

p -adic Jacquet-Langlands functor III

Tobias Schmidt (BUW)
HEGAL mini course part III

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Aim of the talk

- $F/\mathbb{Q}_p$ finite extension
- $\check{F} \hookrightarrow C$ a complete algebraically closed extension
- D the division algebra over F of invariant $1/n$,
- $D^\times$ its units, acting on $\mathbb{P}_C^{n-1}$
- π an admissible smooth $GL_n(F)$ -representation over $\mathbb{F}_p$
- $\mathcal{F}_\pi$ the associated sheaf on $(\mathbb{P}_C^{n-1}/D^\times)_{\text{ét}}$

Relying on the two previous talks, we will explain the proof of the claim: The cohomology group

$$H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)$$

is an admissible smooth $D^\times$ -representation for all i .

Structure of the proof

Let $\mathcal{O}^+$ be the integral structure sheaf of the adic space $\mathbb{P}_C^{n-1}$.
Let K be a compact open subgroup in $D^\times$.

- 1 Step 1: Cohomology of $\mathcal{F}_\pi \otimes \mathcal{O}^+ / p$ on $(\mathbb{P}_C^{n-1} / K)_{\text{ét}}$
- 2 Step 2: Cohomology of $\mathcal{F}_\pi$ on $(\mathbb{P}_C^{n-1} / K)_{\text{ét}}$
- 3 Step 3: Cohomology of $\mathcal{F}_\pi$ on $\mathbb{P}_C^{n-1}$ as $D^\times$ -rep.

Step 1: Cohomology of $\mathcal{F}_\pi \otimes \mathcal{O}^+ / p$ on $(\mathbb{P}_C^{n-1} / K)_{\text{ét}}$

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Reminder from last talk

The period map at level 0

$$\pi_{\text{GH}} : \mathcal{M}_{LT, GL_n(o_F)} \longrightarrow \mathbb{P}_C^{n-1}$$

admits local sections, as a map of adic spaces.

Let $V \subset \mathbb{P}_C^{n-1}$ open affinoid such that π_{GH} has a section on V .

Let $U \subset V$ be a rational subset strictly contained in V .

Lemma

Let $m \geq 0$. There is a compact open subgroup $K_0 \subset D^\times$ stabilizing U, V and the section such that for all $K \subset_o K_0$, the image of

$$H^i((V/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p) \rightarrow H^i((U/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p)$$

is an almost finitely generated o_C -module for all $i = 0, \dots, m$.

Two global corollaries

Corollary 1

Let $m \geq 0$. There is a compact open subgroup $K_0 \subset D^\times$ s.t. for all $K \subset_o K_0$ the cohomology group

$$H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p)$$

is an almost finitely generated $\mathfrak{o}_C$ -module for all $i = 0, \dots, m$.

Proof: Idea

Refine coverings of $\mathbb{P}_C^{n-1}$ with affinoids to which the lemma applies and consider the associated Čech spectral sequences for $\mathcal{F}_\pi \otimes \mathcal{O}^+/p$.

Then "trace almost finiteness results on images of maps through spectral sequences" (cf. last talk) to get finiteness for the abutment.

For this, we need to run through $N \geq m + 2$ spectral sequences.

Proof cont'd

Choose finite affinoid covering

$$\mathbb{P}_C^{n-1} = \cup_{j \in J} U_j$$

and affinoids V_j s.t. $U_j \subset V_j$ is a strictly contained rational subset and $\mathcal{M}_{LT,0} \rightarrow \mathbb{P}_C^{n-1}$ has a section over V_j (cf. last talk).

Find intermediate strict rational subsets

$$U_j = U_j^{(N)} \subset U_j^{(N-1)} \subset \dots \subset U_j^{(1)} = V_j.$$

For any $S \subset J$ put $U_S^{(k)} = \cap_S U_j^{(k)}$. Lemma applies to $U_S^{(k+1)} \subset U_S^{(k)}$

$\implies$ There is a c.o. $K_0 \subset D^\times$ stabilizing all $U_S^{(k)}$ and their sections s.t. for all $K \subset_o K_0$ the image of

$$H^i((U_S^{(k)}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p) \rightarrow H^i((U_S^{(k+1)}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p)$$

is an almost f.g. $\mathcal{O}_C$ -module for all $i = 0, \dots, m$.

For each $k = 1, \dots, N$ have the Čech spectral sequence for the covering $U_j^{(k)}, j \in J$, in $(\mathbb{P}_C^{n-1}/K)_{\text{ét}}$ and the sheaf $\mathcal{G} := \mathcal{F}_\pi \otimes \mathcal{O}^+/p$

$$E_{1,(k)}^{m_1,m_2} = \bigoplus_{S \subset J, |S|=m_1+1} H^{m_2}((U_S^{(k)}/K)_{\text{ét}}, \mathcal{G}) \implies H^{m_1+m_2}((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{G}).$$

The above discussion gives the almost finiteness result on the images of the maps on the E_1 -page for all k .

Tracing it through these N spectral sequences

$$\implies H^{m_1+m_2}((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{G}) \text{ is almost f.g. for } m_1 + m_2 \leq N - 2.$$

Since $m \leq N - 2$, this proves the corollary. □

2nd global corollary

One deduces finiteness unconditionally for all K , and in all cohomological degrees.

Corollary 2

For any compact open subgroup $K \subset D^\times$ the cohomology group

$$H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p)$$

is an almost finitely generated $\mathfrak{o}_C$ -module for all i .

Proof: Fix $m \geq 0$. Corollary 1 $\implies$ there ex. normal $K' \subset_o K$ s.t.

$$H^i((\mathbb{P}_C^{n-1}/K')_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p)$$

is an almost f.g. $\mathfrak{o}_C$ -module for all $i \leq m$.

We want to extend this statement from K' to K .

Restricting the K -action to K' defines a morphism of sites

$$\nu : (\mathbb{P}_C^{n-1}/K')_{\text{ét}} \rightarrow (\mathbb{P}_C^{n-1}/K)_{\text{ét}}$$

where ν^* exact with left inverse $\nu_*(-)^{K/K'}$. For any $\mathcal{G} \in (\mathbb{P}_C^{n-1}/K)_{\text{ét}}^{\sim}$

$$\Gamma((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{G}) = (-)^{K/K'} \circ \Gamma((\mathbb{P}_C^{n-1}/K')_{\text{ét}}, \nu^* \mathcal{G}).$$

Get a convergent "Hochschild-Serre" spectral sequence

$$E_2^{p,q} = H^p(K/K', H^q((\mathbb{P}_C^{n-1}/K')_{\text{ét}}, \mathcal{G})) \implies H^{p+q}((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{G}).$$

Above statement + o_C almost noetherian $\implies$ all $E_2^{p,q}$ are almost f.g.
for $q \leq m$.

All subsequent $E_r^{p,q}$ hence $E_{\infty}^{p,q}$ consist of subquotients $\implies$

$H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{G})$ has filtration by almost f.g. for $i \leq m$.



Step 2: Cohomology of $\mathcal{F}_\pi$ on $(\mathbb{P}_C^{n-1}/K)_{\text{ét}}$

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Proposition

For any compact open subgroup $K \subset D^\times$ the cohomology group

$$H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi)$$

is finite for all i .

Before we come to the proof, two pieces of preparation:

(P1) Let $\mathcal{O}^{b+} = (\varprojlim_{\phi} \mathcal{O}^+ / p)$ be the tilted integral structure sheaf on $(\mathbb{P}_C^{n-1})_{\text{ét}}$, a sheaf of topological perfect flat $\mathcal{O}_{C^b}$ -algebras.

Let $p^b \in \mathcal{O}_{C^b}$ map to p under $\sharp : \mathcal{O}_{C^b} \rightarrow \mathcal{O}_C$. Then

$$\mathcal{O}^{b+}/p^b \simeq \mathcal{O}^+/p \quad \text{and} \quad \mathcal{O}^{b++}/p^b \simeq \mathcal{O}^{++}/p$$

(topologically nilpotent elements).

Preparation cont'd

Let $\mathcal{O}^{b-}$ be quotient of $\mathcal{O}^b := \mathcal{O}^{b+} \otimes_{\mathcal{O}_{C^b}} C^b$ by $\mathcal{O}^{b++}$. It satisfies

$$\mathcal{O}^{b++}/p^b \simeq \mathcal{O}^{b-}[p^b], \quad y \mapsto \frac{y}{p^b}$$

and $\mathcal{O}^{b-}[p^b] \simeq \mathcal{O}^{b-}[(p^b)^{p^k}]$ via Φ^k . This induces an iso. of $\mathbb{F}_p$ -sheaves

$$\varinjlim_{\tilde{\Phi}^{-1}} \mathcal{O}^{++}/p \xrightarrow{\sim} \varinjlim_{\tilde{\Phi}^{-1}} \mathcal{O}^{b-}[(p^b)^k] = \mathcal{O}^{b-}.$$

Rem: (i) The transition map can be made explicit: there is an iso.

$$\mathcal{O}^{++}/p^{1/p} \xrightarrow{\sim} \mathcal{O}^{++}/p, \quad x \mapsto x^p - p^\epsilon x,$$

where $\epsilon = p^{\frac{p-1}{p}}$. Then $\tilde{\Phi}^{-1}$ is the endomorphism of $\mathcal{O}^{++}/p$ coming from the inverse composed with p^ϵ .

(ii) On constant sections one gets $\varinjlim_{\tilde{\Phi}^{-1}} \mathfrak{m}_C/p \xrightarrow{\sim} C^b/\mathfrak{m}_{C^b}$.

Preparation cont'd

(P2) We recall the following lemma. Let $\mathfrak{o} := \mathfrak{o}_{C^b}$ and $\varpi := p^b$.

Lemma

Let M_k be a $\mathfrak{o}/\varpi^k$ -module for any $k \geq 1$, such that:

- M_1 is almost finitely generated;
- there are maps $p_k : M_{k+1} \rightarrow M_k$ and $q_k : M_k \rightarrow M_{k+1}$ such that $p_k q_k$ is multiplication by ϖ , and s.t. the following is exact

$$M_1 \xrightarrow{q_k \cdots q_1} M_{k+1} \xrightarrow{p_k} M_k.$$

- there are isomorphisms compatible with p_k, q_k

$$\Phi_k : M_k \otimes_{\mathfrak{o}/\varpi^k, \Phi} \mathfrak{o}/\varpi^{pk} \simeq M_{pk}.$$

Then there is $r \geq 0$ and almost isomorphisms $M_k \simeq_a (\mathfrak{o}/\varpi^k)^r$ for all k , compatible with the maps p_k, q_k, Φ_k .

Remarks:

- This is Lemma 2.12 from [Scholze, p -adic Hodge theory for rigid-analytic varieties].
- Application of the classification of almost f.g. $\mathcal{O}$ -modules over non-archimedean fields with dense value group $\Gamma \subset \mathbb{R}_{>0}$ (in the special case of algebraically closed fields of characteristic p).
- "compatible with p_k, q_k, Φ_k " means that these maps are taken to projection map, multiplication by ϖ and coordinatewise Frobenius respectively.

Now we start the proof of the proposition. Fix K and i .

Proof:

Claim (a): $H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p) \simeq_a (\mathcal{O}_C/p)^{r_i}$ for some r_i , compatibly with Frobenius Φ .

Proof cont'd

Indeed, we apply the lemma (P2) to the sequence of cohomology groups

$$M_k := H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^{b-}[(p^b)^k]),$$

the necessary maps p_k, q_k, Φ_k coming from the short exact sequences

$$0 \rightarrow \mathcal{O}^{b-}[(p^b)^{k_1}] \rightarrow \mathcal{O}^{b-}[(p^b)^{k_1+k_2}] \rightarrow \mathcal{O}^{b-}[(p^b)^{k_2}] \rightarrow 0$$

as well as the Frobenius isomorphisms

$$\mathcal{O}^{b-}[(p^b)^{k_1}] \xrightarrow{\sim} \mathcal{O}^{b-}[(p^b)^{pk_1}].$$

Note that $\mathcal{O}^{++}/p$ is almost isomorphic to $\mathcal{O}^+/p$, so that

$$M_1 \stackrel{(P1)}{\simeq} H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^{++}/p) \simeq_a H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p)$$

is almost f.g. by Step 1.

Lemma (P2) $\implies$

$$M_1 \simeq_a (o_{C^b}/p^b)^{r_i} \simeq (o_C/p)^{r_i}$$

for some r_i , compatibly with Frobenius Φ .

Claim (b): $H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^{b-}) \simeq (C^b/\mathfrak{m}_{C^b})^{r_i}$, Φ -compatibly

Indeed, $\mathcal{O}^{++}/p = \mathcal{O}^+/p \otimes_{o_C} \mathfrak{m}_C$ and $\mathfrak{m}_C$ is o_C -flat with $\mathfrak{m}_C^2 = \mathfrak{m}_C$

$\implies \otimes_{o_C} \mathfrak{m}_C$ applied to (a) gives Φ -compatible iso

$$H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^{++}/p) \simeq (\mathfrak{m}_C/p)^{r_i}.$$

Φ -compatible implies $\tilde{\Phi}^{-1}$ -compatible. Taking a (filtered) colimit along $\tilde{\Phi}^{-1}$ gives the claim by (P1).

To finish the proof, look at exact Artin-Schreier sequence

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathcal{O}^{b-} \xrightarrow{\Phi-1} \mathcal{O}^{b-} \longrightarrow 0$$

on $(\mathbb{P}_C^{n-1}/K)_{\text{ét}}$,

descended from Artin-Schreier for $\mathcal{O}^b$ and $\mathcal{O}^{b-}$ on $\mathbb{P}_{C,\text{ét}}^{n-1}$.

The induced sequence

$$0 \longrightarrow \mathcal{F}_\pi \longrightarrow \mathcal{F}_\pi \otimes \mathcal{O}^{b-} \xrightarrow{\Phi-1} \mathcal{F}_\pi \otimes \mathcal{O}^{b-} \longrightarrow 0$$

remains exact (at stalks this amounts to tensoring over $\mathbb{F}_p$ with π). In the corresponding long exact cohomology sequence, the parts

$$H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^{b-}) \xrightarrow{\Phi-1} H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi \otimes \mathcal{O}^{b-})$$

are - by claim (b) - of the form

$$(C^b/\mathfrak{m}_{C^b})^{r_i} \xrightarrow{\Phi-1} (C^b/\mathfrak{m}_{C^b})^{r_i}.$$

But the latter map is surjective with kernel $\mathbb{F}_p^{r_i}$!

So $\delta = 0$ in the long exact cohomology sequence and

$$H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi) \simeq \mathbb{F}_p^{r_i}.$$

This is the statement of the key result.

Step 3: Cohomology of $\mathcal{F}_\pi$ on $\mathbb{P}_C^{n-1}$ as $D^\times$ -rep.

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Theorem

The cohomology group

$$H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)$$

is an admissible smooth $D^\times$ -representation for all i .

Proof: Recall from first talk that

$$H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi) = \varinjlim_K H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi).$$

The image of $H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi)$ lies in $H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)^K \implies$
the $D^\times$ -representation on $H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)$ is smooth.

The proof of admissibility is by induction on i using the following two auxiliary results (A1), (A2).

(A1) for any c.o. $K \subset D^\times$ and any $\mathcal{G} \in (\mathbb{P}_C^{n-1}/K)_{\text{ét}}^\sim$, there is a convergent spectral sequence

$$H_{\text{cont}}^p(K, H_{\text{ét}}^q(\mathbb{P}_C^{n-1}, \mathcal{G})) \implies H^{p+q}((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{G}).$$

Here H_{cont}^p denotes continuous group cohomology and each K -module $H_{\text{ét}}^q(\mathbb{P}_C^{n-1}, \mathcal{G})$ is considered discrete.

Indeed, take colimit along normal $K' \subset_o K$ of the Hochschild-Serre spectral sequences from (the proof of) Corollary 2

$$H^p(K/K', H^q((\mathbb{P}_C^{n-1}/K')_{\text{ét}}, \mathcal{G})) \implies H^{p+q}((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{G})$$

and use $\lim_{\rightarrow K'} H^p(K/K', A^{K'}) \simeq H_{\text{cont}}^p(K, A)$ for discrete A .

(A2) If G is a compact p -adic Lie group and V an admissible smooth G -representation over $\mathbb{F}_p$, then $H_{\text{cont}}^i(G, V)$ is finite for all i .

Indeed, G compact $\implies$

cts. group cohomology computes the derived functors $R^i(-)^G$.

Pontryagin duality

$$V \mapsto V^\vee := \text{Hom}_{\text{cont}}(V, \mathbb{Q}/\mathbb{Z})$$

gives anti-equivalence between admissible smooth $V \leftrightarrow$ finitely generated $\mathbb{F}_p[[G]]$ -modules.

The completed group algebra $\mathbb{F}_p[[G]]$ is noetherian with coinvariants $(\mathbb{F}_p[[G]])_G = \mathbb{F}_p$.

Claim (A2) follows from taking a resolution of $V^\vee$ by finitely generated free $\mathbb{F}_p[[G]]$ -modules.

Now we start the proof of the main result using induction on i .

Fix c.o. $K \subset D^\times$. Need to show $H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)^K$ is finite. Consider the contribution in the spectral sequence (A1)

$$E_2^{p,q} = H_{\text{cont}}^p(K, H_{\text{ét}}^q(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)) \implies H^{p+q}((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi)$$

to the abutment at $p + q = i$.

For $i = 0$ have $E_2^{0,0} = E_\infty^{0,0} =$ abutment at 0 so that

$$H_{\text{ét}}^0(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)^K = H^0((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi).$$

This is finite by Step 2. Assume that $i > 0$.

(I) For $q < i$: the $D^\times$ -rep $H_{\text{ét}}^q(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)$ is admissible by induction hypothesis, a fortiori as K -representation $\xrightarrow{(A2)}$

the terms $H_{\text{cont}}^p(K, H_{\text{ét}}^q(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi))$ contribute only finite-dimensional pieces to the abutment at i .

(II) The only other contribution is $q = i$ and $p = 0$, i.e. comes from $H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)^K$. Assume, for a contradiction, it was infinite-dimensional.

On its way to the E_∞ -page it interacts only with terms where $q < i$ and only with finitely many such:

suppose $E_r^{p,q}$, $r \geq 2$ is inf.-dim. and consider $d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$.

$E_r^{p+r, q-r+1}$ is finite since $q - r + 1 < q = i \implies \ker d_r^{p,q}$ is inf.-dim.

and $\text{im } d_r^{p-r, q+r-1} = 0$ since $p - r < 0$

$\implies E_{r+1}^{p,q}$ is inf.-dim. etc.

$\implies$ an infinite-dimensional space survives to E_∞

$\implies$ i.e. a graded piece of $H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi)$ is infinite-dimensional.

But $H^i((\mathbb{P}_C^{n-1}/K)_{\text{ét}}, \mathcal{F}_\pi)$ is finite by Step 2. Contradiction.

This finishes the proof of the main result. □

- One has $H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi) = 0$ for all $i > 2(n-1)$.

This follows from a theorem of Berkovich for the p -cohomological dimension saying that $cd_p X \leq 2 \dim X$ for any qcqs adic space X (locally of finite type) over C .

- The $D^\times$ -representation $H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi)$ is independent of the choice of C , in the sense that $C \hookrightarrow C'$ induces an isomorphism on cohomology.

For this, one uses that

$$H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi) \otimes \mathcal{O}_C/p \longrightarrow H_{\text{ét}}^i(\mathbb{P}_C^{n-1}, \mathcal{F}_\pi \otimes \mathcal{O}^+/p)$$

is an almost isomorphism. This drops out of the proof of the key result by taking a colimit over compact open $K \subset D^\times$.