ARITHMETIC DIFFERENTIAL OPERATORS WITH CONGRUENCE LEVEL STRUCTURES: FIRST RESULTS AND EXAMPLES

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ABSTRACT. This paper is a survey on sheaves of arithmetic differential operators with congruence level on formal schemes. We present first results about these sheaves and discuss some examples.

CONTENTS

1. INTRODUCTION

This note is meant to give a brief account on the theory of arithmetic differential operators with congruence level, as it has been developed in the papers [8, 9, 11, 12], and we discuss a couple of examples. To put the theory into context, consider a smooth formal scheme $\mathfrak{X}_0$ over a complete discrete valuation ring $\mathfrak{o}$ of mixed characteristic $(0,p)$ with uniformizer ϖ. Let $\mathcal{D}^\dagger_{\mathfrak{X}_0}$ be Berthelot's sheaf of arithmetic differential operators (tensored with $\mathbb{Q}$) on $\mathfrak{X}_0$ [3]. The category of coherent $\mathcal{D}^\dagger_{\mathfrak{X}_0}$-modules - together with various subcategories - plays an important role in crystalline and rigid cohomology [2]. Let $k \geqslant 0$ be a non-negative integer. The sheaf $\mathcal{D}^\dagger_{\mathfrak{X}_0,k}$ of arithmetic differential operators of congruence level k is a subring of $\mathcal{D}^\dagger_{\mathfrak{X}_0}$ which is locally generated by divided powers in the vector fields coming from $\varpi^k \mathcal{T}_{\mathfrak{X}_0}$. Here, $\mathcal{T}_{\mathfrak{X}_0}$ denotes the relative tangent sheaf of the formal $\mathfrak{o}$-scheme $\mathfrak{X}_0$. Given an admissible blow-up

$$\mathrm{pr} : \mathfrak{X} \to \mathfrak{X}_0$$

defined by a coherent sheaf of open ideals $\mathcal{I}$ of $\mathcal{O}_{\mathfrak{X}_0}$ containing ϖ^k for some non-negative integer k, the pull-back

$$\mathcal{D}^\dagger_{\mathfrak{X},k} := \mathrm{pr}^* \mathcal{D}^\dagger_{\mathfrak{X}_0,k}$$

is a sheaf of *rings.* Passing to the projective limit in k and to the inductive limit in the tower of *all* admissible formal blow-ups of $\mathfrak{X}_0$ then leads naturally to a well-behaved ring $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$ of analytic infinite order differential operators on the Zariski-Riemann space $\langle\mathfrak{X}_0\rangle$.

The sheaf $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$ is locally a Fréchet-Stein algebra in the sense of Schneider-Teitelbaum [13]. This gives rise to a full abelian subcategory $\mathcal{C}_{\langle\mathfrak{X}_0\rangle}$ of so-called coadmissible $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$-modules which should be viewed as a replacement for the category of coherent D-modules in the complex-analytic setting.

The interest in these constructions comes at least from two sources. On the one hand, the category $\mathcal{C}_{\langle\mathfrak{X}_0\rangle}$ and its equivariant versions figure prominently in the non-archimedean Beilinson-Bernstein localization theory for locally analytic representations of p-adic Lie groups [8, 12]. On the other hand, the new sheaves of type $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$ and $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$ seem to be well-adapted to treat finiteness problems for overconvergent isocrystals arising from wildly ramified coverings or for arithmetic $\mathcal{D}$-modules with essential singularities at the boundary. As an illustration of the first case, we discuss here the Kummer covering $y = x^p$ with exponent p of the multiplicative group. As for the second case, we discuss the $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$-module structure on the open-push forward from the pointed rigid analytic closed unit disc.

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Notations and Conventions. We denote by ϖ a uniformizer of the complete discrete valuation ring $\mathfrak{o}$, and we let $|.|_p$ be the absolute value on $L = Frac(\mathfrak{o})$ which is normalized by $|p|_p = p^{-1}$. We let $\mathbb{F} = \mathfrak{o}/(\varpi)$ be the residue field. Throughout this paper $\mathfrak{S} = \mathrm{Spf}(\mathfrak{o})$. A formal scheme $\mathfrak{X}$ over $\mathfrak{S}$ such that $\varpi\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition and which is locally noetherian is called a $\mathfrak{S}$*-formal scheme.* If the $\mathfrak{S}$-formal scheme $\mathfrak{X}$ is smooth over $\mathfrak{S}$ we denote by $\mathcal{T}_{\mathfrak{X}}$ its relative tangent sheaf. A coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$ is called *open* if for any open $\mathfrak{U} \subset \mathfrak{X}$ the restriction of $\mathcal{I}$ to $\mathfrak{U}$ contains $\varpi^k\mathcal{O}_{\mathfrak{U}}$ (for some $k \in \mathbb{N}$ depending on $\mathfrak{U}$). A scheme which arises from blowing up an open ideal sheaf on $\mathfrak{X}$ will be called *an admissible blow-up* of $\mathfrak{X}$. For an integer $i \geqslant 0$ we also denote X_i the scheme

$$X_i = \mathfrak{X} \times_{\mathfrak{S}} \mathrm{Spec}\left(\mathfrak{o}/\varpi^{i+1}\mathfrak{o}\right) ,$$

where the Cartesian product is taken in the category of locally ringed spaces. Without further mentioning, all occurring modules will be left modules. We let $\mathbb{N} = \{0, 1, 2, ...\}$ be the set of non-negative integers.

2. Arithmetic differential operators with congruence level

Let $\mathfrak{X}_0$ be a smooth $\mathfrak{S}$-formal scheme. For any $m \geqslant 0$ we denote by $\mathcal{D}^{(m)}_{\mathfrak{X}_0}$ the sheaf of arithmetic level m differential operators [3]. Any non-negative integer k defines a sheaf of subalgebras $\mathcal{D}^{(k,m)}_{\mathfrak{X}_0}$ of $\mathcal{D}^{(m)}_{\mathfrak{X}_0}$ whose local description is as follows. Let U be an affine open of $\mathfrak{X}_0$ endowed with étale local coordinates $x_1, \ldots, x_d$ and corresponding derivations $\partial_i = \partial_{x_i}$, $i = 1, \ldots, d$. Denote by $q_\nu = q_\nu^{(m)}$ the quotient of the euclidean division of a natural number ν by p^m and form the differential operator $\partial_l^{\langle \nu \rangle} = q_\nu ! \partial_l^{[\nu]}$. For a multi-index $\underline{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{N}^d$ put $\underline{\partial}^{\langle \underline{\nu} \rangle} = \prod_{l=1}^d \partial_l^{\langle \nu_l \rangle}$. Then the restriction of the sheaf $\mathcal{D}_{\mathfrak{X}_0}^{(k,m)}$ to U is a free $\mathcal{O}_U$-module with basis given by the elements $\varpi^{k|\underline{\nu}|} \underline{\partial}^{\langle \underline{\nu} \rangle}$. In particular,

$$\mathcal{D}_{\mathfrak{X}_0}^{(k,m)}(U) = \left\{ \sum_{\underline{\nu}}^{<\infty} \varpi^{k|\underline{\nu}|} a_{\underline{\nu}} \underline{\partial}^{\langle \underline{\nu} \rangle} \mid a_{\underline{\nu}} \in \mathcal{O}_{\mathfrak{X}_0}(U) \right\} . \tag{2.1}$$

As in the case $k = 0$ these sheaves, as well as their p-adic completions $\widehat{\mathcal{D}}_{\mathfrak{X}_0}^{(k,m)}$, form an inductive system for varying m and we may form the inductive limit (tensored with $\mathbb{Q}$)

$$\mathcal{D}^\dagger_{\mathfrak{X}_0,k} = \varinjlim_m \widehat{\mathcal{D}}_{\mathfrak{X}_0}^{(k,m)} \otimes \mathbb{Q} .$$

This is a sheaf of coherent rings on $\mathfrak{X}_0$ with local description

$$\mathcal{D}^\dagger_{\mathfrak{X}_0,k}(U) = \left\{ \sum_{\underline{\nu}} a_{\underline{\nu}} \underline{\partial}^{[\underline{\nu}]} \mid a_{\underline{\nu}} \in \mathcal{O}_{\mathfrak{X}_0,\mathbb{Q}}(U), \|a_{\underline{\nu}}\| = O(\eta^{|\underline{\nu}|}) \text{ for some } \eta < |\varpi|^k \right\} , \tag{2.2}$$

where $\| \cdot \|$ is any Banach norm on the affinoid algebra $\mathcal{O}_{\mathfrak{X}_0,\mathbb{Q}}(U)$.

Suppose now that

$$\mathrm{pr} : \mathfrak{X} \to \mathfrak{X}_0$$

is an admissible formal blow-up of $\mathfrak{X}_0$ defined by a coherent sheaf of open ideals $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}_0}$ containing ϖ^k for some non-negative integer k. Since the ideal $\mathcal{I}$ is not determined by the blow-up, we denote by $k_{\mathcal{I}}$ the minimal k such that $\varpi^k \in \mathcal{I}$ and put

$$k_{\mathfrak{X}} = \min\{ k_{\mathcal{I}} \mid \mathfrak{X} \text{ is the blowing-up of } \mathcal{I} \text{ on } \mathfrak{X}_0 \}.$$

Theorem 2.3. *The structure sheaf $\mathcal{O}_{\mathfrak{X}}$ admits a natural action of the sheaf of rings $\mathrm{pr}^{-1}\mathcal{D}_{\mathfrak{X}_0}^{(k,m)}$ for $k \geqslant k_{\mathfrak{X}}$.*

For a proof of this result and a more detailed discussion we refer to [9]. The main point is that $\mathrm{pr}^{-1}\mathcal{D}_{\mathfrak{X}_0}^{(k,m)} \otimes \mathbb{Q}$ is independent of k and m and may be viewed as the sheaf of algebraic differential operators on the rigid analytic generic fibre $\mathfrak{X}^{\mathrm{rig}}$ of $\mathfrak{X}$, pushed forward along the specialization map $\mathrm{sp} : \mathfrak{X}^{\mathrm{rig}} \to \mathfrak{X}$. It therefore acts naturally on the sheaf $\mathrm{sp}_* \mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}} = \mathcal{O}_{\mathfrak{X}} \otimes \mathbb{Q}$. A sufficiently small subsheaf $\mathrm{pr}^{-1}\mathcal{D}_{\mathfrak{X}_0}^{(k,m)}$ (one requires $k \geqslant k_{\mathfrak{X}}$) of $\mathrm{pr}^{-1}\mathcal{D}_{\mathfrak{X}_0}^{(k,m)} \otimes \mathbb{Q}$ can then be shown to stabilize the subsheaf $\mathcal{O}_{\mathfrak{X}}$ inside $\mathcal{O}_{\mathfrak{X}} \otimes \mathbb{Q}$.

To give a simple example, we consider the formal affine line $\mathfrak{X}_0 = \mathrm{Spf}(\mathfrak{o}\langle x \rangle)$ and the ideal sheaf $\mathcal{I}$ associated to the open ideal $I = (\varpi, x)$ of the ring $A = \mathfrak{o}\langle x \rangle$. The two sets $U_0 = \mathrm{Spf}(A\langle \frac{x}{\varpi} \rangle)$ and $U_1 = \mathrm{Spf}(A\langle \frac{\varpi}{x} \rangle)$ define an open affine covering of the blow-up $\mathfrak{X}$.

Let ∂ be the derivation belonging to the coordinate x. Then $\partial(\frac{x}{\varpi}) = \frac{1}{\varpi}$ is not an element of $A\langle\frac{x}{\varpi}\rangle$, but only $\varpi\partial(\frac{x}{\varpi}) = 1$ is. In the same manner,

$$\varpi^{\nu}\partial^{\langle\nu\rangle}((\frac{x}{\varpi})^{\nu'})) = \varpi^{\nu}q_{\nu}!\partial^{[\nu]}((\frac{x}{\varpi})^{\nu'})) = q_{\nu}!(\frac{x}{\varpi})^{\nu'-\nu} \in A\langle\frac{x}{\varpi}\rangle$$

for any pair of non-negative integers $\nu' \geqslant \nu$. This defines an action of $\mathrm{pr}^{-1}\mathcal{D}^{(1,m)}_{\mathfrak{X}_0}(U_0)$ on $\mathcal{O}_{\mathfrak{X}}(U_0)$ and finally leads to an action of $\mathrm{pr}^{-1}\mathcal{D}^{(1,m)}_{\mathfrak{X}_0}$ on $\mathcal{O}_{\mathfrak{X}}$.

Coming back to the general case of an admissible blow-up

$$\mathrm{pr} : \mathfrak{X} \to \mathfrak{X}_0,$$

we consider the $\mathcal{O}_{\mathfrak{X}}$-module

$$\mathcal{D}^{(k,m)}_{\mathfrak{X}} := \mathrm{pr}^{*}\mathcal{D}^{(k,m)}_{\mathfrak{X}_0}\ .$$

The theorem implies that $\mathcal{D}^{(k,m)}_{\mathfrak{X}}$ is in fact naturally a sheaf of *rings* on $\mathfrak{X}$ for $k \geqslant k_{\mathfrak{X}}$. As in the case of a smooth formal scheme, we may form the p-adic completion $\widehat{\mathcal{D}}^{(k,m)}_{\mathfrak{X}}$ and pass to the inductive limit over all m. This gives a sheaf of rings

$$\mathcal{D}^{\dagger}_{\mathfrak{X},k} = \varinjlim_{m} \widehat{\mathcal{D}}^{(k,m)}_{\mathfrak{X}} \otimes \mathbb{Q}$$

on $\mathfrak{X}$ for $k \geqslant k_{\mathfrak{X}}$. It follows from 2.2 that its sections over an open affine $U \subset \mathrm{pr}^{-1}(U_0)$ where $U_0 \subset \mathfrak{X}_0$ is endowed with coordinates, admit the description

$$\mathcal{D}^{\dagger}_{\mathfrak{X},k}(U) = \left\{ \sum_{\underline{\nu}} a_{\underline{\nu}}\underline{\partial}^{[\underline{\nu}]} \mid a_{\underline{\nu}} \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U), \|a_{\underline{\nu}}\| = O(\eta^{|\underline{\nu}|}) \text{ for some } \eta < |\varpi|^{k} \right\},$$

where $\|\cdot\|$ is any Banach norm on the affinoid algebra $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U)$. Note that there is a natural inclusion

$$\mathcal{D}^{\dagger}_{\mathfrak{X},k+1}(U) \longrightarrow \mathcal{D}^{\dagger}_{\mathfrak{X},k}(U) \tag{2.4}$$

which is a ring homomorphism.

Remarks 2.5. Let $k \geqslant k_{\mathfrak{X}}$. The sheaves of rings $\mathcal{D}^{(k,m)}_{\mathfrak{X}}, \widehat{\mathcal{D}}^{(k,m)}_{\mathfrak{X}}$ and $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$ enjoy the following properties, cf. [9].

1) The sheaf $\mathcal{D}^{(k,m)}_{\mathfrak{X}}$ admits an order filtration by $\mathcal{O}_{\mathfrak{X}}$-coherent modules with associated graded

$$\mathrm{gr}\left(\mathcal{D}^{(k,m)}_{\mathfrak{X}}\right) \simeq \mathrm{Sym}^{(m)}(\mathcal{T}_{\mathfrak{X},k})$$

isomorphic to the symmetric algebra of level m of the vector bundle $\mathcal{T}_{\mathfrak{X},k} = \varpi^{k}\mathrm{pr}^{*}\mathcal{T}_{\mathfrak{X}_0}$ where $\mathcal{T}_{\mathfrak{X}_0}$ denotes the relative tangent sheaf of $\mathfrak{X}_0$ over $\mathfrak{S}$.

2) The rings $\mathcal{D}^{(k,m)}_{\mathfrak{X}}(U)$ and $\widehat{\mathcal{D}}^{(k,m)}_{\mathfrak{X}}(U)$ are noetherian for any open affine $U \subset \mathfrak{X}$.

3) The sheaves $\mathcal{D}^{(k,m)}_{\mathfrak{X}}$, $\widehat{\mathcal{D}}^{(k,m)}_{\mathfrak{X}}$ and $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$ are sheaves of coherent rings on $\mathfrak{X}$. ■

3. Theorems A and B and an invariance theorem

We keep the notation of the preceding section. Let

$$\mathrm{pr} : \mathfrak{X} \to \mathfrak{X}_0$$

be an admissible formal blow-up of a smooth formal $\mathfrak{S}$-scheme $\mathfrak{X}_0$ and let $k \geqslant k_{\mathfrak{X}}$. Let $U \subset \mathfrak{X}$ be an open formal subscheme. We write $\mathcal{D}^{(k,m)}_{U}$ for the restriction of $\mathcal{D}^{(k,m)}_{\mathfrak{X}}$ to U and similarly for the sheaves $\widehat{\mathcal{D}}^{(k,m)}_{\mathfrak{X}}$ and $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$. If U is affine, we have the following version of Cartan's Theorem A and B.

Proposition 3.1. [9] *Let $U \subset \mathfrak{X}$ be an open affine formal subscheme.*

(i) The functor $\Gamma(U, -)$ gives an equivalence of abelian categories between coherent $\mathcal{D}^{(k,m)}_{U}$-modules and finitely generated $\Gamma(U, \mathcal{D}^{(k,m)}_{U})$-modules. One has $H^i(U, \mathcal{M}) = 0$ for any $i > 0$ and any finitely generated $\mathcal{D}^{(k,m)}_{U}$-module $\mathcal{M}$. The analogous statements hold for the completed sheaf $\widehat{\mathcal{D}}^{(k,m)}_{U}$.

(ii) The functor $\Gamma(U, -)$ gives an equivalence of abelian categories between coherent $\mathcal{D}^{\dagger}_{U,k}$-modules and finitely presented $\Gamma(U, \mathcal{D}^{\dagger}_{U,k})$-modules. One has $H^i(U, \mathcal{M}) = 0$ for any $i > 0$ and any finitely presented $\Gamma(U, \mathcal{D}^{\dagger}_{U,k})$-module $\mathcal{M}$.

As a next result, we will state an invariance theorem which controls the behaviour of the sheaf $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$ when varying the admissible formal blow-up $\mathfrak{X}$ of $\mathfrak{X}_0$. Let therefore $\mathrm{pr}' : \mathfrak{X}' \to \mathfrak{X}_0$ be another admissible formal blow-up of $\mathfrak{X}_0$ and let

$$\pi : \mathfrak{X}' \to \mathfrak{X}$$

be a morphism over $\mathfrak{X}_0$, inducing an isomorphism between the rigid analytic spaces associated to $\mathfrak{X}$ and $\mathfrak{X}'$. It follows from Tate's acyclicity theorem that the ajoint pair of functors (π_*, π^*) induces an equivalence of abelian categories between coherent $\mathcal{O}_{\mathfrak{X}',\mathbb{Q}}$-modules and coherent $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-modules.

The invariance theorem states an analogous property on the level of D-modules. Let $k \geqslant \max\{k_{\mathfrak{X}}, k_{\mathfrak{X}'}\}$. First of all, by definition of the sheaf $\mathcal{D}^{(k,m)}_{\mathfrak{X}}$ and the fact that $(\mathrm{pr}')^* = \pi^* \circ \mathrm{pr}^*$, we have

$$\mathcal{D}^{(k,m)}_{\mathfrak{X}'} \simeq \pi^* \mathcal{D}^{(k,m)}_{\mathfrak{X}} ,$$

and the sheaf $\mathcal{D}^{(k,m)}_{\mathfrak{X}'}$ can therefore be endowed with a natural structure of right $\pi^{-1}\mathcal{D}^{(k,m)}_{\mathfrak{X}}$-module. Passing to p-adic completions and then to the inductive limit over m yields that

$\mathcal{D}^\dagger_{\mathfrak{X}',k}$ is a right $\pi^{-1}\mathcal{D}^\dagger_{\mathfrak{X},k}$-module. For a $\mathcal{D}^\dagger_{\mathfrak{X},k}$-module $\mathcal{M}$, we may therefore define the $\mathcal{D}^\dagger_{\mathfrak{X}',k}$-module

$$\pi^!\mathcal{M} := \mathcal{D}^\dagger_{\mathfrak{X}',k} \otimes_{\pi^{-1}\mathcal{D}^\dagger_{\mathfrak{X},k}} \pi^{-1}\mathcal{M}. \tag{3.2}$$

Into the other direction, we let $\pi_*\mathcal{M}$ denote the ordinary push-forward of $\mathcal{M}$ in the sense of abelian sheaves, for any $\mathcal{D}^\dagger_{\mathfrak{X}',k}$-module $\mathcal{M}$.

Theorem 3.3. [9] *Let $k \geqslant \max\{k_{\mathfrak{X}}, k_{\mathfrak{X}'}\}$. There is a canonical isomorphism of rings $\pi_*\mathcal{D}^\dagger_{\mathfrak{X}',k} \simeq \mathcal{D}^\dagger_{\mathfrak{X},k}$. The adjoint pair of functors $(\pi_*, \pi^!)$ gives an equivalence of abelian categories between the coherent modules over $\mathcal{D}^\dagger_{\mathfrak{X}',k}$ and over $\mathcal{D}^\dagger_{\mathfrak{X},k}$ respectively.*

The theorem implies in particular a canonical ring isomorphism

$$\Gamma(\mathfrak{X}, \mathcal{D}^\dagger_{\mathfrak{X},k}) \simeq \Gamma(\mathfrak{X}_0, \mathcal{D}^\dagger_{\mathfrak{X}_0,k}).$$

An essential point is the fact that $R\pi_*\mathcal{D}^\dagger_{\mathfrak{X}',k} \simeq \mathcal{D}^\dagger_{\mathfrak{X},k}$, which follows via the projection formula at the level of schemes, and from the fact that $R\pi_*\mathcal{O}_{\mathfrak{X}',\mathbb{Q}} \simeq \mathcal{O}_{\mathfrak{X},\mathbb{Q}}$. To show the remaining statements, one reduces to the case of an *affine* formal scheme $\mathfrak{X}_0$. In this case, any coherent $\mathcal{D}^\dagger_{\mathfrak{X}',k}$-module $\mathcal{M}$ admits a finite presentation

$$\left(\mathcal{D}^\dagger_{\mathfrak{X}',k}\right)^r \to \left(\mathcal{D}^\dagger_{\mathfrak{X}',k}\right)^s \to \mathcal{M} \to 0$$

which then gives $R^j\pi_*\mathcal{M} = 0$ for $j > 0$.

As an application of the invariance theorem, the local theorems A and B extend to global statements, provided that the base $\mathfrak{X}_0$ is affine.

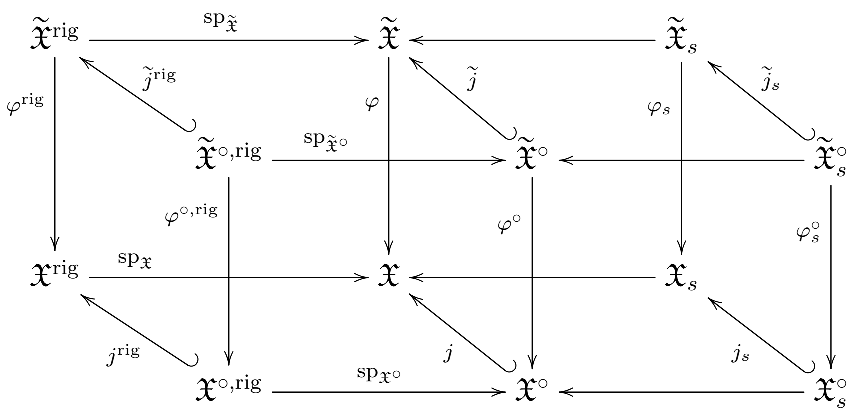
Corollary 3.4. *Let $\mathfrak{X}_0$ be affine. The functor $\Gamma(\mathfrak{X}, -)$ is an equivalence of abelian categories between the categories of coherent modules over $\mathcal{D}^\dagger_{\mathfrak{X},k}$ and over $\Gamma(\mathfrak{X}, \mathcal{D}^\dagger_{\mathfrak{X},k})$ respectively. One has $H^i(\mathfrak{X}, \mathcal{M}) = 0$ for any $i > 0$ and for any coherent $\mathcal{D}^\dagger_{\mathfrak{X},k}$-module $\mathcal{M}$.*

4. Example: the Kummer covering with exponent p

We denote by $\mathfrak{X}$ (resp. $\widetilde{\mathfrak{X}}$) the completion of $\mathrm{Proj}(\mathfrak{o}[x_0, x_1])$ (resp. $\mathrm{Proj}(\mathfrak{o}[y_0, y_1])$ along its special fiber. Let $\varphi : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ be the morphism of formal schemes induced by the homomorphism of $\mathfrak{o}$-algebras $\mathfrak{o}[x_0, x_1] \to \mathfrak{o}[y_0, y_1]$ given by $x_i \mapsto y_i^p$, $i = 0, 1$. Write $\mathfrak{X}_s = \mathrm{Proj}(\mathbb{F}[\overline{x}_0, \overline{x}_1])$ (resp. $\widetilde{\mathfrak{X}}_s = \mathrm{Proj}(\mathbb{F}[\overline{y}_0, \overline{y}_1])$) for the special fiber, and define open subsets $\mathfrak{X}_s^\circ = D_+(\overline{x}_0\overline{x}_1) \subset \mathfrak{X}_s$ (resp. $\widetilde{\mathfrak{X}}_s^\circ = D_+(\overline{y}_0\overline{y}_1) \subset \widetilde{\mathfrak{X}}_s$).

Let $\mathfrak{X}^\circ \subset \mathfrak{X}$ (resp. $\widetilde{\mathfrak{X}}^\circ \subset \widetilde{\mathfrak{X}}$) be the open subscheme whose special fiber is $\mathfrak{X}_s^\circ$ (resp. $\widetilde{\mathfrak{X}}_s^\circ$). On $\mathfrak{X}^\circ$ ($\mathfrak{X}_s^\circ$, $\widetilde{\mathfrak{X}}^\circ$, $\widetilde{\mathfrak{X}}_s^\circ$, resp.) we have the coordinate $x = \frac{x_1}{x_0}$ ($\overline{x} = \frac{\overline{x}_1}{\overline{x}_0}$, $y = \frac{y_1}{y_0}$, $\overline{y} = \frac{\overline{y}_1}{\overline{y}_0}$, resp.), so that $\mathfrak{X}^\circ = \mathrm{Spf}(\mathfrak{o}\langle x, x^{-1}\rangle)$ ($\mathfrak{X}_s^\circ = \mathrm{Spec}(\mathbb{F}[\overline{x}, \overline{x}^{-1}])$, $\widetilde{\mathfrak{X}}^\circ = \mathrm{Spf}(\mathfrak{o}\langle y, y^{-1}\rangle)$, $\widetilde{\mathfrak{X}}_s^\circ = \mathrm{Spec}(\mathbb{F}[y, y^{-1}])$, resp.). Denote by $\mathfrak{X}^{\mathrm{rig}}$ ($\mathfrak{X}^{\circ,\mathrm{rig}}$, $\widetilde{\mathfrak{X}}^{\mathrm{rig}}$, $\widetilde{\mathfrak{X}}^{\circ,\mathrm{rig}}$, resp.) the associated rigid analytic spaces. The morphism φ induces morphisms between these formal schemes and their associated rigid analytic spaces, and we obtain a commutative diagram

(4.1)



where the horizontal arrows on the right are the canonical morphisms (of locally ringed spaces) from the special fiber of a formal scheme into that formal scheme. Those maps are homeomorphisms of the underlying topological spaces which we tacitly identify. Note that φ^{rig} is étale outside the points $y = 0$ and $y = \infty$, that $\varphi^{\circ,\mathrm{rig}}$ is hence étale, but that φ_s and φ_s° are purely inseparable.

As usual we denote the tubular neighborhood $\mathrm{sp}_{\mathfrak{X}}^{-1}(\mathfrak{X}^\circ)$ (resp. $\mathrm{sp}_{\widetilde{\mathfrak{X}}}^{-1}(\widetilde{\mathfrak{X}}^\circ)$) of $\mathfrak{X}^\circ$ (resp. $\widetilde{\mathfrak{X}}^\circ$) by $]\mathfrak{X}_s^\circ[$ (resp. $]\widetilde{\mathfrak{X}}_s^\circ[$). This is nothing else than the rigid analytic space $\mathfrak{X}^{\circ,\mathrm{rig}}$ (resp. $\widetilde{\mathfrak{X}}^{\circ,\mathrm{rig}}$). Put $\widetilde{Z} = \widetilde{\mathfrak{X}}_s \backslash \widetilde{\mathfrak{X}}_s^\circ$ and $Z = \mathfrak{X}_s \backslash \mathfrak{X}_s^\circ$. Let

$$\widetilde{\mathcal{M}} = \mathcal{O}_{\widetilde{\mathfrak{X}},\mathbb{Q}}({}^\dagger \widetilde{Z}) = (\widetilde{j}_s)_*^\dagger \mathcal{O}_{\widetilde{\mathfrak{X}}^{\mathrm{rig}}} := (\mathrm{sp}_{\widetilde{\mathfrak{X}}})_* \widetilde{j}_s^\dagger \mathcal{O}_{\widetilde{\mathfrak{X}}^{\mathrm{rig}}}$$

be the overconvergent constant isocrystal on $\widetilde{\mathfrak{X}}_s^\circ$ (cf. [2, sec. 4], [5, sec. 4.1.6, 4.4] for notation and background). By [2, 4.2.2] the sheaf $\widetilde{\mathcal{M}}$ is a coherent $\mathcal{D}^\dagger_{\widetilde{\mathfrak{X}}}$-module. Our aim is to understand the direct image $\mathcal{M} = (\varphi_s)_* \widetilde{\mathcal{M}}$ as a module over a sheaf of differential operators $\mathcal{D}^\dagger_{\mathfrak{X},k}$ on $\mathfrak{X}$ for a suitable congruence level k.

Remark 4.2. Before proceeding let us compare the situation considered here with the case where the morphism φ is replaced by the morphism $\psi : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ which is given by $\psi^*(x_i) = y_i^n$, where the positive integer n is prime to p. In that case, not only is $\psi^{\circ,\mathrm{rig}} : \widetilde{\mathfrak{X}}^{\circ,\mathrm{rig}} \to \mathfrak{X}^{\circ,\mathrm{rig}}$ étale, but the induced morphism on the special fibers $\psi_s^\circ : \widetilde{\mathfrak{X}}_s^\circ \to \mathfrak{X}_s^\circ$ is étale, too. Hence, in this case, $(\psi_s)_* \mathcal{O}_{\widetilde{\mathfrak{X}},\mathbb{Q}}({}^\dagger \widetilde{Z})$ is an overconvergent F-isocrystal on $\mathfrak{X}_s$, by [14, 4.1.4], [4, Thm. 5]. By [7, 4.3.5] or [10, Prop. 3.1], the $\mathcal{D}^\dagger_{\mathfrak{X}}$-module $(\psi_s)_* \mathcal{O}_{\widetilde{\mathfrak{X}},\mathbb{Q}}({}^\dagger \widetilde{Z})$ is then holonomic, and hence coherent. As we will see below, these results fail to hold for the case of the morphism φ. ■

In order to decompose $\mathcal{M}$ with respect to the action of the covering group of $\widetilde{\mathfrak{X}}^{\circ,\mathrm{rig}}/\mathfrak{X}^{\circ,\mathrm{rig}}$, which is μ_p, we assume that L contains a primitive p-th root of unity ζ_p. It will be clear from the following discussion that we can assume that $\mathbb{Q}_p(\zeta_p)$ is the base field, in which case $\varpi = \zeta_p - 1$ is a uniformizer. The case of an arbitrary finite extension $L/\mathbb{Q}_p(\zeta_p)$ follows then from base change to L. From now on we thus assume $L = \mathbb{Q}_p(\zeta_p)$. For a character $\chi : \mu_p \to L^\times$ we denote by $\mathcal{M}^\chi$ the subsheaf of germs of sections s of $\mathcal{M}$ which have the property that $\zeta_p.s = \chi(\zeta_p) \cdot s$.

Theorem 4.3. *(i) If $\chi(\zeta_p) = \zeta_p^j$, for some $j \in \{0, 1, \dots, p-1\}$, then $\mathcal{M}^\chi$ is a free $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)$-module of rank one generated by y^j.*

(ii) We have $\mathcal{M} = \bigoplus_{\chi \in \mu_p^\vee} \mathcal{M}^\chi$, where $\mu_p^\vee$ is the character group of μ_p.

(iii) For the trivial character $\mathbf{1}$ the sheaf $\mathcal{M}^{\mathbf{1}}$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}}$-module.

(iv) When $\chi \in \mu_p^\vee$ is non-trivial, then the action of the sheaf of finite order differential operators $\mathcal{D}_{\mathbb{P}^1_{\mathfrak{o}},\mathbb{Q}}$ on $\mathcal{M}^\chi$ extends to an action of $\mathcal{D}^\dagger_{\mathfrak{X},p}$ on $\mathcal{M}^\chi$, and $\mathcal{M}^\chi$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X},p}$-module.

Proof. (i) and (ii) : We observe that

$$\begin{aligned} H^0(\mathfrak{X}_s, \mathcal{M}) &= H^0(\widetilde{\mathfrak{X}}_s, \widetilde{\mathcal{M}}) = H^0(\widetilde{\mathfrak{X}}^{\mathrm{rig}}, j_s^\dagger \mathcal{O}_{\widetilde{\mathfrak{X}}^{\mathrm{rig}}}) = H^0(\widetilde{\mathfrak{X}}^{\circ,\mathrm{rig}}, \mathcal{O}_{\widetilde{\mathfrak{X}}^{\mathrm{rig}}})^\dagger \\ &= \Big\{ \textstyle\sum_{i\in\mathbb{Z}} a_i y^i \mid \forall i : a_i \in L,\ \exists r > 1 : |a_i| r^{|i|} \to 0 \text{ as } |i| \to \infty \Big\} \end{aligned}$$

is the space of overconvergent functions on $\widetilde{\mathfrak{X}}^{\circ,\mathrm{rig}} =]\widetilde{\mathfrak{X}}^\circ_s[$. This is a free module of rank p over the ring of overconvergent functions

$$H^0(\mathfrak{X}^{\circ,\mathrm{rig}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}})^\dagger = \Big\{ \sum_{i\in\mathbb{Z}} b_i x^i \mid \forall i : b_i \in L,\ \exists r > 1 : |b_i| r^{|i|} \to 0 \text{ as } |i| \to \infty \Big\}$$

on $\mathfrak{X}^{\circ,\mathrm{rig}} =]\mathfrak{X}^\circ_s[$, and the functions $1, y, \dots, y^{p-1}$ form a basis of $H^0(\widetilde{\mathfrak{X}}^{\circ,\mathrm{rig}}, \mathcal{O}_{\widetilde{\mathfrak{X}}^{\mathrm{rig}}})^\dagger$ as a module over $H^0(\mathfrak{X}^{\circ,\mathrm{rig}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}})^\dagger$. Note that

$$H^0(\widetilde{\mathfrak{X}}^{\circ,\mathrm{rig}}, \mathcal{O}_{\widetilde{\mathfrak{X}}^{\mathrm{rig}}})^\dagger \cdot y^j = \Big(H^0(\widetilde{\mathfrak{X}}^{\circ,\mathrm{rig}}, \mathcal{O}_{\widetilde{\mathfrak{X}}^{\mathrm{rig}}})^\dagger \Big)^\chi = H^0(\mathfrak{X}_s, \mathcal{M}^\chi) \ .$$

The statement about the sheaf $\mathcal{M}^\chi$ follows from these observations, because $\mathcal{M}^\chi$ is obtained from its global sections by localization with respect to the "overconvergent structure sheaf" $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)$, i.e., $\mathcal{M}^\chi(U) = \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)(U) \cdot H^0(\mathfrak{X}_s, \mathcal{M}^\chi)$.

(iii) Follows from (i) since $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X}}$-module, cf. [2, 4.2.2].

(iv) We now assume $\chi(\zeta_p) = \zeta_p^j$ with $0 < j < p$. Let ∂_x (resp. ∂_y) be the derivative with respect to the coordinate x (resp. y). Then $1 = \partial_x x = \partial_x y^p = p y^{p-1} \partial_x y$, so that

$\partial_x y = \frac{1}{py^{p-1}} = \frac{1}{px}y$, and thus $\partial_x y^j = jy^{j-1}\partial_x y = \frac{j}{px}y^j$. It is easy to show by induction that for any $n \in \mathbb{N}$ one has

$$\partial_x^n y^j = \frac{\gamma(j,n)}{p^n x^n} y^j \ , \quad \text{where} \quad \gamma(j,n) = \prod_{i=0}^{n-1}(j - ip) \ . \tag{4.4}$$

Note that $\gamma(j,n)$ is always a p-adic unit. The presence of p^n in the denominator in the fraction on the left of 4.4 shows that $\mathcal{M}^\chi$ can not be a module for $\mathcal{D}^\dagger_{\mathfrak{X}}$, for reasons of convergence. Recall that in the present situation $\varpi = \zeta_p - 1$ is a uniformizer of L, and $|\varpi| = |p|^{\frac{1}{p-1}}$, so that $|\varpi^{pn}| = |p|^n \cdot |p|^{\frac{n}{p-1}}$. We write $\overline{0}$ and $\overline{\infty}$ for the two points in Z where the coordinate $\overline{x}$ takes on the values 0 and ∞, respectively.

Remark. In the following we will consider arithmetic differential operators with congruence level p. If $U \subset \mathfrak{X}_s \backslash \{\overline{\infty}\}$ is open affine, then such a differential operator $P \in \widehat{\mathcal{D}}^{(p,m)}_{\mathfrak{X},\mathbb{Q}}(U)$ can be written as $\sum_{n\geqslant 0} b_n \frac{q_n^{(m)}!}{n!}\varpi^{pn}\partial_x^n$, where $b_n \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U)$ and $\|b_n\| \to 0$ as $n \to \infty$. Here $\|\cdot\|$ can be any L-Banach norm on $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U)$. Because we are working with reduced rings, we can and will equip the ring of rigid analytic functions on the tubular neighborhood $]U[$, which is an affinoid subspace of $\mathfrak{X}^{\mathrm{rig}}$ with its supremum norm, which we denote by $\|\cdot\|_{]U[}$. If we put $\partial_x^{[n]} = \frac{1}{n!}\partial_x^n$, then P assumes the form $\sum_{n\geqslant 0} a_n \varpi^{pn}\partial_x^{[n]}$, where $a_n = q_n^{(m)}!b_n$. For $u \in \mathbb{N}$ let $s_p(u)$ be the sum of the digits of $u \in \mathbb{N}$ in its p-adic expansion. Since

$$v_p(u!) = \frac{u - s_p(u)}{p-1} \ , \tag{4.5}$$

we can conclude that $\|a_n\|_{]U[} = O(\eta^n)$ for any $\eta > |p|^{\frac{1}{(p-1)p^m}}$. This shows that any $P \in \mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(U)$ can be written as $\sum_{n\geqslant 0} a_n \varpi^{pn}\partial_x^{[n]}$, where $a_n \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U)$ and there is $\eta < 1$ such that $\|a_n\|_{]U[} = O(\eta^n)$. An analogous statement is true when $U \subset \mathfrak{X}_s \setminus \{\overline{0}\}$ and when ∂_x is replaced by $\partial_{x^{-1}}$. ■

Step 1: $\mathcal{D}^\dagger_{\mathfrak{X},p}$ acts on $\mathcal{M}^\chi$. We want to show that the natural action of the sheaf of finite order differential operators on $\mathcal{M}^\chi$ extends to an action of $\mathcal{D}^\dagger_{\mathfrak{X},p}$. Let $U \subset \mathfrak{X}_s$ be an open subset, fix $m \in \mathbb{N}$ and consider a differential operator $P \in \widehat{\mathcal{D}}^{(p,m)}_{\mathfrak{X}}(U)$. Set $U_0 = U \setminus \{\overline{\infty}\}$ and $U_\infty = U \setminus \{\overline{0}\}$, so that $U = U_0 \cup U_\infty$. The key point is to check that $(P|_{U_0})\cdot y^j \in \mathcal{M}^\chi(U_0)$ and $(P|_{U_\infty})\cdot y^j \in \mathcal{M}^\chi(U_\infty)$. If this is true, then $P\cdot y^j \in \mathcal{M}^\chi(U)$. Recall that $\overline{x}$ is a coordinate on U_0 and we can write

$$P|_{U_0} = \sum_{n\geqslant 0} a_n \varpi^{pn}\partial_x^{[n]} \ , \tag{4.6}$$

where $a_n \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U_0)$, and there is $\eta < 1$ such that

$$\|a_n\|_{]U_0[} = O(\eta^n) \quad \text{as} \quad n \to \infty \ . \tag{4.7}$$

We apply 4.4 and find that $(P|_{U_0}) \cdot y^j$ can be written formally as a series

$$\left(\sum_{n \geqslant 0} \gamma(j,n) \cdot \frac{a_n}{x^n} \cdot \frac{\varpi^{(p-1)n}}{p^n} \cdot \frac{\varpi^n}{n!}\right) \cdot y^j \ . \tag{4.8}$$

We want to show that 4.8 converges in $\mathcal{M}^\chi(U_0)$. Note that $\frac{\varpi^{(p-1)n}}{p^n}$ is a p-adic unit. It follows from 4.5 that $\frac{\varpi^n}{n!} \in \mathfrak{o}$. If $\overline{0} \notin U_0$, then $\mathcal{M}^\chi(U_0) = \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U_0) \cdot y^j$, and there are no conditions on overconvergence at $\overline{0}$ involved. Thus, 4.7 and 4.8 show that $(P|_{U_0}) \cdot y^j \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U_0) \cdot y^j = \mathcal{M}^\chi(U_0)$. Now suppose that $\overline{0}$ is contained in U_0. Let $\eta < 1$ be as in 4.7, and let $r \in]\eta, 1[$ be any real number. Put $\mathcal{V}_r = \{r \leqslant |x| < 1\} \cup]U_0 \setminus \{\overline{0}\}[$, which is an affinoid subdomain of $\mathfrak{X}^{\rm rig}$ contained in $]U_0[$. Then

$$\left\|\gamma(j,n) \cdot \frac{a_n}{x^n} \cdot \frac{\varpi^{(p-1)n}}{p^n} \cdot \frac{\varpi^n}{n!}\right\|_{\mathcal{V}_r} \leqslant \left\|\frac{a_n}{x^n}\right\|_{\mathcal{V}_r} = O\left(\left(\frac{\eta}{r}\right)^n\right) \ ,$$

which shows that 4.8 converges on $\mathcal{V}_r$. We thus have $(P|_{U_0}) \cdot y^j \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)(U_0) \cdot y^j = \mathcal{M}^\chi(U_0)$. The arguments for $(P|_{U_\infty}) \cdot y^j$ are similar. We note first that $\overline{x}^{-1}$ is a coordinate on U_∞, and $P|_{U_\infty}$ can be expanded in the form $\sum_{n \geqslant 0} b_n \varpi^{pn} \partial_{x^{-1}}^{[n]}$, with $b_n \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U_\infty)$ and $\|b_n\|_{]U[} = O(\eta^n)$ for some $\eta < 1$. The discussion now follows along the same lines as above, and one verifies that $(P|_{U_\infty}) \cdot y^j \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)(U_\infty) \cdot y^j = \mathcal{M}^\chi(U_\infty)$. Hence the natural action of the sheaf of finite order differential operators on $\mathcal{M}^\chi$ extends to an action of $\mathcal{D}^\dagger_{\mathfrak{X},p}$.

Step 2: $\mathcal{M}^\chi$ is generated by y^j as $\mathcal{D}^\dagger_{\mathfrak{X},p}$-module. To this end, let $t \in \mathfrak{X}_s$ be a closed point, and let $U \subset \mathfrak{X}_s$ be an open affine neighborhood of t. Given $f \in \mathcal{M}^\chi(U)$ we want to show that, after possibly shrinking U, there is $P \in \mathcal{D}^\dagger_{\mathfrak{X},p}(U)$ such that $P \cdot y^j = f$. If $t \notin Z$, we may assume $U \cap Z = \varnothing$, in which case $f = h(x)y^j$ with $h(x) \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U)$. Since $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U) \subset \mathcal{D}^\dagger_{\mathfrak{X},p}(U)$ we can take $P = h(x)$. Now suppose $t = \overline{0}$. After possibly shrinking U we may assume $\overline{\infty} \notin U$. Then f can be written as $\left(\sum_{n \geqslant 0} \frac{b_n}{x^n}\right) \cdot y^j$, with $b_n \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U)$, and there is $\eta < 1$ such that $\|b_n\|_{]U[} = O(\eta^n)$. Set

$$a_n = \left(\frac{1}{\gamma(j,n)} \cdot \frac{p^n}{\varpi^{(p-1)n}} \cdot \frac{n!}{\varpi^n}\right) \cdot b_n \ , \tag{4.9}$$

which is an element of $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U)$. Let $\eta_1 < 1$ be any number such that $\eta_1 > \eta$. Formula 4.5 and the estimate $s_p(u) = O(\log_p(u))$ imply that $\|a_n\|_{]U[} = O(\eta_1^n)$. If we now define P by the formula 4.6, then the estimate $\|a_n\|_{]U[} = O(\eta_1^n)$ shows that P lies $\mathcal{D}^\dagger_{\mathfrak{X},p}(U)$. Moreover, it follows from 4.8 that $P \cdot y^j = f$. Analogous arguments apply in the case $t = \overline{\infty}$.

Step 3: $\mathcal{M}^\chi$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X},p}$-module. Put $P_j = px\partial_x - j$. We will show that the following sequence

$$\mathcal{D}^\dagger_{\mathfrak{X},p} \xrightarrow{\beta} \mathcal{D}^\dagger_{\mathfrak{X},p} \xrightarrow{\alpha} \mathcal{M}^\chi \longrightarrow 0 \tag{4.10}$$

is exact, where $\alpha(P) = P \cdot y^j$ and $\beta(P) = P \cdot P_j$. That α is surjective has been shown above in step 2. Formula 4.4 (for $n = 1$) shows that $\alpha \circ \beta = 0$. Let $U \subset \mathfrak{X}_s \setminus \{\overline{\infty}\}$ be an affine open neighborhood of $\overline{0}$, and let $P = \sum_{n\geqslant 0} a_n \varpi^{pn} \partial_x^{[n]} \in \mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(U)$ be a differential operator in the kernel of α. There is $\eta < 1$ such that

$$\|a_n\|_{]U[} = O(\eta^n) \quad \text{as} \quad n \to \infty\ . \tag{4.11}$$

The assumption $\alpha(P) = 0$ is equivalent to the series

$$\sum_{n\geqslant 0} \gamma(j,n) \cdot \frac{a_n}{x^n} \cdot \frac{\varpi^{(p-1)n}}{p^n} \cdot \frac{\varpi^n}{n!} \tag{4.12}$$

converging to zero in $\mathcal{M}^\chi(U) = \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)(U) \cdot y^j$. For $N \in \mathbb{N}$ let

$$S_N = \sum_{n=0}^{N} \gamma(j,n) \cdot \frac{a_n}{x^n} \cdot \frac{\varpi^{(p-1)n}}{p^n} \cdot \frac{\varpi^n}{n!}$$

be the N^{th} partial sum of 4.12. Saying that the S_N converge to zero in $\mathcal{M}^\chi(U)$ means there is $r < 1$ such that, if $\mathcal{V}_r = \{r \leqslant |x| < 1\} \cup\,]U \setminus \{\overline{0}\}[$ (which is an affinoid subdomain of $\mathfrak{X}^{\text{rig}}$), then

$$\|S_N\|_{\mathcal{V}_r} \longrightarrow 0 \quad \text{as} \quad N \to \infty\ . \tag{4.13}$$

We may and will assume that $r > \eta$, where η is as in 4.11. Set $b_0 = -\frac{1}{j} a_0$, and define recursively for $N \geqslant 1$

$$b_N = \frac{N}{j - Np} \frac{px}{\varpi^p} b_{N-1} - \frac{1}{j - Np} a_N\ . \tag{4.14}$$

We claim that the functions b_N are given by the closed formula

$$b_N = -\frac{N!p^N x^N}{\gamma(j, N+1)\varpi^{pN}} \cdot S_N \tag{4.15}$$

Proof of 4.15. This formula is easily verified for $N = 0$. Now suppose that $N \geqslant 1$ and the formula is correct for $N - 1$. The right hand side of 4.15 is equal to

$$\begin{aligned}
& \left[-\tfrac{N!p^N x^N}{\gamma(j,N+1)\varpi^{pN}} \sum_{n=0}^{N-1} \gamma(j,n) \cdot \tfrac{a_n}{x^n} \cdot \tfrac{\varpi^{(p-1)n}}{p^n} \cdot \tfrac{\varpi^n}{n!}\right] - \tfrac{1}{j-Np} a_N \\
= \; & \tfrac{Npx}{(j-Np)\varpi^p} \left[-\tfrac{(N-1)!p^{N-1}x^{N-1}}{\gamma(j,N)\varpi^{p(N-1)}} \sum_{n=0}^{N-1} \gamma(j,n) \cdot \tfrac{a_n}{x^n} \cdot \tfrac{\varpi^{(p-1)n}}{p^n} \cdot \tfrac{\varpi^n}{n!}\right] - \tfrac{1}{j-Np} a_N \\
\overset{(1)}{=} \; & \tfrac{Npx}{(j-Np)\varpi^p} b_{N-1} - \tfrac{1}{j-Np} a_N \overset{(2)}{=} b_N \;,
\end{aligned}$$

where equation (1) holds by induction hypothesis and equation (2) by 4.14. □

Note that each function b_N is contained in $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U)$. Let $\mathcal{V}_r$ be as before. Note that $]U[\setminus \mathcal{V}_r = \{|x| < r\}$, and that $\mathcal{V}_r$ contains the "boundary" $\{|x| = r\}$ of the wide open disc $\{|x| < r\}$. By the maximum principle and 4.15, we have

$$\|b_N\|_{]U[} = \|b_N\|_{\mathcal{V}_r} = \left\| -\frac{N!p^N x^N}{\gamma(j, N+1)\varpi^{pN}} \cdot S_N \right\|_{\mathcal{V}_r} \leqslant \left| \frac{N!}{\varpi^N} \right| \cdot \|S_N\|_{\mathcal{V}_r} \,. \tag{4.16}$$

By 4.13 we know that $\|S_k\|_{\mathcal{V}_r} \to 0$ as $k \to \infty$. This implies

$$\begin{aligned}
\|S_N\|_{\mathcal{V}_r} \; &= \; \lim_{k\to\infty} \|S_k - S_N\|_{\mathcal{V}_r} = \left\| \sum_{n=N+1}^{\infty} \gamma(j,n) \cdot \tfrac{a_n}{x^n} \cdot \tfrac{\varpi^{(p-1)n}}{p^n} \cdot \tfrac{\varpi^n}{n!} \right\|_{\mathcal{V}_r} \\
&\leqslant \; \max_{n>N} \left\| \gamma(j,n) \cdot \tfrac{a_n}{x^n} \cdot \tfrac{\varpi^{(p-1)n}}{p^n} \cdot \tfrac{\varpi^n}{n!} \right\|_{\mathcal{V}_r} \leqslant \; \max_{n>N} \left\| \tfrac{a_n}{x^n} \right\|_{\mathcal{V}_r} \\
&= \; O\left(\left(\tfrac{\eta}{r}\right)^{N+1} \right) \,.
\end{aligned} \tag{4.17}$$

We recall that we have chosen $r \in]\eta, 1[$. Therefore, 4.16 and 4.17 show that for any $\eta_2 \in]\frac{\eta}{r}, 1[$ we have $\|b_N\|_{]U[} = O(\eta_2^N)$. This implies that $Q := \sum_{n\geqslant 0} b_n \varpi^{pn} \partial_x^{[n]}$ is contained in $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(U)$. Note that $\partial_x^{[n]} P_j = p\partial_x^{[n]} x \partial_x - j\partial_x^{[n]} = (n+1)px\partial_x^{[n+1]} + (np-j)\partial_x^{[n]}$. We compute

$$\begin{aligned} QP_j &= \sum_{n\geqslant 0} pxb_n\varpi^{pn}(n+1)\partial_x^{[n+1]} + \sum_{n\geqslant 0}(np-j)b_n\varpi^{pn}\partial_x^{[n]} \\ &= -jb_0 + \sum_{n\geqslant 1}\Big[(np-j)b_n + n\tfrac{px}{\varpi^p}b_{n-1}\Big]\varpi^{pn}\partial_x^{[n]} \\ &= -jb_0 + \sum_{n\geqslant 1}\Big[(np-j)b_n + n\tfrac{px}{\varpi^p}b_{n-1}\Big]\varpi^{pn}\partial_x^{[n]} \\ &= a_0 + \sum_{n\geqslant 1} a_n\varpi^{pn}\partial_x^{[n]} = P\ , \end{aligned}$$

where the first equality in the last row holds by 4.14. The arguments for the case when $U \subset \mathfrak{X}_s \setminus \{\overline{0}\}$ is an affine open neighborhood of $\overline{\infty}$ are analogous to the case just treated. The case when $U \subset Z = \varnothing$ proceeds along the same lines and is even easier. This proves that 4.10 is indeed exact, and $\mathcal{M}^\chi$ is a coherent $\mathcal{D}^\dagger_{\mathfrak{X},p}$-module. □

Proposition 4.18. *(i) When $\chi \in \mu_p^\vee$ is non-trivial, then the action of the sheaf of finite order differential operators $\mathcal{D}_{\mathbb{P}^1_{\mathfrak{o}},\mathbb{Q}}$ on $\mathcal{M}^\chi$ does not extend to an action of $\mathcal{D}^\dagger_{\mathfrak{X}}$ on $\mathcal{M}^\chi$.*

(ii) The action of the sheaf of finite order differential operators $\mathcal{D}_{\mathbb{P}^1_{\mathfrak{o}},\mathbb{Q}}$ on $\mathcal{M}$ does not extend to an action of $\mathcal{D}^\dagger_{\mathfrak{X}}$ on $\mathcal{M}$.

(iii) The action of the sheaf of finite order differential operators $\mathcal{D}_{\mathbb{P}^1_{\mathfrak{o}},\mathbb{Q}}$ on $\mathcal{M}$ extends to an action of $\mathcal{D}^\dagger_{\mathfrak{X},p}$ on $\mathcal{M}$, but $\mathcal{M}$ is not coherent as a $\mathcal{D}^\dagger_{\mathfrak{X},p}$-module.

Proof. (i) This has already been noted in the course of the proof of 4.3 (iv), as a consequence of formula 4.4.

(ii) This is an immediate consequence of (i) because $\mathcal{M}^\chi$ is a direct summand which is stable under $\mathcal{D}_{\mathbb{P}^1_{\mathfrak{o}},\mathbb{Q}}$.

(iii) By 4.3 (iv), the sheaf $\mathcal{D}^\dagger_{\mathfrak{X},p}$ acts on each direct summand $\mathcal{M}^\chi$, when χ is not the trivial character. But if $\chi = \mathbf{1}$ is the trivial character then $\mathcal{M}^{\mathbf{1}}$ is even a module for $\mathcal{D}^\dagger_{\mathfrak{X}}$, and thus, a fortiori, a module over $\mathcal{D}^\dagger_{\mathfrak{X},p} \subset \mathcal{D}^\dagger_{\mathfrak{X}}$. But for $\mathcal{M}$ to be coherent over $\mathcal{D}^\dagger_{\mathfrak{X},p}$, it is necessary (and sufficient) that $\mathcal{M}^{\mathbf{1}} = \mathcal{M}/\bigoplus_{\chi\neq\mathbf{1}}\mathcal{M}^\chi$ is coherent over $\mathcal{D}^\dagger_{\mathfrak{X},p}$. If $\mathcal{M}^{\mathbf{1}}$ would be coherent over $\mathcal{D}^\dagger_{\mathfrak{X},p}$, it would be of finite type. We will show, however, that no finite set of germs of $\mathcal{M}^{\mathbf{1}}$ at $\overline{0}$ can generate the stalk $\mathcal{M}^{\mathbf{1}}_{\overline{0}}$ as a module over the stalk of $\mathcal{D}^\dagger_{\mathfrak{X},p}$ at $\overline{0}$. Let U be an open affine neighborhood of $\overline{0}$. For $r \in]0,1[$ put $\mathcal{V}_r = \{r \leqslant |x| < 1\} \cup]U \setminus \{\overline{0}\}[$. Let $f \in \mathcal{M}^{\mathbf{1}}(U) = \mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)(U)$. Then there is $r \in]0,1[$ such that $f \in \mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{V}_r)$. We claim that

$$\|\partial_x^{[n]} f\|_{\mathcal{V}_r} = O(r^{-n}) \ . \qquad (4.19)$$

Proof of 4.19. We recall this familiar argument for the sake of completeness. For this purpose we may work with the $\overline{L}$-valued points of $\mathcal{V}_r$. Note first that $U = \mathfrak{X}_s \setminus \{\overline{\infty}, s_1, \ldots, s_g\}$ for a finite set of closed points $\{s_1, \ldots, s_g\} \subset \mathfrak{X}_s \setminus Z$. Therefore,

$$]U \setminus \{\overline{0}\}[= \mathfrak{X}^{\mathrm{rig}} \setminus \Big( ]\overline{\infty}[\cup \bigcup_{i=1}^{g}]\{s_i\}[\Big) .$$

If $a \in \mathcal{V}_r(\overline{L}) \subset U(\overline{L})$ is such that $|a| = 1$, then a is not contained in any of the residue discs $]\{s_i\}[$. Thus, for any $\rho < 1$ the rigid analytic disc $\mathbb{B}(a, \rho) = \{|x - a| \leqslant \rho\}$ is contained in $\mathcal{V}_r$. Similarly, if $a \in \mathcal{V}_r(\overline{L})$ is such that $r \leqslant |a| < 1$, then $\mathbb{B}(a, \rho)$ is contained in $\mathcal{V}_r$ if and only if $\rho < |a|$. Fix $a \in \mathcal{V}_r(\overline{L})$ and expand f as a power series around a, i.e., $f(x) = \sum_{i \geqslant 0} (\partial_x^{[i]} f)(a)(x - a)^i$, which is convergent on $\mathbb{B}(a, \rho)$ for any $\rho < |a|$. Then

$$\|f\|_{\mathbb{B}(a,\rho)} = \max_{i \geqslant 0} \{ \ |(\partial_x^{[i]} f)(a)| \cdot \rho^i \ \} . \tag{4.20}$$

Noting that $\partial_x^{[i]} \partial_x^{[n]} = \binom{n+i}{i} \partial_x^{[n+i]}$, we see that 4.20 implies that

$$\|\partial_x^{[n]} f\|_{\mathbb{B}(a,\rho)} \leqslant \|f\|_{\mathbb{B}(a,\rho)} \cdot \rho^{-n} .$$

Letting $\rho \uparrow |a|$ we find that the supremum norm of $\partial_x^{[n]} f$ on $\{|x - a| < |a|\}$ is bounded by $\|f\|_{\mathbb{B}(a,\rho)} \cdot |a|^{-n}$, from which we obtain 4.19 when we take $|a| = r$ to be minimal among all $a \in \mathcal{V}_r(\overline{L})$. □

Now write $P \in \mathcal{D}^{\dagger}_{\mathfrak{X},p}(U)$ as $P = \sum_{n \geqslant 0} a_n \varpi^{pn} \partial_x^{[n]}$, with $a_n \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(U)$, and such that there is $\eta < 1$ satisfying $\|a_n\|_{]U[} = O(\eta^n)$. If $r \geqslant |\varpi^p| = |p|^{\frac{p}{p-1}}$, then the estimate 4.19 implies that $P \cdot f = \sum_{n \geqslant 0} a_n \varpi^{pn} \partial_x^{[n]} f$ converges in the space of rigid analytic functions on $\mathcal{V}_r$. Suppose now $(f_1)_{\overline{0}}, \ldots, (f_\nu)_{\overline{0}}$ are finitely many elements of the stalk $\mathcal{M}^{\mathbf{1}}_{\overline{0}}$. Then there is an open neighborhood U of $\overline{0}$ such that each $(f_i)_{\overline{0}}$ is represented by $f_i \in \mathcal{M}^{\mathbf{1}}(U)$, and there is $r \in [|\varpi|^p, 1[$ such that each f_i is a rigid analytic function on $\mathcal{V}_r = \{r \leqslant |x| < 1\} \cup]U \setminus \{\overline{0}\}[$. By what we have just seen, any element of the stalk represented by $f = P_1 \cdot f_1 + \ldots + P_\nu \cdot f_\nu$, with $P_i \in \mathcal{D}^{\dagger}_{\mathfrak{X},p}(U)$ is thus represented by a rigid analytic function on $\mathcal{V}_r$ (i.e., with the same "radius of overconvergence"). But not every element in this stalk can be represented by a function of this kind. □

5. The sheaves $\mathcal{D}_{\mathfrak{X},\infty}$ and $\mathcal{D}_{\langle \mathfrak{X}_0 \rangle}$

We come back to the general situation of sections 2 and 3 and let

$$\mathrm{pr} : \mathfrak{X} \to \mathfrak{X}_0$$

be an admissible formal blow-up of a smooth formal scheme $\mathfrak{X}_0$. In the following k stands for an integer bigger or equal to $k_{\mathfrak{X}}$. We then have the sheaf of coherent rings $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$ on $\mathfrak{X}$, together with morphisms $\mathcal{D}^{\dagger}_{\mathfrak{X},k+1} \to \mathcal{D}^{\dagger}_{\mathfrak{X},k}$ which are given locally by the natural inclusions (2.4). These data form a projective system and we denote by

$$\mathcal{D}_{\mathfrak{X},\infty} := \varprojlim_{k} \mathcal{D}^{\dagger}_{\mathfrak{X},k} \tag{5.1}$$

its projective limit. For every open subset $U \subset \mathfrak{X}$, the sheaf of rings $\mathcal{D}_{\mathfrak{X},\infty}$ satifies $\mathcal{D}_{\mathfrak{X},\infty}(U) = \varprojlim_k \mathcal{D}^{\dagger}_{\mathfrak{X},k}(U)$.

Proposition 5.2.

(i) For every affine open subset $U \subset \mathfrak{X}$ the algebra $\mathcal{D}_{\mathfrak{X},\infty}(U)$ is a Fréchet-Stein algebra in the sense of [13].

(ii) Let $U \subset \mathfrak{X}_0$ be an open affine subset endowed with étale coordinates $x_1, \ldots, x_d$ and $\partial_1, \ldots, \partial_d$ the corresponding derivations. Then, for any affine open $V \subset \mathrm{pr}^{-1}(U)$ we have

$$\mathcal{D}_{\mathfrak{X},\infty}(V) = \left\{ \sum_{\underline{\nu}} a_{\underline{\nu}} \underline{\partial}^{[\underline{\nu}]} \,\middle|\, a_{\underline{\nu}} \in \mathcal{O}_{\mathfrak{X},\mathbb{Q}}(V)\,,\, \|a_{\underline{\nu}}\| = O(\eta^{|\underline{\nu}|}) \text{ for all } \eta > 0 \right\}\,, \tag{5.3}$$

where $\|\cdot\|$ is any Banach algebra norm on $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}(V)$.

(iii) Let $\mathfrak{X}' \to \mathfrak{X}_0$ be another admissible formal blow-up, and let $\pi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism over $\mathfrak{X}_0$. Then the canonical isomorphisms $\pi_* \mathcal{D}^{\dagger}_{\mathfrak{X}',k} \simeq \mathcal{D}^{\dagger}_{\mathfrak{X},k}$, for $k \geqslant \max\{k_{\mathfrak{X}'}, k_{\mathfrak{X}}\}$ give rise to a canonical isomorphism

$$\pi_* \mathcal{D}_{\mathfrak{X}',\infty} \simeq \mathcal{D}_{\mathfrak{X},\infty}\,. \tag{5.4}$$

For every affine open subset $U \subset \mathfrak{X}$ we have, according to the preceding proposition, the abelian category of coadmissible $\mathcal{D}_{\mathfrak{X},\infty}(U)$-modules $\mathcal{C}_{\mathcal{D}_{\mathfrak{X},\infty}(U)}$, cf. [13].

Definition 5.5. A $\mathcal{D}_{\mathfrak{X},\infty}$-module $\mathcal{M}$ is coadmissible if and only if there is a projective system $(\mathcal{M}_k, \mathcal{M}_{k+1} \to \mathcal{M}_k)_{k \geqslant k_{\mathfrak{X}}}$, where $\mathcal{M}_k$ is a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$-module, such that

(i) the transition map $\mathcal{M}_{k+1} \to \mathcal{M}_k$ is $\mathcal{D}^{\dagger}_{\mathfrak{X},k+1}$-linear and the induced $\mathcal{D}^{\dagger}_{\mathfrak{X},k}$-linear morphism

$$\mathcal{D}^{\dagger}_{\mathfrak{X},k} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X},k+1}} \mathcal{M}_{k+1} \xrightarrow{\simeq} \mathcal{M}_k \tag{5.6}$$

is an isomorphism, and

(ii) $\mathcal{M} \simeq \varprojlim_k \mathcal{M}_k$ as $\mathcal{D}_{\mathfrak{X},\infty}$-module.

We denote by

$$\mathcal{C}_{\mathfrak{X}} \subseteq \mathrm{Mod}(\mathcal{D}_{\mathfrak{X},\infty})$$

the full subcategory of coadmissible $\mathcal{D}_{\mathfrak{X},\infty}$-modules in the category of all $\mathcal{D}_{\mathfrak{X},\infty}$-modules.

Proposition 5.7. *The category $\mathcal{C}_{\mathfrak{X}}$ is abelian. Let $\mathfrak{X}' \to \mathfrak{X}_0$ be another admissible formal blow-up, and let $\pi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism over $\mathfrak{X}_0$. The canonical isomorphism $\pi_*\mathcal{D}_{\mathfrak{X}',\infty} \simeq \mathcal{D}_{\mathfrak{X},\infty}$ induces an equivalence of abelian categories*

$$\pi_* : \mathcal{C}_{\mathfrak{X}'} \xrightarrow{\simeq} \mathcal{C}_{\mathfrak{X}} \ .$$

The proof of the proposition relies on the invariance theorem and the following versions of theorem A and B.

Theorem 5.8. *(Theorem A for coadmissible modules on $\mathfrak{X}$) Let $U \subset \mathfrak{X}$ be an affine open subset. Then the global sections functor $\Gamma(U,-)$ induces an equivalence of categories*

$$\Gamma(U,-) : \mathcal{C}_U \xrightarrow{\simeq} \mathcal{C}_{\mathcal{D}_{\mathfrak{X},\infty}(U)} \ .$$

Theorem 5.9. *(Theorem B for coadmissible modules on $\mathfrak{X}$) One has $H^i(\mathfrak{X},\mathcal{M}) = 0$ for every $i > 0$ and for every coadmissible $\mathcal{D}_{\mathfrak{X},\infty}$-module $\mathcal{M}$.*

For the proofs of these two theorems we refer to [9].

We now explain how these sheaves lead to a sheaf on the Zariski-Riemann space of $\mathfrak{X}_0$. Let $\mathcal{F}_{\mathfrak{X}_0}$ be the set of all admissible formal blow-ups $\mathfrak{X} \to \mathfrak{X}_0$. This is a directed partially ordered set and the projective limit

$$\langle \mathfrak{X}_0 \rangle = \varprojlim_{\mathfrak{X} \in \mathcal{F}_{\mathfrak{X}_0}} \mathfrak{X}$$

is the Zariski-Riemann space associated with $\mathfrak{X}_0$. There is a canonical equivalence of categories between the abelian sheaves on $\langle \mathfrak{X}_0 \rangle$ and the abelian sheaves on the rigid analytic space $\mathfrak{X}_0^{\mathrm{rig}}$. For this and other basic properties of $\langle \mathfrak{X}_0 \rangle$ we refer to [6, 9.3].

For $\mathfrak{X} \in \mathcal{F}_{\mathfrak{X}_0}$ we denote the canonical projection map $\langle \mathfrak{X}_0 \rangle \to \mathfrak{X}$ by $\mathrm{sp}_{\mathfrak{X}}$. Assume $\mathfrak{X}' \geq \mathfrak{X}$. The canonical isomorphism $(\pi_{\mathfrak{X}',\mathfrak{X}})_*\mathcal{D}_{\mathfrak{X}',\infty} \simeq \mathcal{D}_{\mathfrak{X},\infty}$, together with the adjunction map $\pi^{-1}_{\mathfrak{X}',\mathfrak{X}} \circ (\pi_{\mathfrak{X}',\mathfrak{X}})_* \to \mathrm{id}$ gives rise to a canonical map

$$\varphi_{\mathfrak{X},\mathfrak{X}'} : \pi^{-1}_{\mathfrak{X}',\mathfrak{X}}\mathcal{D}_{\mathfrak{X},\infty} = \pi^{-1}_{\mathfrak{X}',\mathfrak{X}}(\pi_{\mathfrak{X}',\mathfrak{X}})_*\mathcal{D}_{\mathfrak{X}',\infty} \longrightarrow \mathcal{D}_{\mathfrak{X}',\infty}.$$

Applying the functor $\mathrm{sp}_{\mathfrak{X}'}^{-1}$ and using $\mathrm{sp}_{\mathfrak{X}} = \pi_{\mathfrak{X}',\mathfrak{X}} \circ \mathrm{sp}_{\mathfrak{X}'}$ results in a map

$$\mathrm{sp}_{\mathfrak{X}}^{-1}\mathcal{D}_{\mathfrak{X},\infty} \longrightarrow \mathrm{sp}_{\mathfrak{X}'}^{-1}\mathcal{D}_{\mathfrak{X}',\infty}.$$

We obtain in this way an inductive system $(\mathrm{sp}_{\mathfrak{X}}^{-1}\mathcal{D}_{\mathfrak{X},\infty})_{\mathfrak{X}\in\mathcal{F}_{\mathfrak{X}_0}}$, and we form its inductive limit

$$\mathcal{D}_{\langle \mathfrak{X}_0 \rangle} := \varinjlim_{\mathfrak{X}} \mathrm{sp}_{\mathfrak{X}}^{-1}\mathcal{D}_{\mathfrak{X},\infty} \ .$$

We view $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$ as a sheaf of (infinite order) analytic differential operators on $\langle\mathfrak{X}_0\rangle$ or $\mathfrak{X}_0^{\mathrm{rig}}$. A first important step is to single out a suitable abelian category of $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$-modules replacing the classical category of coherent D-modules.

Definition 5.10. A $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$-module $\mathcal{M}$ is called *coadmissible* if there is a family $(\mathcal{M}_{\mathfrak{X}}, \psi^{\mathcal{M}}_{\mathfrak{X},\mathfrak{X}'})$ of coadmissible $\mathcal{D}_{\mathfrak{X},\infty}$-modules $\mathcal{M}_{\mathfrak{X}}$, for all $\mathfrak{X} \in \mathcal{F}_{\mathfrak{X}_0}$, together with an isomorphism

$$\psi^{\mathcal{M}}_{\mathfrak{X}',\mathfrak{X}} : (\pi_{\mathfrak{X}',\mathfrak{X}})_*\mathcal{M}_{\mathfrak{X}'} \xrightarrow{\simeq} \mathcal{M}_{\mathfrak{X}} ,$$

of $\mathcal{D}_{\mathfrak{X},\infty}$-modules, whenever we have $\mathfrak{X}' \geq \mathfrak{X}$ in $\mathcal{F}_{\mathfrak{X}_0}$. This system of modules and isomorphisms is required to satisfy the following conditions:

(i) Whenever $\mathfrak{X}'' \geq \mathfrak{X}' \geq \mathfrak{X}$ in $\mathcal{F}_{\mathfrak{X}_0}$ the following transitivity condition holds :

$$\psi^{\mathcal{M}}_{\mathfrak{X}',\mathfrak{X}} \circ (\pi_{\mathfrak{X}',\mathfrak{X}})_*(\psi^{\mathcal{M}}_{\mathfrak{X}'',\mathfrak{X}'}) = \psi^{\mathcal{M}}_{\mathfrak{X}'',\mathfrak{X}} .$$

(ii) $\mathcal{M}$ is isomorphic to the inductive limit $\varinjlim_{\mathfrak{X}} \mathrm{sp}_{\mathfrak{X}}^{-1}\mathcal{M}_{\mathfrak{X}}$ as $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$-module.

Note that the transition morphism $\mathrm{sp}_{\mathfrak{X}}^{-1}\mathcal{M}_{\mathfrak{X}} \to \mathrm{sp}_{\mathfrak{X}'}^{-1}\mathcal{M}_{\mathfrak{X}'}$ in the inductive limit in (ii) is defined as above, starting from the isomorphism $(\pi_{\mathfrak{X}',\mathfrak{X}})_*\mathcal{M}_{\mathfrak{X}'} \simeq \mathcal{M}_{\mathfrak{X}}$, using the adjunction map and finally applying the functor $\mathrm{sp}_{\mathfrak{X}'}^{-1}$.

We denote by

$$\mathcal{C}_{\langle\mathfrak{X}_0\rangle} \subseteq \mathrm{Mod}(\mathcal{D}_{\langle\mathfrak{X}_0\rangle})$$

the full subcategory of coadmissible $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$-modules in the category of all $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$-modules.

Proposition 5.11. *The category $\mathcal{C}_{\langle\mathfrak{X}_0\rangle}$ is abelian. Let $\mathfrak{X} \in \mathcal{F}_{\mathfrak{X}_0}$. One has an equivalence of categories*

$$(\mathrm{sp}_{\mathfrak{X}})_* : \mathcal{C}_{\langle\mathfrak{X}_0\rangle} \xrightarrow{\simeq} \mathcal{C}_{\mathfrak{X}} .$$

This is an application of the invariance property 5.7 of the category $\mathcal{C}_{\mathfrak{X}}$. We refer to [9] for the proof. The proposition implies, in particular, that there is a canonical isomorphism of sheaves of rings

$$\Gamma(\langle\mathfrak{X}\rangle, \mathcal{D}_{\langle\mathfrak{X}_0\rangle}) \simeq \Gamma(\mathfrak{X}, \mathcal{D}_{\mathfrak{X},\infty})$$

for any $\mathfrak{X} \in \mathcal{F}_{\mathfrak{X}_0}$.

We have the following Theorem A and B for coadmissible $\mathcal{D}_{\langle\mathfrak{X}_0\rangle}$-modules, cf. [9].

Theorem 5.12. *(i) Let $U \subset \mathfrak{X}$ be an open affine and let $\langle U\rangle = \mathrm{sp}_{\mathfrak{X}}^{-1}(U)$ be its Zariski-Riemann space. The global sections functor $\Gamma(\langle U\rangle, -)$ fournishes an equivalence of categories*

$$\Gamma(\langle U\rangle, -) : \mathcal{C}_{\langle U\rangle} \xrightarrow{\simeq} \mathcal{C}_{\mathcal{D}_{U,\infty}(U)} .$$

(ii) One has $H^i(\langle\mathfrak{X}_0\rangle, \mathcal{M}) = 0$ for every $i > 0$ and every $\mathcal{M} \in \mathcal{C}_{\langle\mathfrak{X}_0\rangle}$.

6. Example: some coadmissible modules with singularities on the unit disc

6.1. *The direct image* $j_*\mathcal{O}_{X^\circ}$. We let $\mathfrak{X} = \mathrm{Spf}(\mathfrak{o}\langle x\rangle)$, and write X for the associated Zariski-Riemann space $\langle\mathfrak{X}\rangle$ which we identify with $\mathrm{Sp}(L\langle x\rangle)$, the rigid analytic unit disc over F. We thus write $\mathcal{D}_X$ for $\mathcal{D}_{\langle\mathfrak{X}\rangle}$. Set $X^\circ = X \setminus \{x = 0\}$, and let $j : X^\circ \to X$ be the inclusion.

Proposition 6.2. *(i)* $j_*\mathcal{O}_{X^\circ}$ *is a coadmissible* $\mathcal{D}_X$*-module.*

(ii) $H^0(X^\circ, \mathcal{O}_{X^\circ}) = H^0(X, j_*\mathcal{O}_{X^\circ})$ *is a coadmissible* $H^0(X, \mathcal{D}_X)$*-module.*

Proof. (i) is a consequence of (ii) because of 5.12.

(ii) The space $M := H^0(X^\circ, \mathcal{O}_{X^\circ})$ consists of all functions $f(x) = \sum_{n\in\mathbb{Z}} a_n x^n$, with coefficients $a_n \in L$, and where $\lim_{|n|\to\infty} |a_n| r^n = 0$ for any $r \in]0, 1]$. For a given $r \in]0, 1]$, we consider the annulus $\mathcal{V}_r = \{r \leqslant |x| \leqslant 1\} \subset X$. The space M is given the structure of a Fréchet space via the supremum norms $\|\cdot\|_{\mathcal{V}_r}$ for $r \in]0, 1]$. Furthermore, $D := H^0(X, \mathcal{D}_X)$ consists of all differential operators $P = \sum_{n\geqslant 0} c_n \partial_x^{[n]}$, with coefficients $c_n \in \mathcal{O}(X) = L\langle x\rangle$, and such that

$$|c_n\|_X = O(\eta^n) \quad \text{for any} \quad \eta > 0 \,. \tag{6.3}$$

Put $P_0 = x\partial_x + 1$. We claim that the following sequence is exact:

$$D \xrightarrow{\beta} D \xrightarrow{\alpha} M \longrightarrow 0 \,, \tag{6.4}$$

where $\alpha(P) = P \cdot \frac{1}{x}$, and $\beta(Q) = QP_0$. Because $P_0 \cdot \frac{1}{x} = 0$ we have $\alpha \circ \beta = 0$. Moreover, $\partial_x^{[n]}(\frac{1}{x}) = \frac{(-1)^n}{x^{n+1}}$. Then, given $f(x) = \sum_{n\in\mathbb{Z}} a_n x^n \in M$, we set $c_0 = c_0(x) = \sum_{n=-1}^{\infty} a_n x^{n+1}$, and for $n \geqslant 1$ we put $c_n = (-1)^n a_{-n-1}$. Then $P := \sum_{n\geqslant 0} c_n \partial_x^{[n]}$ is in D, and $P \cdot \frac{1}{x} = f(x)$. This shows that α is surjective. Now suppose $P = \sum_{n\geqslant 0} c_n \partial_x^{[n]}$ is in D, and

$$P \cdot \frac{1}{x} = \sum_{n\geqslant 0} (-1)^n \frac{c_n}{x^{n+1}} = 0 \,,$$

where the convergence takes place in M, i.e., if we set $S_N = \sum_{n=0}^{N} (-1)^n \frac{c_n}{x^{n+1}}$, then, for any $r \in]0, 1]$ we have $\lim_{N\to\infty} \|S_N\|_{\mathcal{V}_r} = 0$. It follows from 6.3 that for fixed $r \in]0, 1]$ one has for any $\eta > 0$ that $\left\|\frac{c_n}{x^{n+1}}\right\|_{\mathcal{V}_r} = O(\eta^n)$. Therefore,

$$\|S_N\|_{\mathcal{V}_r} = \lim_{k\to\infty} \|S_k - S_N\|_{\mathcal{V}_r} = \left\| \sum_{n\geqslant N+1} (-1)^n \frac{c_n}{x^{n+1}} \right\|_{\mathcal{V}_r} = O(\eta^{N+1}) \,.$$

For $n \geqslant 0$ set $b_n = \frac{(-1)^n x^{n+1}}{n+1} \cdot S_n$. Note that each b_n is in $\mathcal{O}(X)$, and $\|b_n\|_X \leqslant \|b_n\|_{\mathcal{V}_1} \leqslant \frac{1}{|n+1|}\|S_n\|_{\mathcal{V}_1} = O(\eta^{n+1})$, for any $\eta > 0$. Put $Q := \sum_{n\geqslant 0} b_n \partial_x^{[n]}$, which is thus an element of D. We have $\partial_x^{[n]} x\partial_x = x(n+1)\partial_x^{[n+1]} + n\partial_x^{[n]}$, and therefore

$$\begin{aligned} QP_0 &= \textstyle\sum_{n\geqslant 0} b_n\partial_x^{[n]} x\partial_x + \sum_{n\geqslant 0} b_n\partial_x^{[n]} \\ &= b_0 + \textstyle\sum_{n\geqslant 0}\Big(xb_n(n+1)\partial_x^{[n+1]} + b_n n\partial_x^{[n]}\Big) + \sum_{n\geqslant 1} b_n\partial_x^{[n]} \\ &= b_0 + \textstyle\sum_{n\geqslant 0} xb_n(n+1)\partial_x^{[n+1]} + \sum_{n\geqslant 1} b_n(n+1)\partial_x^{[n]} \\ &= b_0 + \textstyle\sum_{n\geqslant 1}\Big(xb_{n-1}n + b_n(n+1)\Big)\partial_x^{[n]} \;=\; P\,. \end{aligned}$$

This shows that 6.4 is indeed exact, and M is a D-module of finite presentation, and hence coadmissible by [13, 3.4]. □

Remark 6.5. Assertion (i) of 6.2 is a special case of the more general result [1, corollary in sec. 10.4].

6.6. *Direct image under the Kummer map with exponent p.* Let $\widetilde{X} = \mathrm{Sp}(L\langle y\rangle)$ be another rigid analytic unit disc, set $\widetilde{X}^\circ = \widetilde{X} \setminus \{y = 0\}$, and denote by $\widetilde{j} : \widetilde{X}^\circ \to \widetilde{X}$ the open embedding. Let $\varphi : \widetilde{X} \to X$ be defined by $\varphi^*(x) = y^p$. This morphism is étale over X° and ramified over 0. In analogy with the situation studied in section 4, we consider the sheaf $\mathcal{M} = \varphi_*\widetilde{j}_*\mathcal{O}_{\widetilde{X}^\circ}$. The same formulas and techniques that we have used in section 4 apply in the present setting to analyze this sheaf, with some simplifications, however, in certain places. Carrying out those computations and arguments gives the following result, which shows that the situation simplifies when we pass to the sheaf of infinite order differential operators $\mathcal{D}_X$. For a character $\chi : \mu_p \to L^\times$ we denote by $\mathcal{M}^\chi$ the subsheaf of germs of sections s which have the property that $\zeta_p.s = \chi(\zeta_p) \cdot s$.

Theorem 6.7. *(i) If $\chi(\zeta_p) = \zeta_p^j$, for some $j \in \{0, 1, \ldots, p-1\}$, then $\mathcal{M}^\chi = j_*\mathcal{O}_{X^\circ} \cdot y^j$, i.e., it is a free module of rank one over $j_*\mathcal{O}_{X^\circ}$ generated by the global section y^j.*

(ii) We have $\mathcal{M} = \bigoplus_{\chi\in\mu_p^\vee} \mathcal{M}^\chi$, where $\mu_p^\vee$ is the character group of μ_p.

(iii) For the trivial character $\mathbf{1}$ the sheaf $\mathcal{M}^{\mathbf{1}}$ is canonically isomorphic to $j_*\mathcal{O}_{X^\circ}$, and the action of the sheaf of finite order algebraic differential operators extends to an action of $\mathcal{D}_X$ on $\mathcal{M}^{\mathbf{1}}$, which makes this sheaf a coadmissible $\mathcal{D}_X$-module (cf. 6.2).

(iv) If $\chi(\zeta_p) = \zeta_p^j$, for some $j \in \{1, \ldots, p-1\}$, then the action of the sheaf of finite order algebraic differential operators extends to an action of $\mathcal{D}_X$ on $\mathcal{M}^\chi$, and $\mathcal{M}^\chi$ has the following presentation

$$\mathcal{D}_X \xrightarrow{\beta} \mathcal{D}_X \xrightarrow{\alpha} \mathcal{M}^{\chi} \longrightarrow 0$$

as a $\mathcal{D}_X$-module, where $\alpha(P) = P \cdot y^j$ and $\beta(Q) = QP_j$, where $P_j = px\partial_x - j$ is as in 4.3.

(v) For each $\chi \in \mu_p^\vee$ the sheaf $\mathcal{M}^\chi$ is a coadmissible $\mathcal{D}_X$-module, and $\mathcal{M}$ is hence a coadmissible $\mathcal{D}_X$-module.

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