nr}})$. We summarize the situation by the following diagram in which the upper horizontal arrows are open immersions of rigid analytic spaces, and where the vertical arrows are specialization morphisms

(7.2.7)
$$\begin{array}{ccccc}
\Sigma_0(n) & \longrightarrow &]\widehat{\Sigma}_0(n)_s^c[& \longrightarrow & (\mathfrak{X}_n^{\mathrm{nr}})^{\mathrm{rig}} \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{\Sigma}_0(n)_s & \longrightarrow & \widehat{\Sigma}_0(n)_s^c & \overset{\simeq}{\longrightarrow} & \mathfrak{X}_{n,s}^{\mathrm{nr}}
\end{array}$$

7.3. The starting point: the affinoid $\Sigma_1(0)$.

The case when $n = 0$ is somewhat easier to deal with because $\widehat{\Sigma}_1(0)_s^c$ is smooth, since it is a disjoint union of $q - 1$ copies of the curve in 7.1.4. Moreover, the morphism $\widehat{f}_{0,s} : \widehat{\Sigma}_1(0)_s \to \widehat{\Sigma}_0(0)_s$ is smooth too. It is because of these simplifications that we treat this case beforehand. Let $\mathscr{D}_{\mathfrak{X}_0^{\mathrm{nr}},\mathbb{Q}}^{\dagger}$ be the sheaf of arithmetic differential operators on the smooth formal scheme $\mathfrak{X}_0^{\mathrm{nr}}$.

Proposition 7.3.1. *The space* $H^0\left(\Sigma_1(0), \mathcal{O}\right)^{\dagger}$ *is a finitely presented* $\mathcal{D}^{\mathrm{an}}(\mathbb{G}(0)^{\circ})^{\mathrm{nr}}$-*module.*

Proof. Recall the sheaf $\mathscr{F}_0 = (\mathrm{sp}_{\widehat{\Sigma}_1})_* v_0^{\dagger} \mathcal{O}_{\Sigma_1}$ introduced in 7.1.7. This sheaf on $\widehat{\Sigma}_1(0)_s^c$ depends only on $\widehat{\Sigma}_1(0)_s$ which is a disjoint union of affine Deligne-Lusztig curves $xy^q - x^q y = 1$. By [18, sec. 2], $\mathscr{F}_0$ is an overconvergent F-isocrystal on the pair $(\widehat{\Sigma}_{1,s}(0), \widehat{\Sigma}_1(0)_s^c)$, where we use here the notation of [43]. It is overconvergent along $Z_{1,0} = \widehat{\Sigma}_1(0)_s^c \setminus \widehat{\Sigma}_1(0)_s$. Consider the diagram

(7.3.2)
$$\begin{array}{ccccc}
\widehat{\Sigma}_1(0)_s & \longrightarrow & \widehat{\Sigma}_1(0)_s^c & \longrightarrow & \widehat{\Sigma}_1 \\
\downarrow{\scriptstyle \widehat{f}_{0,s}} & & \downarrow{\scriptstyle \widehat{f}_{0,s}^c} & & \downarrow{\scriptstyle \widehat{f}} \\
\widehat{\Sigma}_0(0)_s & \longrightarrow & \widehat{\Sigma}_0(0)_s^c & \longrightarrow & \widehat{\Sigma}_0
\end{array}$$

cf. 7.1.5. By [43, Thm. 4.1.4], the direct image $(\widehat{f}_0^c)_*\mathscr{F}_0$ is an overconvergent F-isocrystal on the pair $(\widehat{\Sigma}_0(0)_s, \widehat{\Sigma}_0(0)_s^c)$. The preceding diagram 7.3.2 corresponds to the diagram in [43, Thm. 4.1.1]. Note that $\widehat{\Sigma}_0(0)_s^c$ is isomorphic to the special fiber of $\mathfrak{X}_0^{nr}$, which is projective line over the residue field of $\mathfrak{o}_K$ (compare the discussion in section 7.2.5). We can thus consider $(\widehat{f}_0^c)_*\mathscr{F}_0$ as an overconvergent isocrystal on $(\mathfrak{X}_{0,s}^{nr,\circ}, \mathfrak{X}_{0,s}^{nr})$. In particular, it carries the structure of a module over $\mathscr{D}_{\mathfrak{X}_0^{nr},\mathbb{Q}}^\dagger$.

By [8, Thm. 4.3.5] or [28, Prop. 3.1], the $\mathscr{D}_{\mathfrak{X}_0^{nr},\mathbb{Q}}^\dagger$-module $(\widehat{f}_0^c)_*\mathscr{F}_0$ on $\mathfrak{X}_{0,s}^{nr}$, is holonomic, and, in particular, coherent. By the main result of [23] (or by 4.3.3), the space of global sections of this sheaf, namely

$$H^0\left(\mathfrak{X}_{0,s}^{nr}, (\widehat{f}_0^c)_*\mathscr{F}_0\right) = H^0\left(\Sigma_1(0), \mathcal{O}\right)^\dagger,$$

is a finitely presented $H^0(\mathfrak{X}_{0,s}^{nr}, \mathscr{D}_{\mathfrak{X}_0^{nr},\mathbb{Q}}^\dagger)$-module. Using 3.3.7 we see that it is a finitely presented $\mathcal{D}^{an}(\mathbb{G}(0)^\circ)_{\theta_0}^{nr}$-module. $\qquad\square$

7.4. **Extending the arguments to $\Sigma_1(n)$ for all n.** In this section we prove 7.2.1 for all n. To simplify notation we will occasionally write $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$ instead of $\widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^\dagger$. We recall that the special fiber $\widehat{\Sigma}_0(n)_s^c$ of $\widehat{\Sigma}_0(n)^c$ is canonically isomorphic to the special fiber of $\mathfrak{X}_n^{nr}$, cf. 7.2.5.

7.4.1. *Three types of closed points.* We want to show that $(\widehat{f}_n^c)_*\mathscr{F}_n$ is a coherent $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$-module on $\mathfrak{X}_n^{nr}$. Once we have this result at hand we can apply 4.3.1 to deduce that

$$H^0(\mathfrak{X}_{n,s}^{nr}, (\widehat{f}_n^c)_*\mathscr{F}_n) = H^0(\Sigma_1(n), \mathcal{O})^\dagger$$

is a coherent $\mathcal{D}^{an}(\mathbb{G}(n)^\circ)_{\theta_0}$-module and we are done. In order to show that $(\widehat{f}_n^c)_*\mathscr{F}_n$ is a coherent $\widetilde{\mathscr{D}}_{n,\mathbb{Q}}^\dagger$-module, we distinguish the following types of closed points on $\mathfrak{X}_{n,s}^{nr}$:

(a) Points x which lie on the smooth locus of $\mathfrak{X}_{n,s}^{nr}$, but not in $\mathfrak{X}_n(\mathbb{F}_q)$.

(b) Points x which lie on the singular locus of $\mathfrak{X}_{n,s}^{nr}$.

(c) Points x which lie on the smooth locus of $\mathfrak{X}_{n,s}^{nr}$ and in $\mathfrak{X}_n(\mathbb{F}_q)$. These are exactly the smooth $\mathbb{F}_q$-rational points of $\mathfrak{X}_n$, the set of which we denote by $\mathfrak{X}_n^{sm}(\mathbb{F}_q)$.

For any open subscheme U of $\mathbb{X}_n$ we let $\mathcal{D}_{U^{\log}}^{(m)}$ denote the usual sheaf of logarithmic arithmetic differential operators of level m on U as considered in [33],[26]. We write $\mathscr{D}_{U^{\log}}^{(m)}$ for its p-adic completion and $\mathscr{D}_{U^{\log},\mathbb{Q}}^\dagger$ for the union of the $\mathscr{D}_{U^{\log},\mathbb{Q}}^{(m)}$. We use the same notations for an open formal subscheme U of $\mathfrak{X}_n$, that is, we write $\mathcal{D}_{U^{\log}}^{(m)}$ for the usual sheaf of logarithmic arithmetic differential operators of level m on U etc. Finally, we

denote the $\mathcal{O}_{\mathfrak{X}_n}$-module given as the base change of $\widetilde{\mathcal{D}}_{n,k}^{(m)}$ along the morphism of ringed spaces $\mathfrak{X}_n \subset \mathbb{X}_n$ by the same symbol $\widetilde{\mathcal{D}}_{n,k}^{(m)}$, when no confusion is likely. It is a sheaf of rings on $\mathfrak{X}_n$.

Proposition 7.4.2. *Let U be an open formal subscheme of $\mathfrak{X}_n$ and let $\mathring{\mathscr{E}}$ be a $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$-module which is coherent as an $\mathcal{O}_U$-module. Then:*

(i) $\mathring{\mathscr{E}}$ is coherent over $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$,

(ii) the canonical homomorphism

$$\mathring{\mathscr{E}} \xrightarrow{\simeq} \widetilde{\mathscr{D}}_{n,n}^{(m)}|_U \otimes_{\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U} \mathring{\mathscr{E}}$$

given by $a \mapsto 1 \otimes a$ is an isomorphism.

Proof. We follow the arguments of [6, Prop. 3.1.3]. We may assume that U is affine and such that the ring $\widetilde{\mathcal{D}}_{n,n}^{(m)}(U) := \Gamma(U, \widetilde{\mathcal{D}}_{n,n}^{(m)})$ is noetherian, cf. Prop. 3.2.1. Since $\mathring{\mathscr{E}}$ is coherent over $\mathcal{O}_U$, there is a surjection $\mathcal{O}_U^{\oplus r} \to \mathring{\mathscr{E}}$ and hence a surjection $(\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U)^{\oplus r} \to \mathring{\mathscr{E}}$. Let $\mathcal{N}$ be its kernel. The subsheaf $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U \subset \mathcal{D}_{U^{\log}}^{(m)}$ has the induced order filtration $F_i(\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U)$, $i \geqslant 0$ which is a filtration by coherent $\mathcal{O}_U$-submodules. Letting $\mathcal{N}_i$ be the kernel of the morphism $(F_i(\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U))^{\oplus r} \to \mathring{\mathscr{E}}$, it follows that $\mathcal{N}$ has a filtration by coherent $\mathcal{O}_U$-modules. Since $\widetilde{\mathcal{D}}_{n,n}^{(m)}(U)$ is noetherian, there is a surjection $(\widetilde{\mathcal{D}}_{n,n}^{(m)}(U))^{\oplus s} \to \Gamma(U, \mathcal{N})$. Since $\Gamma(U, .)$ commutes with inductive limits, one obtains from this a surjection $(\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U)^{\oplus s} \to \mathcal{N}$. The $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$-module $\mathring{\mathscr{E}}$ has therefore a finite presentation. Since $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$ is a sheaf of coherent rings [5, (3.1.1)], any module of finite presentation is coherent. This shows (i).

The map in (ii) is injective, since the module structure on $\mathring{\mathscr{E}}$ extends by continuity from $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$ to $\widetilde{\mathscr{D}}_{n,n}^{(m)}|_U$. We put $\partial := \varpi^n \partial_x$, where x is the standard coordinate on $\mathbb{X}_0$, so that $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$ is generated over $\mathcal{O}_U$ by $\partial^{\langle d \rangle} := \frac{q_d^{(m)}!}{d!} \partial^d$ for $d \geqslant 0$. In particular, $F_i(\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U) = \sum_{d \leqslant i} a_d \partial^{\langle d \rangle}$ with local sections a_d of $\mathcal{O}_U$. Put $\partial' := \partial^{\langle p^m \rangle}$. Given an integer $r < p^m$, one has $\partial^{\langle r \rangle} \partial^{\langle q \rangle} = u_{q,r} \partial^{\langle p^m q + r \rangle}$ where $u_{q,r}$ is a p-adic unit. This implies that any operator $P \in \widetilde{\mathscr{D}}_{n,n}^{(m)}|_U$ can be written in the form $P = \sum_k B_k \partial'^k$ where $B_k \in \widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$ is an operator of order $< p^m$ and B_k tends p-adically to zero for $k \to \infty$. Let e be a local section of $\mathring{\mathscr{E}}$. Since $\mathring{\mathscr{E}}$ is coherent over $\mathcal{O}_U$, the $\mathcal{O}_U$-submodule generated by the elements $\partial'^k.e$ for $k \geqslant 0$ is generated by finitely many of those elements. Thus, there is a number ℓ and sections $a_0, ..., a_{\ell-1}$ of $\mathcal{O}_U$ and a relation

$$(\partial'^\ell - \sum_{j < \ell} a_j \partial'^j).e = 0.$$

It follows as in the proof of [6, 3.1.3] that for any element B_k there are operators $R_k \in \widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$ of order $< p^m \ell$ and $Q_k \in \widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$ such that

$$B_k \partial'^k = Q_k(\partial'^\ell - \sum_{j<\ell} a_j \partial'^j) + R_k$$

with Q_k and R_k tending p-adically to zero if B_k tends p-adically to zero. It follows that $P \otimes e = 1 \otimes (\sum_k R_k).e$ in $\widetilde{\mathscr{D}}_{n,n}^{(m)}|_U \otimes_{\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U} \mathring{\mathscr{E}}$ which proves the surjectivity in (ii).   □

Corollary 7.4.3. *Let U be an open formal subscheme of $\mathfrak{X}_n$ and let $\mathring{\mathscr{E}}$ be coherent $\mathcal{O}_U$-module. Put $\mathscr{E} := \mathring{\mathscr{E}} \otimes \mathbb{Q}$. Then:*

(i) If $\mathring{\mathscr{E}}$ is a $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$-module for some m then the canonical homomorphism

$$\mathscr{E} \xrightarrow{\simeq} \widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^{(m)}|_U \otimes_{\widetilde{\mathcal{D}}_{n,n,\mathbb{Q}}^{(0)}|_U} \mathscr{E}$$

is an isomorphism and $\mathscr{E}$ is coherent over $\widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^{(m)}|_U$.

(ii) If $\mathring{\mathscr{E}}$ is a $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$-module for any m then the canonical homomorphism

$$\mathscr{E} \xrightarrow{\simeq} \widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^\dagger|_U \otimes_{\widetilde{\mathcal{D}}_{n,n,\mathbb{Q}}^{(0)}|_U} \mathscr{E}$$

is an isomorphism and $\mathscr{E}$ is coherent over $\widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^\dagger|_U$.

Proof. Since $\widetilde{\mathcal{D}}_{n,n}^{(m)}$ and $\widetilde{\mathcal{D}}_{n,n}^{(0)}$ agree upon tensoring with $\mathbb{Q}$, the isomorphism in claim (i) follows from tensoring the isomorphism in part (ii) of 7.4.2 with $\mathbb{Q}$. Taking the limit over m gives the isomorphism in claim (ii). It remains to prove the assertions about coherence. According to part (i) of 7.4.2, the module $\mathring{\mathscr{E}}$ is coherent over $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$. Passing to an open subspace of U we may therefore assume that there is a finite presentation

$$(\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U)^{\oplus s} \to (\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U)^{\oplus r} \to \mathring{\mathscr{E}} \to 0 \ .$$

Tensoring with $\mathbb{Q}$ gives a presentation of $\mathscr{E}$ over $\widetilde{\mathcal{D}}_{n,n,\mathbb{Q}}^{(0)}|_U$. Tensoring the latter with $\widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^{(m)}|_U$ or $\widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^\dagger|_U$ and gives a finite presentation of $\mathscr{E}$ over $\widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^{(m)}|_U$ or $\widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^\dagger|_U$. Since the latter sheaves are sheaves of coherent rings, any module of finite presentation is coherent. This completes the proof.   □

Let $\mathcal{R} \subset \mathfrak{o}$ be any system of representatives for $\mathfrak{o}/\varpi\mathfrak{o}$ and put $\mathcal{R}_\infty = \mathcal{R} \cup \{\infty\}$. Let $1 \leqslant \nu \leqslant n$. Each vector $\underline{a} \in \mathcal{R}_\infty \times \mathcal{R}^{\nu-1}$ determines an outermost component of $\mathbb{X}_n$ and each $a_n \in \mathcal{R}$ determines a smooth $\mathbb{F}_q$-rational point in this component. Proceeding as in the case $\mathfrak{o} = \mathbb{Z}_p$ [33, 4.3] we may construct the analogous open subscheme $\mathbb{X}_{\underline{a}}^{(\nu)}$ of $\mathbb{X}_n^\circ$, endowed with a local coordinate $x_{\underline{a}}^{(\nu-1)}$, and the smooth open subscheme $\mathbb{D}_{\underline{a},a_n}^{(n)}$ in the outermost component of $\mathbb{X}_n$ belonging to $\underline{a}$, endowed with a local coordinate $x_{\underline{a},a_n}^{(n)}$. The

latter component intersects the remaining components in a unique singular point given locally by an equation $x_{\underline{a}}^{(\nu-1)} z_{\underline{a}}^{(\nu)} = p$ where $z_{\underline{a}}^{(\nu)}$ is a local coordinate on the component. The open sets $\mathbb{X}_{\underline{a}}^{(\nu)}\left[\frac{1}{x_{\underline{a}}^{(\nu-1)}}\right]$ together with the $\mathbb{D}_{\underline{a},a_n}^{(n)}$ form an open covering of the smooth locus of $\mathbb{X}_n$.

Lemma 7.4.4. *Suppose U is an open subspace of $\mathbb{X}_n$ which is smooth. The inclusion $\mathcal{D}_{U^{\log}}^{(m)} \subset \mathcal{D}_U^{(m)}$ is an equality.*

Proof. The logarithmic tangent sheaf $\mathcal{T}_{\mathbb{X}_n^{\log}}$ is generated on $\mathbb{X}_{\underline{a}}^{(\nu)}$ freely by $x_{\underline{a}}^{(\nu-1)} \partial_{x_{\underline{a}}^{(\nu-1)}}$. It therefore coincides with the usual tangent sheaf on $\mathbb{X}_{\underline{a}}^{(\nu)}[\frac{1}{x_{\underline{a}}^{(\nu-1)}}]$. Similarly on $\mathbb{D}_{\underline{a},a_n}^{(n)}$. Hence $\mathcal{T}_{U^{\log}} = \mathcal{T}_U$ which implies the claim. $\qquad\square$

Lemma 7.4.5. *One has $\widetilde{\mathcal{D}}_{n,k}^{(m)} \subset \mathcal{D}_{\mathbb{X}_n^{\log}}^{(m)}$ for any $k \geqslant n$. For $k = n$ this inclusion becomes an equality over each $\mathbb{D}_{\underline{a},a_n}^{(n)}$.*

Proof. By definition one has $\widetilde{\mathcal{T}}_{n,k} \subseteq \mathcal{T}_{\mathbb{X}_n^{\log}}$ and this implies $\widetilde{\mathcal{D}}_{n,k}^{(m)} \subseteq \mathcal{D}_{\mathbb{X}_n^{\log}}^{(m)}$. Now suppose $k = n$ and $U = \mathbb{D}_{\underline{a},a_n}^{(n)}$. Since U is smooth, $\mathcal{D}_{U^{\log}}^{(m)} = \mathcal{D}_U^{(m)}$ is generated over $\mathcal{O}_U$ by $\frac{q_d^{(m)}!}{d!}\partial_{x_{\underline{a},a_n}^{(n)}}^d$ whereas the subsheaf $\widetilde{\mathcal{D}}_{n,n}^{(m)}|_U$ is generated over $\mathcal{O}_U$ by $\frac{q_d^{(m)}!}{d!}\varpi^{dn}\partial_x^d$ where x is the standard coordinate on $\mathbb{X}_0$. But according to equation (24) in the proof of [33, Prop. 5.2(a)], we have $\partial_{x_{\underline{a},a_n}^{(n)}} = \varpi^n \partial_x$. This implies the claim. $\qquad\square$

7.4.6. *Coherence at points of type (a).* By definition of the sheaf $(\widehat{f}_n^c)_* \mathscr{F}_n$ we have for its restriction to the open subset $\hat{\Sigma}_0(n)_s \subset \mathfrak{X}_{n,s}^{\mathrm{nr}}$ that

$$\left((\widehat{f}_n^c)_* \mathscr{F}_n\right)\Big|_{\hat{\Sigma}_0(n)} = (\mathrm{sp}_{\hat{\Sigma}_0(n)})_*(f_n)_* \mathcal{O}_{\Sigma_1(n)}$$

where f_n is the restriction of f to $\Sigma_1(n) = f^{-1}(\Sigma_0(n))$. The points of type (a) lie in the smooth locus $U := \hat{\Sigma}_0(n)^{\mathrm{sm}}$ of $\hat{\Sigma}_0(n)$. Let $V := \hat{f}_n^{-1}(U)$. The morphism

$$\hat{f}_n : V \longrightarrow U$$

induced by $\hat{f}_n$ is smooth and proper. Under these assumptions, the coherent $\mathcal{O}_{U,\mathbb{Q}}$-module $\mathscr{E} := \mathbb{Q} \otimes (\hat{f}_n)_* \mathcal{O}_V$ is a convergent isocrystal over (the closed fibre of) U [29, 3.7]. According to [6, 3.1.2], there exists, for every m, a $\mathscr{D}_U^{(m)}$-module $\mathring{\mathscr{E}}$ which is coherent over $\mathcal{O}_U$ together with an $\mathscr{D}_{U,\mathbb{Q}}^{(m)}$-linear isomorphism $\mathbb{Q} \otimes \mathring{\mathscr{E}} \simeq \mathscr{E}$. Via the natural inclusion $\widetilde{\mathscr{D}}_{n,n}^{(m)}|_U \subseteq \mathscr{D}_U^{(m)}$, cf. 7.4.4 and 7.4.5, we may endow $\mathring{\mathscr{E}}$ with a $\widetilde{\mathscr{D}}_{n,n}^{(m)}$-module structure and apply part (ii) of 7.4.3. Thus, $\mathscr{E}$ is a coherent $\widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^{\dagger}$-module. It follows that $\left((\widehat{f}_n^c)_* \mathscr{F}_n\right)\Big|_U$ is a coherent $\widetilde{\mathscr{D}}_{n,n,\mathbb{Q}}^{\dagger}$-module, too. This proves our claim for points of type (a).

7.4.7. *Coherence at points of type (b).* We have

$$\left((\hat{f}_n^c)_* \mathscr{F}_n\right)\Big|_{\hat{\Sigma}_0(n)} = (\mathrm{sp}_{\hat{\Sigma}_0(n)})_* (f_n)_* \mathcal{O}_{\Sigma_1(n)}$$

as explained above. Since points of type (b) are singular, we will make use of logarithmic structures. Let k be the residue field of $\mathfrak{o}_K$ equipped with the trivial log structure. The morphism

$$\hat{f}_{n,s} : \hat{\Sigma}_1(n)_s \longrightarrow \hat{\Sigma}_0(n)_s$$

is a finite flat morphism of degree $|\mathbb{F}_q^\times|$. We equip source and target with their fine log structures coming from the singular loci [25, 3.7(2)]. In particular, source and target are log smooth over k and $\hat{f}_{n,s}$ is a log morphism. The target $\hat{\Sigma}_0(n)_s$ is covered by (the base change to k of) the reductions of the $\mathbb{X}_{\underline{a}}^{(\nu)}$ for $1 \leqslant \nu \leqslant n$ and $\underline{a} \in \mathcal{R}_\infty \times \mathcal{R}^{\nu-1}$. On the reduction of $\mathbb{X}_{\underline{a}}^{(\nu)}$ we have local coordinates $x_{\underline{a}}^{(\nu-1)}$ and $z_{\underline{a}}^{(\nu)}$ and a unique double point at $x_{\underline{a}}^{(\nu-1)} z_{\underline{a}}^{(\nu)} = 0$. The log structure is therefore given by the monoid formed by the powers $(x_{\underline{a}}^{(\nu-1)})^{n_1}(z_{\underline{a}}^{(\nu)})^{n_2}$ with $n_i \in \mathbb{N}$. We now use the notation introduced in the discussion around (7.1.4). So let $[x : y]$ be projective coordinates on the $x_{\underline{a}}^{(\nu-1)}$-component such that $x_{\underline{a}}^{(\nu-1)} = \frac{x}{y}$. Writing $s = \frac{z}{y}$ it follows from this discussion that

$$C \cap \hat{f}_{n,s}^{-1}(\{y \neq 0\}) = \{s^{q+1} = x_{\underline{a}}^{(\nu-1)}(1 - (x_{\underline{a}}^{(\nu-1)})^{q-1})\}.$$

A similar description holds on the $z_{\underline{a}}^{(\nu)}$-component. Hence there is an étale chart for $\hat{f}_{n,s}$ over the reduction of $\mathbb{X}_{\underline{a}}^{(\nu)}$ in which the corresponding map of monoids (or rather of their group completions) is given by multiplication with $(q + 1)$ on $\mathbb{Z}^{\oplus 2}$, cf. [16, p. 477]. The latter map is therefore injective and the order of its cokernel is prime to p. According to [25, 3.5] and [25, 4.1(1)iv], the morphism $\hat{f}_{n,s}$ is therefore log smooth and integral.

In the following we will make use of the work of Shiho on the logarithmic version of the convergent topos in characteristic p [30]. To make the comparison easier for the reader, we adapt some notation. Let $V := \mathfrak{o}_K$ and equip Spf V with its log structure coming from the closed point of $Spec V$. Let X_i be $\hat{\Sigma}_i(n)_s$ for $i = 0, 1$. Following [40, 2.1.3] we denote by $(X_i/V)_{\mathrm{conv,Zar}}^{\log}$ the log convergent site of X_i over Spf V with respect to the Zariski topology and by $\mathcal{K}_{X_1/V}$ the isocrystal

$$T \mapsto \Gamma(T, \mathcal{O}_T) \otimes \mathbb{Q}$$

on $(X_1/V)_{\mathrm{conv,Zar}}^{\log}$. Since X_0 is log smooth over k, the morphism $\hat{f}_{n,s}$ has even log smooth parameter in the sense of [39, 3.4]. We may therefore apply [39, 4.10] to $\hat{f}_{n,s}$ and $\mathcal{K}_{X_1/V}$. Hence, for each number $q \geqslant 0$, there is a unique isocrystal $\mathcal{F}^q$ on $(X_0/V)_{\mathrm{conv,Zar}}^{\log}$ whose value on the enlargement $\hat{\Sigma}_0(n)$ is given by the relative log analytic cohomology $R^q(f_s)_{X_1/\hat{\Sigma}_0(n),\mathrm{an}*}$ of $\mathcal{K}_{X_1/V}$ (as defined in [39, 4.1]). By definition, the latter analytic cohomology is computed on the level of rigid analytic generic fibres via the log de Rham

complex. Since the rigid analytic morphism f is étale, one finds $\mathcal{F}^0(\hat{\Sigma}_0(n)) = f_*\mathcal{O}_{\Sigma_1(n)}$ and $\mathcal{F}^q(\hat{\Sigma}_0(n)) = 0$ for $q > 0$.

The isocrystal $\mathcal{F}^0$ admits an interpretation via tubular neighbourhoods, as in the non-logarithmic setting [4, 2.2.5]. In particular, $\mathcal{F}^0$ induces a convergent logarithmic connection ∇ on $f_*\mathcal{O}_{\Sigma_1(n)}$ and $\mathscr{E} := (\mathrm{sp}_{\hat{\Sigma}_0(n)})_* f_*\mathcal{O}_{\Sigma_1(n)}$, cf. [39, 2.2.7] and [39, p.67]. We claim that the actions of the logarithmic derivations induced by ∇ extend to an action of the sheaf of logarithmic arithmetic differential operators $\mathscr{D}^\dagger_{\hat{\Sigma}_0^{\log},\mathbb{Q}}$ on $\mathscr{E}$. Since sections of $\mathscr{D}^\dagger_{\hat{\Sigma}_0^{\log},\mathbb{Q}}$ locally admit convergent power series expansions [26, 2.3.2.C.], it suffices to verify the appropriate convergence condition for the Taylor series of ∇ [6, (3.0.1.1)]. But this convergence condition follows from the fibration lemma [39, 2.31] applied to the projection p_1, as in the non-logarithmic setting [4, 2.2.13].

The $\mathscr{D}^\dagger_{\hat{\Sigma}_0^{\log},\mathbb{Q}}$-module $\mathscr{E}$ is coherent over $\mathcal{O}_{\hat{\Sigma}_0(n),\mathbb{Q}}$ and therefore coherent over $\mathscr{D}^\dagger_{\hat{\Sigma}_0^{\log},\mathbb{Q}}$, cf. [10, 3.1]. It follows from the proof of loc.cit. that there is for every m, a $\mathscr{D}^{(m)}_{\hat{\Sigma}_0^{\log}}$-module $\mathring{\mathscr{E}}$ which is coherent over $\mathcal{O}_{\hat{\Sigma}_0(n)}$ together with an $\mathscr{D}^{(m)}_{\hat{\Sigma}_0^{\log},\mathbb{Q}}$-linear isomorphism $\mathbb{Q} \otimes \mathring{\mathscr{E}} \simeq \mathscr{E}$. Via the natural inclusion $\widetilde{\mathscr{D}}^{(m)}_{n,n}|_{\hat{\Sigma}_0} \subseteq \mathscr{D}^{(m)}_{\hat{\Sigma}_0^{\log}}$, cf. 7.4.5, we may endow $\mathring{\mathscr{E}}$ with a $\widetilde{\mathscr{D}}^{(m)}_{n,n}$-module structure and apply part (ii) of 7.4.3. Thus, $\mathscr{E}$ is a coherent $\widetilde{\mathscr{D}}^\dagger_{n,n,\mathbb{Q}}$-module. This proves our claim for points of type (b).

7.4.8. *An alternative argument valid when the ramification index is small.* Suppose the ramification index of $L/\mathbb{Q}_p$ is $\leqslant p-1$. The fact that $(\widehat{f}^c_n)_* \mathscr{F}_n$ is a coherent $\widetilde{\mathscr{D}}^\dagger_{n,n,\mathbb{Q}}$-module at points of type (a) and (b) can then be seen explicitly as follows. The open subset $\hat{\Sigma}_0(n)$ is covered by open formal subschemes U equal to (the base change to $\mathfrak{o}_K$ of) the formal completions $\hat{\mathbb{X}}^{(\nu)}_{\underline{a}}$ for $1 \leqslant \nu \leqslant n$ and $\underline{a} \in \mathcal{R}_\infty \times \mathcal{R}^{\nu-1}$. Over U the sheaf $\widetilde{\mathcal{D}}^{(m)}_{n,n}$ is generated by $\frac{q^{(m)}_d!}{d!}\varpi^{dn}\partial^d_x$ where x is the standard coordinate on $\mathbb{X}_0$. According to [5, 2.4.3] any element $P \in \widetilde{\mathscr{D}}^\dagger_{n,n}$ has therefore a unique expression over U as infinite series in the form $P = \sum_{d \geqslant 0} a_d (\varpi^n \partial_x)^d / d!$ where the coefficients $a_d \in \mathcal{O}(U)$ have the property that there exist real constants $c > 0, \eta < 1$ (depending on P) with $\|a_d\| < c\eta^d$ for all d. Here $\|.\|$ denotes some Banach norm on the affinoid algebra $A := \mathcal{O}(U)_{\mathbb{Q}}$. We denote by $\|.\|$ also a Banach norm on the finite A-module $B := \mathscr{E}(U)$. We write $\mathcal{A} := \mathcal{O}(U)$ and $\mathcal{B} := \mathring{\mathscr{E}}(U)$. Making U smaller if necessary we may assume that $\mathcal{B}$ is a finite free $\mathcal{A}$-module of rank $t = |\mathbb{F}^\times_{p^2}|$. More precisely, there is a unit $\Pi \in \mathcal{A}^\times$ such that $\mathcal{B} = \mathcal{A}[X]/1 - X^t\Pi$ [42]. The powers Π^j for $j = 0, ..., t-1$ induce a decomposition

$$\mathcal{B} \simeq \oplus_{j=0}^{t-1} \mathcal{A}(j)$$

as $\mathcal{A}$-modules where $\mathcal{A}(j)$ equals $\mathcal{A}$. Similarly for the A-module B. Since the morphism f is finite étale, we have the Gauss-Manin connection on the étale A-algebra B. A short calculation along the lines of [18, 2.(I)] shows that it respects the direct sum decomposition $B \simeq \oplus_j A(j)$ and is given on the j-th summand as

$$f \mapsto d(f) - jt^{-1}f dlog(\Pi)$$

for $f \in A$. Here, $d : A \to \Omega_{A/K}$ is the canonical derivation of the K-algebra A and $dlog(\Pi) := d(\Pi)/\Pi$. The induced action of a derivation $\partial \in Der_K(A)$ on the j-th factor $A(j)$ is therefore given by

$$(7.4.9) \qquad\qquad f \mapsto \partial.f := \partial(f) - jt^{-1}f\partial(\Pi)/\Pi.$$

To obtain from this an action of an infinite sum like P on B, it suffices to verify for any $f \in B$ and any $\eta < 1$ the convergence property

$$(7.4.10) \qquad\qquad \left\| \left(\frac{(\varpi^n \partial_x)^d}{d!} \right).f \right\| \eta^d \to 0$$

for $d \to \infty$ [6, 3.1.1]. The $GL_2(\mathfrak{o})$-equivariance of the situation allows us to assume that the affinoid $Sp(A) = \mathrm{sp}^{-1}(U) \subset \mathbb{P}_K^{1,an}$ equals

$$\{z \in \mathbb{P}_K^{1,an} : |z| \leqslant 1\} - \bigcup_{i=0,\ldots,q-1} (B_1(i) \cup B_{1/q}(qi)) - B_{1/q}(0)$$

where $B_\delta(z_0)$ is the open ball of radius δ centered at z_0 and where the affine coordinate z on $\mathbb{P}_K^{1,an}$ is induced by $x_{\underline{a}}^{(\nu-1)}$ [41, p. 649]. According to the Mittag-Leffler decomposition [17, Prop. 2.2.6], we may assume that f is a holomorphic function on the complement in $\mathbb{P}_K^{1,an}$ of *one* of the above open discs. We may assume that the radius of the open disc is 1 (otherwise we work with the coordinate $z = z_{\underline{a}}^{(\nu)}$) and may thus write $f(z) = \sum_{n \leqslant 0} a_n(z - z_0)^n$ with $a_n \in K$ such that $\lim_{n \to \infty} |a_{-n}| = 0$. Moreover, $\|f\| = \max_{n \leqslant 0} |a_n|$ and so $\|\partial_z(f)\| \leqslant \|f\|$. According to the proof of [42, Cor. 6] we may assume that Π is a polynomial in z with coefficients in $\mathfrak{o}_K$ and so $\|\partial_z(\Pi)\| \leqslant \|\Pi\|$. All in all, this implies $\|\partial_z.f\| \leqslant \|f\|$ according to (7.4.9). On the other hand, by the equation (5.1.3) in the proof of [33, Prop. 5.1.2], we have $\partial_z := \partial_{x_{\underline{a}}^{(\nu-1)}} = \varpi^{\nu-1}\partial_x$ and therefore $\varpi^n \partial_x = a\varpi \partial_z$ with a p-adic integer a. The convergence property 7.4.10 follows now from the fact $\frac{|\varpi^d|}{d!}\eta^d \to 0$. The latter fact holds since $|\varpi|\eta < p^{-\frac{1}{p-1}}$ (it is here where we use the assumption on the ramification index of L).

7.4.11. *Coherence at points of type (c).* We recall that the special fibre of $\widehat{\Sigma}_0(n)^c$ is canonically isomorphic to the special fibre of $\mathfrak{X}_n^{\mathrm{nr}}$, cf. the discussion in section 7.2.5. Let $\widehat{\Sigma}_{0,s}(v) \subset \widehat{\Sigma}_0(n)_s^c \simeq \mathfrak{X}_{n,s}^{\mathrm{nr}}$ be one of the 'outermost' irreducible components of $\mathfrak{X}_{n,s}$. These are indexed by the vertices v of $\mathcal{T}$ which have distance n to v_0. Let $\widehat{\Sigma}_{0,s}(v)^\circ$ be the smooth locus of $\widehat{\Sigma}_{0,s}(v)$. Then $\widehat{\Sigma}_{0,s}(v)^\circ$ is isomorphic to an affine line over the residue field of $\mathfrak{o}_K$. Let $\widehat{\Sigma}_{0,s}(v)^{\circ\circ} \subset \widehat{\Sigma}_{0,s}(v)^\circ$ be the complement of the $\mathbb{F}_q$-rational points of $\widehat{\Sigma}_{0,s}(v)^\circ$. Put

$$\Sigma_1(v) = f^{-1}\left(\mathrm{sp}_{\widehat{\Sigma}_0}^{-1}\left(\widehat{\Sigma}_{0,s}(v)^{\circ\circ}\right)\right) .$$

The étale covering

$$f|_{\Sigma_1(v)} : \Sigma_1(v) \longrightarrow \mathrm{sp}_{\widehat{\Sigma}_0}^{-1}\left(\widehat{\Sigma}_{0,s}(v)^{\circ\circ}\right)$$

is isomorphic to the covering $\Sigma_1(0) \to \Sigma_0(0)$. Let $\widehat{\Sigma}_{1,s}(v) \subset \widehat{\Sigma}_{1,s}$ be the image of $\Sigma_1(v)$ under the specialization map. Denote here by the superscript "c" the topological closure[8]. Note that $\widehat{\Sigma}_{0,s}(v)^c$ is equal to $\left[\widehat{\Sigma}_{0,s}(v)^{\circ\circ}\right]^c$ and is isomorphic to a projective line over the residue field of $\mathfrak{o}_K$. Then we have a commutative diagram

(7.4.12)
$$
\begin{array}{ccccc}
\widehat{\Sigma}_{1,s}(v) & \longrightarrow & \widehat{\Sigma}_{1,s}(v)^c & \longrightarrow & \widehat{\Sigma}_1 \\
\downarrow{\scriptstyle \widehat{f}(v)_s} & & \downarrow{\scriptstyle \widehat{f}(v)_s^c} & & \downarrow{\scriptstyle \widehat{f}} \\
\widehat{\Sigma}_{0,s}(v)^{\circ\circ} & \longrightarrow & \widehat{\Sigma}_{0,s}(v)^c & \longrightarrow & \widehat{\Sigma}_0
\end{array}
$$

where the vertical morphisms on the left and in the middle are those induced by $\widehat{f}$. This diagram is analogous to the diagram 7.3.2. By [43, Thm. 4.1.4], the sheaf $(\widehat{f}(v)^c)_* \mathscr{F}_n$ is an overconvergent F-isocrystal on the pair

$$\left(\widehat{\Sigma}_{0,s}(v)^{\circ\circ}, \widehat{\Sigma}_{0,s}(v)^c\right) .$$

By [8, Thm. 4.3.5] or [28, Prop. 3.1], the $\mathscr{D}_{\widehat{\Sigma}_{0,s}(v)^c,\mathbb{Q}}^\dagger$-module $(\widehat{f}(v)^c)_* \mathscr{F}_n$ is holonomic, and, in particular, coherent. Here, $\mathscr{D}_{\widehat{\Sigma}_{0,s}(v)^c,\mathbb{Q}}^\dagger$ is the usual sheaf of arithmetic differential operators on $\widehat{\Sigma}_{0,s}(v)^c$.

---

[8]Earlier, when we considered the formal scheme $\widehat{\Sigma}_0(n)$, we denoted by $\widehat{\Sigma}_0(n)^c$ the completion along the tolopological closure of $\widehat{\Sigma}_0(n)$ in $\widehat{\Sigma}_0$. The special fiber of $\widehat{\Sigma}_0(n)^c$ is simply the closure of the special fiber of $\widehat{\Sigma}_0(n)$ in $\widehat{\Sigma}_{0,s}$. In this sense the notation here is consistent with the earlier notation.

We now use the fact that the sheaf $\widetilde{\mathscr{D}}^\dagger_{n,n,\mathbb{Q}}$ on $\mathfrak{X}_n$, when restricted to $\widehat{\Sigma}_{0,s}(v) \subset \widehat{\Sigma}_0(n)^c_s \simeq \mathfrak{X}^{\mathrm{nr}}_{n,s}$ is isomorphic to $\mathscr{D}^\dagger_{\widehat{\Sigma}_{0,s}(v),\mathbb{Q}}$, cf. 7.4.4 and 7.4.5. Therefore, $(\widehat{f}(v)^c)_*(\mathscr{F}_n)|_{\widehat{\Sigma}_{0,s}(v)^c}$ is a coherent module over $\widetilde{\mathscr{D}}^\dagger_{n,n,\mathbb{Q}}|_{\widehat{\Sigma}_{0,s}(v)^c}$. In particular, if x is a smooth $\mathbb{F}_q$-rational point of $\mathfrak{X}_n$, which necessarily lies on $\widehat{\Sigma}_{0,s}(v)$ for some vertex v which has distance n to v_0, the sheaf $(\widehat{f}^c_n)_*(\mathscr{F}_n)$ is coherent over $\widetilde{\mathscr{D}}^\dagger_{n,n,\mathbb{Q}}$ at x. This proves our claim for points of type (c). $\qquad\square$

7.5. $\mathbb{G}(n-1)^\circ$-analytic vectors in the dual of $H^0(\Sigma_1(n), \mathcal{O})^\dagger$. We abbreviate

$$R := D(G_0, K), \ R_n := D(\mathbb{G}^\circ(n), G_0)^{\mathrm{nr}} \ \text{and} \ E_n := H^0(\Sigma_1(n), \mathcal{O})^\dagger \ .$$

The rigid-analytic space Σ_1 is quasi-Stein with defining affinoid covering given by the $\Sigma_1(n)$. This implies that the restriction map $E_n \to E_{n-1}$ has dense image. Hence, the induced homomorphism of finitely generated R_{n-1}-modules

$$h_n : R_{n-1} \otimes_{R_n} E_n \longrightarrow E_{n-1}$$

has dense image. On the other hand, this image is closed by 5.1.1, and so h_n is surjective. We now show that h_n is injective. To start with, the restriction map $E_n \to E_{n-1}$ is injective. Since the natural ring homomorphism $R \to R_n$ is flat, the map

$$R_{n-1} \otimes_R E_n \longrightarrow R_{n-1} \otimes_R E_{n-1}$$

is injective. Hence, the injectivity of h_n follows if we can show that, for each n, the natural map

$$w_n : R_n \otimes_R E_n \xrightarrow{\ \simeq\ } E_n$$

is a bijection. Since w_n is obviously surjective, we will prove its injectivity in the following. Since the group G_{n+1} acts $\mathbb{G}(n)^\circ$-analytically on the strong dual $(E_n)'_b$, we may dualise the isomorphism in [14, 3.4.5] and obtain the isomorphism

$$\mathcal{D}^{\mathrm{an}}(\mathbb{G}(n)^\circ) \widehat{\otimes}_{D(G_{n+1})} E_n \xrightarrow{\ \simeq\ } E_n \ .$$

The latter factors through the natural surjection

$$\widehat{w}_n : R_n \widehat{\otimes}_R E_n \to E_n$$

which must then be an isomorphism, too. It therefore remains to see that the natural map

$$\iota : R_n \otimes_R E_n \longrightarrow R_n \widehat{\otimes}_R E_n$$

is injective (and hence bijective). Fix a set of generators $e_1, ..., e_s$ for the R_n-module E_n. Consider the topological R_n-module $M := R_n \otimes_{K,\pi} E_n$. Its underlying topological space is Hausdorff [36, 17.5] and has a completion which is of compact type [14, 1.1.32]. Let N be its closed R_n-submodule generated by the finitely many elements $1 \otimes \delta_g e_j - \delta_g \otimes e_j$ where g runs through a finite set of topological generators for the group G_0. There is a natural quotient map $q : M/N \to R_n \otimes_R E_n$. On the other hand, the natural map $\iota' : M/N \to \widehat{M/N}$ of M/N into its Hausdorff completion is injective. Arguing as in [35, 3.4], the completion $\widehat{M/N}$ is the same as $R_n \hat{\otimes}_R E_n$. We obtain an injective map $\iota' : M/N \to R_n \hat{\otimes}_R E_n$ which is seen to equal $\iota \circ q$. Hence ι must be injective, too.

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