Analytic vectors in continuous p-adic representations

Tobias Schmidt

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Abstract

Given a compact p-adic Lie group G over a finite unramified extension $L/\mathbb{Q}_p$ let $G_{L/\mathbb{Q}_p}$ be the product over all Galois conjugates of G. We construct an exact and faithful functor from admissible G-Banach space representations to admissible locally L-analytic $G_{L/\mathbb{Q}_p}$-representations that coincides with passage to analytic vectors in case $L = \mathbb{Q}_p$. On the other hand, we study the functor "passage to analytic vectors" and its derived functors over general basefields. As an application we compute the higher analytic vectors in certain locally analytic induced representations.

1 Introduction

Recently, Schneider and Teitelbaum initiated a systematic study of continuous representations of p-adic Lie groups into p-adic topological vector spaces (cf. [ST1-6]). A central result in this theory is that in case of a compact group G over $\mathbb{Q}_p$ the algebra of locally analytic distributions on G is a faithfully flat extension of the algebra of continuous distributions. As a consequence, passage to analytic vectors constitutes an exact and faithful functor $F_{\mathbb{Q}_p}$ from admissible Banach space G-representations to admissible locally analytic G-representations. Due to its properties $F_{\mathbb{Q}_p}$ is a basic tool in a possible classification of admissible topologically irreducible (unitary) Banach space representations which is of particular interest in the realm of the p-adic Langlands programme. It is therefore a natural question (raised by J. Teitelbaum, cf. [T]) how to correctly generalize the above results to groups over arbitrary base fields $\mathbb{Q}_p \subseteq L$.

Given a compact locally L-analytic group G simple examples show that the naive analogues of the above results do not hold (for example, F_L is not exact and often zero). The reason, as we believe, is that the notion of a K-valued locally L-analytic function depends on embedding the base field L into the coefficient field K. Consequently, we introduce, at least in case $L/\mathbb{Q}_p$ Galois, the various restrictions of scalars G_σ of G via $\sigma \in Gal(L/\mathbb{Q}_p)$ together with their function spaces. Let $G_{L/\mathbb{Q}_p} := \prod_\sigma G_\sigma$. Denoting as usual by $D^c(.,K)$ and $D(.,K)$ continuous and locally analytic K-valued distributions respectively we construct a ring extension

$$D^c(G,K) \to D(G_{L/\mathbb{Q}_p},K)$$

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which reduces to the former in case $L = \mathbb{Q}_p$ and is faithfully flat in case $L/\mathbb{Q}_p$ is unramified (Thm. 4.6). To obtain from this a well-behaved generalization of $F_{\mathbb{Q}_p}$ we introduce for any Banach space G-representation V the subspace $V_{\sigma-an}$ of σ-analytic vectors whose formation is functorial in V. Denoting by $\text{Ban}_G^{adm}(K)$ and $\text{Rep}_K^a(G)$ the abelian categories of admissible Banach space and locally analytic representations of G over K respectively we construct a functor

$$F : \text{Ban}_G^{adm}(K) \to \text{Rep}_K^a(G_{L/\mathbb{Q}_p})$$

that enjoys, in case $L/\mathbb{Q}_p$ unramified, the following properties (Thm. 4.7): it is exact and faithful and coincides with $F_{\mathbb{Q}_p}$ in case $L = \mathbb{Q}_p$. Given $V \in \text{Ban}_G^{adm}(K)$ the representation $F(V)$ is strongly admissible. Viewed as a $G_{\sigma^{-1}}$-representation $F(V)$ contains $V_{\sigma-an}$ as a closed subrepresentation and functorial in V. The results obtained so far generalize two main theorems of Schneider and Teitelbaum (cf. [S], Thm. 4.2/3) to unramified extensions $L/\mathbb{Q}_p$.

The functor F_L (over a general finite extension $L/\mathbb{Q}_p$) being nevertheless an important construction we continue our work by studying its derived functors. More generally, for an arbitrary locally L-analytic group G we study the functor "passage to L-analytic vectors" $F_{\mathbb{Q}_p}^L$ from admissible locally $\mathbb{Q}_p$-analytic representations to admissible locally L-analytic representations. Then $F_L = F_{\mathbb{Q}_p}^L \circ F_{\mathbb{Q}_p}$ if G is compact and we may deduce left-exactness of F_L. It is unclear at present whether the categories of admissible locally analytic representations have enough injective objects. Nevertheless, we prove that $F_{\mathbb{Q}_p}^L$ extends to a cohomological δ-functor $R^iF_{\mathbb{Q}_p}^L$ between admissible representations vanishing in degrees $i > ([L : \mathbb{Q}_p] - 1)\dim_L G$. The functors $R^iF_{\mathbb{Q}_p}^L$ turn out to be certain Ext-groups and satisfy $R^iF_L = R^iF_{\mathbb{Q}_p}^L \circ F_{\mathbb{Q}_p}$ with R^iF_L the right-derived functors of F_L.

As an application we study the interaction of the δ-functor $R^iF_{\mathbb{Q}_p}^L$ with locally analytic induction. Let $P \subseteq G$ be a closed subgroup (satisfying a mild extra condition). For all $i \geq 0$ we obtain (Thm. 7.5)

$$R^iF_{\mathbb{Q}_p}^L \circ \text{Ind}_{P_0}^{G_0} = \text{Ind}_P^G \circ R^iF_{\mathbb{Q}_p}^L$$

as functors on finite dimensional locally $\mathbb{Q}_p$-analytic P-representations. Here, $(.)_0$ refers to the underlying $\mathbb{Q}_p$-analytic group. In case that G equals the L-points of a quasi-split connected reductive group over L and P a parabolic subgroup we deduce from this an explicit formula for the higher analytic vectors in principal series representations of G.

Notations. Let $|.|$ be the p-adic absolute value of $\mathbb{C}_p$ normalized by $|p| = p^{-1}$. Let $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ be complete intermediate fields with respect to $|.|$ where $L/\mathbb{Q}_p$ is a finite extension of degree n and K is discretely valued. Let $\mathfrak{o} \subseteq L$ be the valuation ring. G always denotes a locally L-analytic group with Lie algebra $\mathfrak{g}$. Their restriction of scalars to $\mathbb{Q}_p$ are denoted by G_0 and $\mathfrak{g}_0$. For any field F denote by Vec_F the category of F-vector spaces. For any ring R denote by $\mathcal{M}(R)$ the category of right modules. Let $\kappa = 1$ or 2 if p is odd or even respectively. We refer to [NVA] for all notions from non-archimedean functional analysis.

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2 Fréchet-Stein algebras

In this section we recall and discuss two classes of Fréchet-Stein algebras: distribution algebras and hyperenveloping algebras. For a detailed account on abstract Fréchet-Stein algebras as well as distribution algebras as their first examples we refer to [ST5]. For all basic theory on uniform pro-p groups we refer to [DDMS]. Throughout this work all indices r are supposed to satisfy the technical conditions $r \in p^{\mathbb{Q}}$, $p^{-1} < r < 1$ and $r \notin \{p^{\frac{-1}{p^h - p^{h-1}}}, h \in \mathbb{N}\}$.

A K-Fréchet algebra A is called (two-sided) *Fréchet-Stein* if there is a sequence $q_1 \leq q_2 \leq ...$ of algebra norms on A defining its Fréchet topology and such that for all $m \in \mathbb{N}$ the completion A_m of A with respect to q_m is a left and right noetherian K-Banach algebra and a flat left and right A_{m+1}-module via the natural map $A_{m+1} \to A_m$.

Theorem 2.1 (Schneider-Teitelbaum) *Given a compact locally L-analytic group G the algebra $D(G,K)$ of K-valued locally analytic distributions on G is Fréchet-Stein.*

This is [ST5], Thm. 5.1. We recall the construction thereby fixing some notation: choose a normal open subgroup $H_0 \subseteq G_0$ which is a uniform pro-p group. Choose a minimal set of ordered generators $h_1, ..., h_d$ for H_0. The bijective global chart $\mathbb{Z}_p^d \to H_0$ for the manifold H_0 given by

$$(x_1, ..., x_d) \mapsto h_1^{x_1} ... h_d^{x_d} \tag{1}$$

induces a topological isomorphism $C^{an}(H_0, K) \simeq C^{an}(\mathbb{Z}_p^d, K)$ for the locally convex spaces of K-valued locally analytic functions. In this isomorphism the right-hand side is a space of classical Mahler series and the dual isomorphism $D(H_0, K) \simeq D(\mathbb{Z}_p^d, K)$ therefore realizes $D(H_0, K)$ as a space of noncommutative power series. More precisely, putting $b_i := h_i - 1 \in \mathbb{Z}[G]$, $\mathbf{b}^\alpha := b_1^{\alpha_1} ... b_d^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$ the Fréchet space $D(H_0, K)$ equals all convergent series

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha \tag{2}$$

with $d_\alpha \in K$ such that the set $\{|d_\alpha| r^{\kappa|\alpha|}\}_\alpha$ is bounded for all r. The family of norms $\|.\|_r$ defined via $\|\lambda\|_r := \sup_\alpha |d_\alpha| r^{\kappa|\alpha|}$ defines the Fréchet topology. They are multiplicative and the corresponding completions $D_r(H_0, K)$ are K-Banach algebras exhibiting a Fréchet-Stein structure on $D(H_0, K)$. Choose representatives $g_1, ..., g_r$ for the cosets in $H \backslash G$ and define on $D(G_0, K) = \oplus_i D(H_0, K)\, g_i$ the norms $|| \sum_i \lambda_i\, g_i ||_r := \max_i ||\lambda_i||_r$. The completions $D_r(G_0, K)$ are the desired Banach algebras for $D(G_0, K)$. Finally, $D(G, K)$ is equipped with the corresponding quotient norms $\|.\|_{\bar{r}}$ coming from the quotient map $\iota' : D(G_0, K) \to D(G, K)$. The latter arises as the dual map to the embedding

$$\iota : C^{an}(G, K) \subseteq C^{an}(G_0, K). \tag{3}$$

Passing to the norm completions $D_r(G, K)$ yields the appropriate Banach algebras.

We mention another important feature of $D(G, K)$ in case G_0 is a uniform pro-p group. Each algebra $D_r(G_0, K)$ carries the filtration defined by the additive subgroups

$$F_r^s D_r(G_0, K) := \{\lambda \in D_r(G_0, K),\ ||\lambda||_r \leq p^{-s}\}$$

and $F_r^{s+} D_r(G_0, K)$ (defined as $F_r^s D_r(G_0, K)$ via replacing $\leq$ by $<$) for $s \in \mathbb{R}$. Put

$$gr_r^{\cdot}\, D_r(G_0, K) := \oplus_{s\in\mathbb{R}}\, F_r^s D_r(G_0, K)/F_r^{s+} D_r(G_0, K)$$

for the associated graded ring. Given $\lambda \in F_r^s D_r(G_0, K) \setminus F_r^{s+} D_r(G_0, K)$ we denote by $\sigma(\lambda) = \lambda + F_r^{s+} D_r(G_0, K) \in gr_r^{\cdot}\, D_r(G_0, K)$ the *principal symbol* of λ. Note that $D_r(G, K) \simeq D_r(G_0, K)/I_r$ is endowed with the quotient filtration where $I := \ker\, D(G_0, K) \to D(G, K)$ and I_r denotes the closure of $I \subseteq D_r(G_0, K)$. These filtrations are exhaustive, separated, complete and quasi-integral (in the sense of [ST5], §1). For $\lambda \neq 0$ in $D_r(G, K)$ the principal symbol $\sigma(\lambda) \in gr_r^{\cdot}\, D_r(G, K)$ is defined analogously.

Theorem 2.2 (Schneider-Teitelbaum) *If G_0 is a d-dimensional locally $\mathbb{Q}_p$-analytic group and uniform pro-p then there is an isomorphism of $gr^{\cdot} K$-algebras*

$$gr_r^{\cdot}\, D_r(G_0, K) \xrightarrow{\sim} (gr^{\cdot} K)[X_1, ..., X_d],\ \ \sigma(b_i) \mapsto X_i.$$

This is [loc.cit.], Thm. 4.5. Since $gr_r^{\cdot}\, D_r(G, K)$ equals a quotient of $gr_r^{\cdot}\, D_r(G_0, K)$ each $D_r(G, K)$ is a complete filtered ring with noetherian graded ring and hence, is a Zariski ring (cf. [LVO], II.2.2.1).

Working over the base field L we shall need to impose a mild additional condition on uniform subgroups. Let G be a d-dimensional locally L-analytic group whose underlying $\mathbb{Q}_p$-analytic group is uniform. Any minimal ordered set of generators for G_0 defines a global chart $\mathbb{Z}_p^{nd} \to G_0$ and hence, determines a $\mathbb{Q}_p$-basis of $\mathfrak{g}_0$. Note that $\mathfrak{g}_0 \simeq \mathfrak{g}_L$ canonically over $\mathbb{Q}_p$ ([B-VAR], 5.14.5). We call G *uniform** if the generators can be chosen in such a way that this basis has the form $v_i\mathfrak{x}_j$ for a $\mathbb{Z}_p$-basis $v_1 = 1, v_2, ..., v_n$ of $\mathfrak{o}_L$ and an L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}_L$. By [Sch], Cor. 4.4 each locally L-analytic group has a fundamental system of neighbourhoods of the identity consisting of normal uniform* subgroups (note that being uniform* implies the condition (L) used in [loc.cit.])

We turn to another closely related class of Fréchet-Stein algebras. Let G be a locally L-analytic group of dimension d. Let $U(\mathfrak{g})$ be the enveloping algebra and let $C_1^{an}(G, K)$ be the stalk at $1 \in G$ of the sheaf of K-valued locally L-analytic functions on G. It is a topological algebra with augmentation whose underlying locally convex K-vector space is of compact type (for the basic properties of such spaces we refer to [ST2], §1). Denote by $U(\mathfrak{g}, K) := C_1^{an}(G, K)'_b$ its strong dual, the *hyperenveloping algebra* (cf. [P], §8). The notation reflects that, up to isomorphism, $U(\mathfrak{g}, K)$ depends only on $\mathfrak{g}$. It is a topological algebra with augmentation on a nuclear Fréchet space. There is a canonical algebra embedding $U(\mathfrak{g}) \subseteq U(\mathfrak{g}, K)$ with dense image compatible with the augmentations. The formation of $C_1^{an}(G, K)$ and $U(\mathfrak{g}, K)$ (as locally convex topological algebras) is functorial in G and converts direct products into (projectively) completed tensor products taken over K. Dualizing the strict surjection $C^{an}(G, K) \to C_1^{an}(G, K)$ yields an injective continuous algebra map $U(\mathfrak{g}, K) \to D(G, K)$ which is a topological embedding with closed image.

Theorem 2.3 (Kohlhaase) *Suppose G_0 is a uniform pro-p group. Denote by $h_{11}, ..., h_{nd}$ a minimal set of ordered generators and put $b_{ij} = h_{ij} - 1$. Denote by $U_r(\mathfrak{g}, K)$ the closure of $U(\mathfrak{g}, K) \subseteq D_r(G, K)$. There is a number $\epsilon(r, p) \in \mathbb{N}$ depending only on r and p such that the (left or right) $U_r(\mathfrak{g}, K)$-module $D_r(G, K)$ is finite free on the basis $\mathcal{R} := \{\mathbf{b}^\alpha, \ \alpha_{ij} < \epsilon(r, p) \text{ for all } (i, j) \in \{1, ..., n\} \times \{1, ..., d\}\}$. Letting $\mathfrak{x}_1, ..., \mathfrak{x}_d$ be an L-basis of $\mathfrak{g}$ and $\mathfrak{X}^\beta := \mathfrak{x}_1^{\beta_1} \cdots \mathfrak{x}_d^{\beta_d}$ one has as K-vector spaces*

$$U_r(\mathfrak{g}, K) = \{ \sum_{\beta \in \mathbb{N}_0^d} d_\beta \mathfrak{X}^\beta, \ d_\beta \in K, \ ||d_\beta \mathfrak{X}^\beta||_{\bar{r}} \to 0 \text{ for } |\beta| \to \infty \}$$

where the power series expansions are uniquely determined. The Banach algebras $U_r(\mathfrak{g}, K)$ exhibit $U(\mathfrak{g}, K)$ as a Fréchet-Stein algebra.

Proof: This is extracted from (the proof of) [Ko1], Thm. 1.4.2. The number $\epsilon(r, p)$ equals the (unique) value of t where the supremum $\sup_{t \in \mathbb{N}} |1/t| r^{\kappa t}$ is attained. □

In [loc.cit.] the noetherian and the flatness property of the family $U_r(\mathfrak{g}, K)$ is immediately deduced from the commutative diagram

$$\begin{array}{ccc} U_r(\mathfrak{g}, K) & \longrightarrow & U_{r'}(\mathfrak{g}, K) \\ \downarrow & & \downarrow \\ D_r(G, K) & \longrightarrow & D_{r'}(G, K) \end{array}$$

for $r' \leq r$ in which the lower horizontal arrow is a flat map between noetherian rings and the vertical arrows are, by the first statement in the theorem, finite free ring extensions. We also remark that the first statement in the theorem in case $L = \mathbb{Q}_p$ is due to H. Frommer ([F], 1.4 Lem. 3, Cor. 1/2/3).

Proposition 2.4 *Suppose G_0 is uniform* and let $\mathfrak{x}_1, ..., \mathfrak{x}_d$ be the corresponding basis of $\mathfrak{g}_L$. Endowing the extension $U_r(\mathfrak{g}, K) \subseteq D_r(G, K)$ with the $\|.\|_{\bar{r}}$-norm filtration the map $gr_r^{\cdot} U_r(\mathfrak{g}, K) \to gr_r^{\cdot} D_r(G, K)$ is finite free on the basis $\sigma(\mathcal{R})$. Moreover, $gr_r^{\cdot} U_r(\mathfrak{g}, K)$ equals a polynomial ring in $\sigma(\mathfrak{x}_1), ..., \sigma(\mathfrak{x}_d)$ and $||.||_{\bar{r}}$ is multiplicative on $U_r(\mathfrak{g}, K)$. For any $\lambda = \sum_{\beta \in \mathbb{N}_0^d} d_\beta \mathfrak{X}^\beta \in U_r(\mathfrak{g}, K)$ one has*

$$||\lambda||_{\bar{r}} = \sup_\beta |d_\beta| c_r^{|\beta|}$$

where $c_r \in \mathbb{R}_{>0}$ depends only on r and p.

Proof: Let $v_1, ..., v_n$ be a corresponding $\mathbb{Z}_p$-basis of $\mathfrak{o}_L$ for the uniform* group G. Then $h_{ij} := \exp(v_i \mathfrak{x}_j)$ are a minimal set of topological generators for G. Put as usual $b_{ij} := h_{ij} - 1$. Now $v_i \mathfrak{x}_j = \log(1 + b_{ij})$ is a $\mathbb{Q}_p$-basis for $\mathfrak{g}_0$ and a short calculation yields $\sigma(v_i \mathfrak{x}_j) = \sigma(b_{ij})^{p^h}$ with h depending only on r and p. Hence, Thm. 2.2 translates the map $gr_r^{\cdot} U_r(\mathfrak{g}_0, K) \to gr_r^{\cdot} D_r(G_0, K)$ into the inclusion $(gr^{\cdot} K)[X_{11}^{p^h}, ..., X_{nd}^{p^h}] \subseteq (gr^{\cdot} K)[X_{11}, ..., X_{nd}]$. For $L = \mathbb{Q}_p$ we obtain from this all statements together with the fact that the $U_r(\mathfrak{g}_0, K)$-module basis $\mathcal{R}$ for $D_r(G_0, K)$ is in fact orthogonal with respect to $||.||_r$. By [Sch], Lem. 5.3/Prop. 5.5 the graded ideal $gr_r^{\cdot} I_r$ where $I_r = \ker(D_r(G_0, K) \to D_r(G, K))$ is generated by the elements $X_{ij}^{p^h} - \bar{v}_i X_{1j}^{p^h}$ where $\bar{v}_i \in gr^{\cdot} K$ equals the residue class of v_i. By similar arguments the same holds true for $gr^{\cdot}_r J_r$ where $J_r = \ker\ (U_r(\mathfrak{g}_0, K) \rightarrow U_r(\mathfrak{g}, K))$. It follows that $I_r = \oplus_{g \in \mathcal{R}} J_r\, g$. By orthogonality the quotient norm on $U_r(\mathfrak{g}, K)$ with respect to $||.||_r$ and $U_r(\mathfrak{g}_0, K) \rightarrow U_r(\mathfrak{g}, K)$ equals precisely $\|.\|_{\bar{r}}$. In other words, $gr^{\cdot}_r U_r(\mathfrak{g}_0, K)/gr J_r \simeq gr^{\cdot}_r U_r(\mathfrak{g}, K)$ and since $gr^{\cdot}_r I_r \cap gr^{\cdot}_r U_r(\mathfrak{g}_0, K) = gr^{\cdot}_r J_r$ the first statement follows. Now $gr^{\cdot}_r U_r(\mathfrak{g}_0, K)/gr J_r$ is readily seen to be a polynomial ring in the residue classes of the $X_{1j}^{p^h}$, $j = 1, ..., d$ which correspond to $\sigma(\mathfrak{x}_j) \in gr^{\cdot}_r U_r(\mathfrak{g}, K)$. This implies that $\|.\|_{\bar{r}}$ is multiplicative on $U_r(\mathfrak{g}, K)$, that the topological K-basis $\mathfrak{X}^\beta$, $\beta \in \mathbb{N}_0^d$ for $U_r(\mathfrak{g}, K)$ is in fact an orthogonal basis with respect to $\|.\|_{\bar{r}}$ and that $c_r := ||\mathfrak{x}_j||_{\bar{r}} = ||\mathfrak{x}_j||_r = ||\log(1 + b_{1j})||_r = \sup_{t \in \mathbb{N}} |1/t| r^{\kappa t}$ depends only on r and p. □

Next we prove a proposition on the compatibility of two Fréchet-Stein structures. This will be used in the proof of Thm. 7.5.

Lemma 2.5 *Let G be a compact locally L-analytic group of dimension d and $P \subseteq G$ a closed subgroup of dimension $l \leq d$. There is an open normal subgroup $G' \subseteq G$ with the following properties: it is uniform* with respect to bases $\mathfrak{x}_1, ..., \mathfrak{x}_d$ and $v_1, ..., v_n$. Furthermore, $P' := P \cap G'$ is uniform* with respect to the bases $\mathfrak{x}_1, ..., \mathfrak{x}_l$ and $v_1, ..., v_n$.*

Proof: Denote the Lie algebras of G resp. P by $\mathfrak{g}_L$ resp. $\mathfrak{p}_L$. Denote by G_0, P_0 the underlying locally $\mathbb{Q}_p$-analytic groups. Applying [DDMS], Prop. 3.9, Thm. 4.2/4.5 we see that P contains a uniform subgroup P_1 such that every open normal subgroup of P lying in P_1 is uniform itself. After this preliminary remark we choose, according to [Sch], Cor. 4.4 and (the proof of) [loc.cit.], Prop. 4.3, a locally L-analytic group G' open normal in G with the following properties: it is uniform* with an L-basis $\mathfrak{x}_1, ..., \mathfrak{x}_d$ of $\mathfrak{g}_L$ such that $\mathfrak{x}_1, ..., \mathfrak{x}_l$ is an L-basis of $\mathfrak{p}_L$. Furthermore, we may arrange that $P' := G' \cap P \subseteq P_1$. Since $\mathfrak{x}_1, ..., \mathfrak{x}_l$ is an L-basis of $\mathfrak{p}_L$ and exp may be viewed an exponential map for P the nl elements $\exp(v_i \mathfrak{x}_j)$, $i = 1, ..., n$, $j = 1, ..., l$ are part of a minimal generating system for G' and lie in $G' \cap P = P'$. Since they are pairwise different modulo G'^p, hence modulo P'^p it follows from $\dim_L P' = l$ that they form a minimal generating system for the uniform group P'. Thus, P' is uniform* with the required bases. □

Proposition 2.6 *Let G be a compact locally L-analytic group. There is a family of norms $(||.||_r)$ on $D(G_0, K)$ with the following properties: it defines the Fréchet-Stein structure on $D(G_0, K)$ as well as on the subalgebra $D(P_0, K)$. For each r the completion $D_r(G_0, K)$ is flat as a $D_r(P_0, K)$-module. The family of quotient norms $(\|.\|_{\bar{r}})$ defines the Fréchet-Stein structure on $D(G, K)$ as well as on the subalgebra $D(P, K)$. For each r the completion $D_r(G, K)$ is flat as a $D_r(P, K)$-module.*

Proof: Apply the preceding lemma to $P \subseteq G$ to find an open normal subgroup $G' \subseteq G$ which is uniform* with respect to bases $\mathfrak{x}_j$ and v_i. The nd elements $\exp(v_i \mathfrak{x}_j)$ are then topological generators for G' where the first nl elements $(l := \dim_L P)$ generate the uniform group $P' := P \cap G'$. Endow $D(G_0, K)$ and

$D(G,K)$ with the Fréchet-Stein structures constructed in the beginning of this section and restrict the norms to $D(P_0,K)$ and $D(P,K)$. Then [ST6], Prop. 6.2. yields all statements over $\mathbb{Q}_p$. By definition, the quotient map $D(G_0,K) \rightarrow D(G,K)$ restricts to the quotient map $D(P_0,K) \rightarrow D(P,K)$ whence it is easy to see that the restricted norms $\|.\|_{\bar{r}}$ on $D(P,K)$ equal the quotient norms. Hence they realize a Fréchet-Stein structure on $D(P,K)$ and only the last claim remains to be justified. By the argument given at the end of [loc.cit.] it suffices to prove it for the pair $P' \subseteq G'$. Applying Prop. 2.4 to $D(P',K)$ and $D(G',K)$ we obtain a commutative diagram of commutative algebras

$$\begin{array}{ccc} gr^{\cdot}_r\, U_r(\mathfrak{p},K) & \longrightarrow & gr^{\cdot}_r\, U_r(\mathfrak{g},K) \\ \downarrow & & \downarrow \\ gr^{\cdot}_r\, D_r(P',K) & \longrightarrow & gr^{\cdot}_r\, D_r(G',K) \end{array}$$

in which the vertical arrows are finite free ring extensions on bases $\sigma(\mathcal{R}(\mathfrak{p}))$ resp. $\sigma(\mathcal{R}(\mathfrak{g}))$. By our assumptions and by the explicit shape of these bases there is a set $S \subseteq \sigma(\mathcal{R}(\mathfrak{g}))$ such that $\sigma(\mathcal{R}(\mathfrak{g})) = \{s\,t, s \in S,\ t \in \sigma(\mathcal{R}(\mathfrak{p}))\}$. It follows that the map of $gr^{\cdot}_r\, D_r(P',K)$-modules

$$\oplus_{g \in S}\ gr^{\cdot}_r\, D_r(P',K) \otimes_{gr^{\cdot}_r\, U_r(\mathfrak{p},K)} gr^{\cdot}_r\, U_r(\mathfrak{g},K) \rightarrow gr^{\cdot}_r\, D_r(G',K)$$

induced by $(\lambda \otimes \mu)_g \mapsto \sum_g \lambda\, \mu\, g$ is bijective. Using [ST5], Prop. 1.2 we are hence reduced to prove the flatness of the upper horizontal arrow. But this equals the inclusion of a polynomial ring over $gr^{\cdot} K$ in l variables into one of d variables (again by Prop. 2.4) which is clearly flat. $\square$

In case $L = \mathbb{Q}_p$ this result is precisely [ST6], Prop. 6.2. The proof of our proposition was simplified by a remark of J. Kohlhaase.

We finish this section with some results on the Lie algebra cohomology of $U(\mathfrak{g},K)$. Recall the homological standard complex of free $U(\mathfrak{g})$-modules $U(\mathfrak{g}) \otimes_L \bigwedge^{\cdot} \mathfrak{g}$ whose differential is given via

$$\partial(\lambda \otimes \mathfrak{x}_1 \wedge ... \wedge \mathfrak{x}_q) = \textstyle\sum_{s<t} (-1)^{s+t} \lambda \otimes [\mathfrak{x}_s, \mathfrak{x}_t] \wedge \mathfrak{x}_1 \wedge ... \wedge \widehat{\mathfrak{x}_s} \wedge ... \wedge \widehat{\mathfrak{x}_t} \wedge ... \wedge \mathfrak{x}_q$$

$$+ \textstyle\sum_s (-1)^{s+1} \lambda \mathfrak{x}_s \otimes \mathfrak{x}_1 \wedge ... \wedge \widehat{\mathfrak{x}_s} \wedge ... \wedge \mathfrak{x}_q.$$

Composing with the augmentation $U(\mathfrak{g}) \rightarrow L$ yields a finite free resolution of the $U(\mathfrak{g})$-module L and $H^*(\mathfrak{g},V) := h^*(\mathrm{Hom}_L(\bigwedge^{\cdot} \mathfrak{g}, V))$ resp. $H_*(\mathfrak{g},V) := h_*(V \otimes_L \bigwedge^{\cdot} \mathfrak{g})$ as objects in Vec_L for any $\mathfrak{g}$-module V. Assume $V \in Vec_K$ is nuclear Fréchet or of compact type such that $\mathfrak{g}$ acts by continuous K-linear operators. Endow each $V \otimes_L \bigwedge^q \mathfrak{g}$ resp. $\mathrm{Hom}_L(\bigwedge^q \mathfrak{g}, V)$ with the projective tensor product topology resp. the strong topology. The obvious map $V'_b \otimes_L \bigwedge^q \mathfrak{g} \rightarrow (\mathrm{Hom}_L(\bigwedge^q \mathfrak{g}, V))'_b$ is a topological isomorphism and identifies $V'_b \otimes_L \bigwedge^{\cdot} \mathfrak{g}$ with the strong dual of $\mathrm{Hom}_L(\bigwedge^{\cdot} \mathfrak{g}, V)$ (e.g. [P], 1.4). Endow $H^*(\mathfrak{g},V)$ resp. $H_*(\mathfrak{g},V)$ always with the induced topologies.

Lemma 2.7 *Let V be a nuclear Fréchet space with continuous $\mathfrak{g}$-action. Suppose the differential in $V \otimes_L \bigwedge^{\cdot} \mathfrak{g}$ is strict. There are isomorphisms of locally convex K-vector spaces*

$$H^*(\mathfrak{g}, V'_b) \simeq H_*(\mathfrak{g}, V)'_b \tag{4}$$

natural in V.

Proof: Since $V \otimes_L \dot{\bigwedge}\mathfrak{g}$ consists of Fréchet spaces the differential has closed image. By [ST2], Thm. 1.1, Prop. 1.2 the complex $(V \otimes_L \dot{\bigwedge}\mathfrak{g})'_b$ consists of spaces of compact type and has a strict differential with closed image. Thus, we may substitute in the proof of [Ko2], Lem. 3.6 all weak topologies by the strong topologies and obtain K-linear bijections $H_*(\mathfrak{g}, V)'_b = (h_*(V \otimes_L \dot{\bigwedge}\mathfrak{g}))'_b \simeq h^*((V \otimes_L \dot{\bigwedge}\mathfrak{g})'_b)$ which are readily seen to be topological. Since $\mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}, V'_b)$ consists of spaces of compact type the remark preceding the lemma implies $(V \otimes_L \dot{\bigwedge}\mathfrak{g})'_b \simeq \mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}, V'_b)$ topologically whence the claim. □

Proposition 2.8 *One has* $\oplus_* H_*(\mathfrak{g}, U(\mathfrak{g}, K)) = H_0(\mathfrak{g}, U(\mathfrak{g}, K)) = K$.

Proof: Over the complex numbers this follows from [P], Thm. 8.6. In our setting our results allow to give a proof along the lines of [ST6], Prop. 3.1. The case $* = 0$ is clear. Now being a Fréchet-Stein algebra (Thm.2.3) the topology on $U(\mathfrak{g}, K)$ is nuclear Fréchet and the differential in $U(\mathfrak{g}, K) \otimes_L \dot{\bigwedge}\mathfrak{g}$ is strict ([ST5], §3). By Lem. 2.7 it suffices to prove $H^*(\mathfrak{g}, C_1^{an}(G, K)) = 0$ for $* > 0$. By [BW], VII.1.1. the complex $\mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}, C_1^{an}(G, K))$ equals (up to sign) the stalk at $1 \in G$ of the deRham complex of K-valued global locally-analytic differential forms on the manifold G. By the usual Poincaré lemma the latter is acyclic. □

3 Continuous representations and analytic vectors

We recall some definitions and results from continuous representation theory relying on [S]. We introduce the notion of analytic vector and prove some basic properties.

A *locally analytic G-representation* is a barrelled locally convex Hausdorff K-vectorspace V equipped with a G-action via continuous operators such that for all $v \in V$ the orbit map $o_v : G \to V,\ g \mapsto g^{-1}v$ lies in $C^{an}(G, V)$, the space of V-valued locally analytic functions on G. With continuous K-linear G-maps these representations form a category $\mathrm{Rep}_K(G)$. Endowing $C^{an}(G, V)$ with the left regular action $((g.f)(h) = f(g^{-1}h))$ yields $C^{an}(G, V) \in \mathrm{Rep}_K(G)$ and a G-equivariant embedding $o : V \to C^{an}(G, V),\ v \mapsto o_v$. Let $\mathrm{Rep}_K^a(G) \subseteq \mathrm{Rep}_K(G)$ be the full subcategory of *admissible* representations. Denote the abelian category of coadmissible modules by $\mathcal{C}_G$. There is an anti-equivalence $\mathrm{Rep}_K^a(G) \simeq \mathcal{C}_G$ via $V \mapsto V'_b$. In particular, any $M \in \mathcal{C}_G$ has a nuclear Fréchet topology (the *canonical* topology). If G is compact $\mathcal{C}_G$ contains all finitely presented modules. In this case, a $V \in \mathrm{Rep}_K^a(G)$ such that V' is finitely generated is called *strongly admissible*.

Now let G be compact and $K/\mathbb{Q}_p$ be finite. A *Banach space representation* of G is a K-Banach space V with a linear action of G such that $G \times V \to V$ is continuous. Let $D^c(G, K)$ be the algebra of continuous K-valued distributions on G. Denote by $\mathrm{Ban}_G^{adm}(K)$ the abelian category of admissible representations and by $\mathcal{M}^{fg}(D^c(G, K))$ the finitely generated modules. There is an anti-equivalence $\mathrm{Ban}_G^{adm}(K) \simeq \mathcal{M}^{fg}(D^c(G, K))$ via $V \mapsto V'$. In particular,

$\mathrm{Ban}_G^{adm}(K)$ has enough injective objects. If $V \in \mathrm{Ban}_G^{adm}(K)$ then $v \in V$ is called a *locally L-analytic* vector if the orbit map $g \mapsto gv$ lies in $C^{an}(G, V)$. The subspace $V_{an} \subseteq V$ consisting of all these vectors has an induced continuous G-action and is endowed with the subspace topology arising from the embedding $o : V_{an} \to C^{an}(G, V)$. By [E1], Prop. 2.1.26 the inclusion $C(G, K) \subseteq C^{an}(G, K)$ is continuous and dualizes therefore to an algebra map $D^c(G, K) \to D(G, K)$.

Theorem 3.1 (Schneider-Teitelbaum) *Let G be compact and $K/\mathbb{Q}_p$ be finite. Suppose that $L = \mathbb{Q}_p$. The map*

$$D^c(G, K) \longrightarrow D(G, K) \tag{5}$$

is faithfully flat. Given $V \in \mathrm{Ban}_G^{adm}(K)$ the representation V_{an} is a strongly admissible locally analytic representation and $V_{an} \subseteq V$ is norm-dense. The functor $F_{\mathbb{Q}_p} : V \mapsto V_{an}$ between $\mathrm{Ban}_G^{adm}(K)$ and $\mathrm{Rep}_K^a(G)$ is exact. The dual functor equals base extension.

This is [S], Thm. 4.2/3. Given the exactness statement $V_{an} \subseteq V$ being dense is equivalent to the functor being faithful. However, in general, the functor is not full (cf. [E2], end of §3). Furthermore, if $L \neq \mathbb{Q}_p$, it is generally not exact and can be zero on objects. For example (cf. [E1], §3) let $G = (\mathfrak{o}_L, +)$ and suppose $\psi : G \to K$ is $\mathbb{Q}_p$-linear but not L-linear. The two-dimensional representation of G given by the matrix $\begin{pmatrix} 1 & \psi \\ & 1 \end{pmatrix}$ is an extension of the trivial representation by itself but not locally L-analytic.

To study F_L we have to introduce another functor. Let G be an *arbitrary* locally L-analytic group and $\mathbb{Q}_p \subseteq K$ be discretely valued. Given $V \in \mathrm{Rep}_K(G_0)$ we call $v \in V$ a *locally L-analytic vector* if $o_v \in C^{an}(G_0, V)$ lies in the subspace $C^{an}(G, V)$. Denote the space of these vectors, endowed with the subspace topology from V, by V_{an}. Since translation on G is locally L-analytic V_{an} has an induced continuous G-action. In the following we will show that the correspondance $V \mapsto V_{an}$ induces a functor

$$F_{\mathbb{Q}_p}^L : \mathrm{Rep}_K^a(G_0) \to \mathrm{Rep}_K^a(G).$$

Given $V \in \mathrm{Rep}_K(G)$ the Lie algebra $\mathfrak{g}$ acts on V via continuous endomorphisms

$$\mathfrak{x}v := \frac{d}{dt}\exp(t\mathfrak{x})v|_{t=0}$$

for $\mathfrak{x} \in \mathfrak{g},\ v \in V$. Denote by $C_1^{an}(G, K)$ the local ring at $1 \in G$ as introduced before. Given $f \in C^{an}(G, K)$ denote its image in $C_1^{an}(G, K)$ by $[f]$. Viewing $\mathfrak{g}_0$ as point derivations on $C_1^{an}(G_0, K)$ restricting derivations to $C_1^{an}(G, K)$ induces a map $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0 \to \mathfrak{g}$. Denote its kernel by $\mathfrak{g}^0$. Let $(\mathfrak{g}^0)$ denote the two-sided ideal generated by $\mathfrak{g}^0$ inside $U(\mathfrak{g}_0, K)$ as well as in $D(G_0, K)$. It equals the kernel of the quotient maps $U(\mathfrak{g}_0, K) \to U(\mathfrak{g}, K)$ as well as $D(G_0, K) \to D(G, K)$ (by straighforward generalizations of [Sch], Lem. 5.1)

Lemma 3.2 *An element $f \in C^{an}(G_0, K)$ is locally L-analytic at $1 \in G$ if and only if the space of derivations $\mathfrak{g}^0$ annihilates $[f]$.*

Proof: The function f is locally L-analytic at 1 if and only if this is true for $[f]$. By Thm. 2.3 $U(\mathfrak{g}_0, K)$ is Fréchet-Stein. Hence $(\mathfrak{g}^0) \subseteq U(\mathfrak{g}_0, K)$ being finitely generated is closed. By [B-TVS], IV.2.2 Corollary the natural map $C_1^{an}(G_0,K)/C_1^{an}(G,K) \to (\mathfrak{g}^0)'$ is an isomorphism whence the claim follows. $\Box$

Lemma 3.3 *Given $V \in \mathrm{Rep}_K(G_0)$ of compact type one has $V_{an} = V^{\mathfrak{g}^0}$ as subspaces of V. In particular, $V_{an} \subseteq V$ is closed.*

Proof: This follows also from [E1], Prop. 3.6.19 but we give a proof in the present language. We may assume that G is compact. Suppose first that $V = C^{an}(G_0,K)$. The inclusion $V_{an} \subseteq V^{\mathfrak{g}^0}$ is clear from Lem. 3.2. Let $f \in V^{\mathfrak{g}^0}$ and $g \in G$. Denote by Ad the adjoint action of G. Since $\mathfrak{g}^0$ is $L \otimes_{\mathbb{Q}_p} \mathrm{Ad}(g)$-stable the identity $g\mathfrak{x}g^{-1}.v = \mathrm{Ad}(g)\mathfrak{x}.v$ for $v \in V$ implies that $g.f$ (left regular action) lies in $V^{\mathfrak{g}^0}$ whence is locally analytic at $1 \in G$ by Lem. 3.2. This settles the case $V = C^{an}(G_0,K)$. For general $V \in \mathrm{Rep}_K(G_0)$ of compact type equipping $C^{an}(G_0,K)\hat{\otimes}_K V$ with the diagonal action (trivial on the second factor) the topological vector space isomorphism $C^{an}(G_0,K)\hat{\otimes}_K V \xrightarrow{\sim} C^{an}(G_0,V)$ ([E1], Prop. 2.1.28) becomes G-equivariant. By continuity one obtains $C^{an}(G_0,V)^{\mathfrak{g}^0} =$ closure of $C^{an}(G_0,K)^{\mathfrak{g}^0} \otimes_K V$ which, by the first step, equals $C^{an}(G,V)$. Hence, for $v \in V$ we have $\mathfrak{g}^0\, v = 0 \Leftrightarrow \mathfrak{g}^0\, o_v = 0 \Leftrightarrow o_v \in C^{an}(G,V)$ whence $V^{\mathfrak{g}^0} = V_{an}$. $\Box$

We remark that, by definition, admissible locally analytic representations are, in particular, vector spaces of compact type.

Proposition 3.4 *The correspondance $V \mapsto V_{an}$ induces a left exact functor*

$$F^L_{\mathbb{Q}_p} : \mathrm{Rep}^a_K(G_0) \to \mathrm{Rep}^a_K(G).$$

The dual functor equals base extension.

Proof: By Lem. 3.3 $V_{an} \subseteq V$ is closed and hence of compact type. If $H \subseteq G$ is a compact open subgroup then since $D(H_0,K)/(\mathfrak{g}^0) = D(H,K)$ the strong dual $(V_{an})'_b$ lies in $\mathcal{C}_{H_0} \cap \mathcal{M}(D(H,K)) = \mathcal{C}_H$. Thus, $V_{an} \in \mathrm{Rep}^a_K(G)$. It is immediate that $V \mapsto V_{an}$ is a functor which is left exact (Lem. 3.3). Putting $\mathfrak{h} := \mathfrak{g}^0$ in Prop. 5.5 below the last claim follows. $\Box$

Corollary 3.5 *Let G be compact and $K/\mathbb{Q}_p$ be finite. One has $F_L = F^L_{\mathbb{Q}_p} \circ F_{\mathbb{Q}_p}$ whence F_L is left exact and the dual functor equals base extension.*

Proof: Given $V \in \mathrm{Ban}^{adm}_G(K)$ the identity $F_L(V) = F^L_{\mathbb{Q}_p} \circ F_{\mathbb{Q}_p}(V)$ as abstract $K[G]$-modules is clear from the definitions. Since $C^{an}(G,V) \subseteq C^{an}(G_0,V)$ is a closed topological embedding (by a straightforward generalization of [ST4], Lem. 1.2) the topology on the space $F_L(V) = F_{\mathbb{Q}_p}(V) \cap C^{an}(G,V)$ coincides with the one induced by $F_{\mathbb{Q}_p}(V)$. By definition this equals the topology of $F^L_{\mathbb{Q}_p} \circ F_{\mathbb{Q}_p}(V)$. The last claim follows from Thm. 3.1 and the last proposition by associativity of the tensor product. $\Box$

Remarks:

1. Since F_L is not exact we obtain from the corollary that as a rule, the map $D^c(G,K) \to D(G,K)$ is not flat for $L \neq \mathbb{Q}_p$ (but see Thm. 4.6). In view of the characterization as certain Lie invariants (Lem. 3.3) one may ask whether flatness holds when G is semisimple. This is answered negatively by Cor. 7.8 below.

2. On certain interesting subcategories of $\mathrm{Rep}_K^a(G_0)$ the functor $F_{\mathbb{Q}_p}^L$ may very well be exact. To give an example recall that $V \in \mathrm{Rep}_K^a(G)$ is called *locally* $U(\mathfrak{g})$*-finite* if for all $x \in V_b'$ the orbit $U(\mathfrak{g})x$ is contained in a finite dimensional K-subspace of V_b'. These representations are studied in [ST1]. Let $\mathrm{Rep}_K^{a,f}(G)$ denote the full abelian subcategory of $\mathrm{Rep}_K^a(G)$ consisting of these representations. We claim: if G is semisimple passage to analytic vectors is an *exact* functor $\mathrm{Rep}_K^{a,f}(G_0) \to \mathrm{Rep}_K^{a,f}(G)$. Indeed: let $V \in \mathrm{Rep}_K^{a,f}(G_0)$. It suffices to see that $H^1(\mathfrak{g}^0, V) = 0$. The $\mathfrak{g}^0$-module V_b' is a direct limit of finite dimensional ones W. Since $\mathfrak{g}_0$ and thus $\mathfrak{g}^0$ are semisimple Lie algebras the first Whitehead lemma together with Lem. 2.7 for $* = 1$ yields $H_1(\mathfrak{g}^0, W) = 0$. Since $H_1(\mathfrak{g}^0, .)$ commutes with direct limits using Lem. 2.7 again we obtain $H^1(\mathfrak{g}^0, V) = 0$.

4 σ-analytic vectors

In this section G denotes a compact d-dimensional locally L-analytic group. Under the assumption that $L/\mathbb{Q}_p$ is unramified we will prove a generalization to Thm. 3.1.

So let us first **assume** that $L/\mathbb{Q}_p$ is Galois. We start with a result on the Fréchet-Stein structure on $U(\mathfrak{g}, K)$. Given $\sigma \in Gal(L/\mathbb{Q}_p)$ let G_σ be the scalar restriction of G via $\sigma : L \to L$ ([B-VAR], 5.14.1). It is a compact locally L-analytic group. Denote by $\mathfrak{g}_\sigma$ its Lie algebra. Of course, $(G_\sigma)_0 = G_0$, $(\mathfrak{g}_\sigma)_0 = \mathfrak{g}_0$ since σ is $\mathbb{Q}_p$-linear. Put $G_{L/\mathbb{Q}_p} := \prod_\sigma G_\sigma$ and $\mathfrak{g}_{L/\mathbb{Q}_p}$ for its Lie algebra. There is a commutative diagram of locally convex K-vector spaces

$$\begin{array}{ccc} C^{an}(G_{L/\mathbb{Q}_p}, K) & \xrightarrow{\Delta^* \circ \iota} & C^{an}(G_0, K) \\ {\scriptstyle [.]}\downarrow & & \downarrow{\scriptstyle [.]} \\ C_1^{an}(G_{L/\mathbb{Q}_p}, K) & \xrightarrow{\Delta^* \circ \iota} & C_1^{an}(G_0, K) \end{array}$$

where the horizontal arrows are induced functorially from the diagonal map $\Delta : G_0 \mapsto (\prod_\sigma G_\sigma)_0$ together with the canonical embedding ι (cf. (3)).

Lemma 4.1 *The lower horizontal map is bijective.*

Proof: We may assume that G admits a global chart ϕ, that $L = K$ and that $\dim_L G = 1$. Using ϕ and the induced global chart for G_σ resp. G_0 we arrive at a map $C_1^{an}(\prod_\sigma L_\sigma, L) \to C_1^{an}(\mathbb{Q}_p^n, L) \simeq C_1^{an}(L^n, L)$ where the first map depends on a choice of $\mathbb{Q}_p$-basis v_i of L and the second identification as rings is the obvious one. Tracing through the definitions shows that this map is induced by the L-linear isomorphism $\prod_\sigma L_\sigma \simeq L \otimes_{\mathbb{Q}_p} L \simeq \oplus_i L v_i \simeq L^n$ and hence, is bijective. $\square$

Passing to strong duals yields a commutative diagramm of topological algebras

$$\begin{array}{ccc} U(\mathfrak{g}_0, K) & \xrightarrow{\sim} & U(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \\ {\scriptstyle [.]'}\downarrow & & {\scriptstyle [.]'}\downarrow \\ D(G_0, K) & \xrightarrow{\iota' \circ \Delta_*} & D(G_{L/\mathbb{Q}_p}, K). \end{array}$$

We abbreviate in the following $\varphi := \iota' \circ \Delta_*$.

Lemma 4.2 *Suppose G is uniform*. For each r sufficiently close to 1 the above diagram extends to a commutative diagram*

$$\begin{array}{ccc} U_r(\mathfrak{g}_0, K) & \longrightarrow & U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \\ \downarrow & & \downarrow \\ D_r(G_0, K) & \xrightarrow{\varphi_r} & D_r(G_{L/\mathbb{Q}_p}, K) \end{array}$$

of Banach algebras where the upper horizontal map is injective, norm-decreasing and has dense image.

Proof: Let $||.||_r$ be a fixed norm on $D(G_0, K) = D((G_\sigma)_0, K)$ and let $\|.\|_{\bar{r}}^{\sigma}$ be the quotient norm on $D(G_\sigma, K)$ under $\iota'_\sigma : D((G_\sigma)_0, K) \to D(G_\sigma, K)$. Let $\mathfrak{x}_j$ and v_i be bases for the uniform* group G. In particular $\mathfrak{x}_1, ..., \mathfrak{x}_d$ is an L-basis for $\mathfrak{g}$ and $v_1 = 1, ..., v_n$ is a $\mathbb{Z}_p$-basis for $\mathfrak{o}_L$. The elements $h_{ij} := \exp(v_i \mathfrak{x}_j)$ are a minimal set of topological generators for G_0. Putting $b_{ij} := h_{ij} - 1$ we obtain from (2) that $D(G_0, K)$ consists of certain series $\lambda = \sum_{\alpha \in \mathbb{N}_0^{nd}} d_\alpha \mathbf{b}^\alpha$ where $||\lambda||_r = \sup_\alpha |d_\alpha| r^{\kappa\alpha}$. The product version of these considerations yields norms $||.||_r^{(\sigma)}$ resp. quotient norms $\|.\|_{\bar{r}}^{(\sigma)}$ on $D(\prod_\sigma G_0, K)$ resp. $D(G_{L/\mathbb{Q}_p}, K)$. Now φ is induced from Δ and $D((\prod_\sigma G_\sigma)_0, K) \to D(\prod_\sigma G_\sigma, K)$ where the second map is certainly norm-decreasing with respect to $||.||_r^{(\sigma)}$ and $\|.\|_{\bar{r}}^{(\sigma)}$. Furthermore $\Delta_*(b_{ij}) = \Delta(h_{ij}) - 1$. Since all nd elements $\Delta(h_{ij})$ are pairwise different modulo the first step $(\prod_\sigma G_0)^p$ in the lower p-series of the uniform group $\prod_\sigma G_0$ the discussion in [DDMS], 4.2 shows that they may be completed to a minimal ordered system of generators for $\prod_\sigma G_0$. Since the norm $||.||_r^{(\sigma)}$ does not depend on a particular choice of such system (cf. discussion in [ST5] after Thm. 4.10) one obtains $||\Delta(h_{ij}) - 1||_r^{(\sigma)} = ||(h_{ij}, 1, ...) - 1||_r^{(\sigma)} = ||h_{ij} - 1||_r$ whence it easily follows that Δ_* is an isometry. Then φ is norm-decreasing with respect to $||.||_r$ and $\|.\|_{\bar{r}}^{(\sigma)}$ which yields the completed diagram and its commutativity. It is clear that the upper horizontal map is norm-decreasing with dense image whence it remains to establish injectivity. We first show that the inverse map

$$\varphi^{-1} : U(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \to U(\mathfrak{g}_0, K)$$

is norm-decreasing when both sides are given suitable norm topologies. By Thm. 2.3 the rings $U_r(\mathfrak{g}_0, K)$ resp. $U_r(\mathfrak{g}_\sigma, K)$ are certain noncommutative power series rings in the "variables" $\partial_{ij} := v_i \mathfrak{x}_j \in \mathfrak{g}_0$ resp. $\partial_{\sigma,j} := \mathfrak{x}_j \in \mathfrak{g}_\sigma$. More precisely, $U_r(\mathfrak{g}_\sigma, K)$ consists of all formal series $\sum_{\beta \in \mathbb{N}_0^d} d_\beta \partial_\sigma^\beta$ where $d_\beta \in K$, $\partial_\sigma^\beta := \partial_{\sigma,1}^{\beta_1} \cdots \partial_{\sigma,d}^{\beta_d}$ and $||d_\beta \partial_\sigma^\beta||_{\bar{r}}^{\sigma} \to 0$ for $|\beta| \to \infty$. By Prop. 2.4 $\|.\|_{\bar{r}}^{\sigma}$ is multiplicative, the topological K-basis ∂_σ^β for $U_r(\mathfrak{g}_\sigma, K)$ is even orthogonal with respect to $\|.\|_{\bar{r}}^{\sigma}$ and $||\partial_{\sigma,j}||_{\bar{r}}^{\sigma} = ||\partial_{1j}||_r = c_r$. Given a generic element of $U(\mathfrak{g}_{L/\mathbb{Q}_p}, K)$, say $\lambda := \sum_{\beta\in\mathbb{N}_0^{nd}} d_\beta \prod_\sigma \partial_\sigma^{\beta_{\sigma\cdot}}$ with $d_\beta \in K$, $\partial_\sigma^{\beta_{\sigma\cdot}} = \partial_{\sigma,1}^{\beta_{\sigma,1}} \cdots \partial_{\sigma,d}^{\beta_{\sigma,d}}$ we have $||\lambda||_{\bar{r}}^{(\sigma)} = \sup_\beta |d_\beta|(c_r)^{|\beta|}$ since the elements $\prod_\sigma \partial_\sigma^{\beta_{\sigma\cdot}}$ are orthogonal with respect to $||.||_{\bar{r}}^{(\sigma)}$ and this latter norm is multiplicative. After these remarks consider the map $L \otimes_{\mathbb{Q}_p} L \to \prod_\sigma L_\sigma$. Let $\sum_{a_\sigma, b_\sigma} a_\sigma \otimes b_\sigma$ be the inverse image of $1 \in L_\sigma$ where we may assume that $b_\sigma \in \mathfrak{o}_L$. Choosing $b_\sigma^{(i)} \in \mathbb{Z}_p$ such that $\sum_i b_\sigma^{(i)} v_i = b_\sigma$ we put

$$s := \sup_{\sigma, a_\sigma, b_\sigma, i} |a_\sigma b_\sigma^{(i)}|.$$

We have by definition of the map φ that

$$\varphi^{-1}(\partial_{\sigma,j}) = \sum_{a_\sigma, b_\sigma^{(i)}, i} a_\sigma b_\sigma^{(i)} \partial_{ij}.$$

Hence,

$$||\varphi^{-1}(\partial_{\sigma,j})||_r = \sup_i |\sum_{a_\sigma, b_\sigma^{(i)}, i} a_\sigma b_\sigma^{(i)}| c_r \leq s c_r$$

using that the ∂_{ij} are orthogonal. Given $\lambda \in U(\mathfrak{g}_{L/\mathbb{Q}_p}, K)$ as above define another norm on $U(\mathfrak{g}_{L/\mathbb{Q}_p}, K)$ via

$$||\lambda||_{(\bar{r})}^{(\sigma)} := \sup_\beta |d_\beta|(s c_r)^{|\beta|}$$

and let $U_{(r)}(\mathfrak{g}_{L/\mathbb{Q}_p}, K)$ be the completion. We obtain

$$||\varphi^{-1}(\lambda)||_r \leq \sup_\beta |d_\beta| \prod_{\sigma,j} ||\varphi^{-1}(\partial_{\sigma,j})||_r^{\beta_{\sigma,j}} = \sup_\beta |d_\beta|(s c_r)^{|\beta|} = ||\lambda||_{(\bar{r})}^{(\sigma)}$$

using that $||.||_r$ is multiplicative. Thus φ^{-1} is norm-decreasing with respect to the indicated norms. Now suppose r is sufficiently close to 1. Then there exists $r' \leq r$ such that $s c_{r'} \leq c_r$ (note that $c_r \uparrow \infty$ for $r \uparrow 1$) whence $U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \subseteq U_{(r')}(\mathfrak{g}_{L/\mathbb{Q}_p}, K)$. A simple approximation argument shows that the map

$$U_r(\mathfrak{g}_0, K) \xrightarrow{\varphi_r} U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \xrightarrow{\subseteq} U_{(r')}(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \xrightarrow{\varphi_{r'}^{-1}} U_{r'}(\mathfrak{g}_0, K)$$

equals the inclusion $U_r(\mathfrak{g}_0, K) \subseteq U_{r'}(\mathfrak{g}_0, K)$. This finishes the proof. □

For the rest of this section we assume $L/\mathbb{Q}_p$ to be **unramified**.

Lemma 4.3 *Let $x \in \mathfrak{o}_L^\times$ be a lift of a primitive element $\bar{x}$ for the residue field extension of $L/\mathbb{Q}_p$. Suppose G is uniform* with a $\mathbb{Z}_p$-basis $v_1 = 1, ..., v_n$ of $\mathfrak{o}_L$ of the form $v_i = x^{i-1}$. In the situation of the last lemma the map*

$$\varphi_r : U_r(\mathfrak{g}_0, K) \to U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K)$$

is an isometry.

Proof: Source and target of our map are filtered through the respective norm and it suffices to see that the associated graded map $\dot{gr_r}\, \varphi_r : \dot{gr_r}\, U_r(\mathfrak{g}_0, K) \to \dot{gr_r}\, U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K)$ is injective. We use the notation of the preceding proof. Let

X_{ij} resp. $X_{\sigma,j}$ be the principal symbol of ∂_{ij} resp. $\partial_{\sigma,j}$. Then $gr^{\cdot}_r\, U_r(\mathfrak{g}_0, K) = (gr^{\cdot} K)[X_{11}, ..., X_{nd}]$ and $U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K) = (gr^{\cdot} K)[X_{\sigma_1,1}, ..., X_{\sigma_n,d}]$. Denote for each $\sigma \in Gal(L/\mathbb{Q}_p)$ by $\bar{\sigma}$ the induced Galois automorphism on residue fields. Let F_j be the ring homomorphism

$$F_j : (gr^{\cdot} K)[X_{1j}, ..., X_{nj}] \longrightarrow (gr^{\cdot} K)[X_{\sigma_1,j}, ..., X_{\sigma_n,j}],\ X_{ij} \mapsto \sum_{\sigma} \bar{\sigma}^{-1}\bar{v}_i.X_{\sigma,j}.$$

Then $gr^{\cdot}_r\, \varphi$ equals, in the obvious sense, $F_1 \otimes_{gr^{\cdot} K} ... \otimes_{gr^{\cdot} K} F_d$ whence, by induction, it suffices to prove bijectivity of F_1. Now F_1 respects the grading by total degree whence it is enough to prove bijectivity on homogeneous components. Since each latter is free of finite rank over the principal ideal domain $gr^{\cdot} K$ it suffices to prove surjectivity in each component or, since F_1 is a ring homomorphism, in the degree 1 component.But the representing matrix $(a_{ij})_{i,j=1,...,n}$ of the degree 1 part of F_1 with respect to the $gr^{\cdot} K$-bases $X_i, i = 1, ..., n$ resp. $X_\sigma, \sigma \in Gal(L/\mathbb{Q}_p)$ on source resp. target has the shape $a_{ij} = (\bar{\sigma}_i^{-1}\bar{x})^{j-1}$ and hence determinant $\prod_{j>i}(\bar{\sigma}_j^{-1}\bar{x} - \bar{\sigma}_i^{-1}\bar{x})$. Since $L/\mathbb{Q}_p$ is unramified and $\bar{x}$ generates the residue extension this determinant is a nonzero element of the residue field of L whence a unit in $gr^{\cdot} K$. $\square$

Lemma 4.4 *Let r be sufficiently close to 1. The ring extension*

$$\varphi_r : D_r(G_0, K) \longrightarrow D_r(G_{L/\mathbb{Q}_p}, K)$$

is faithfully flat.

Proof: We prove only the left version (the right version follows similarly). Suppose $H \subseteq G$ is a normal open subgroup which is uniform*. Endow $D(G, K)$ with the Fréchet-Stein structure induced by $D(H, K)$ as explained in section 2. One obtains a commutative diagram

$$\begin{array}{ccc} D_r(H_0, K) & \longrightarrow & D_r(H_{L/\mathbb{Q}_p}, K) \\ \downarrow & & \downarrow \\ D_r(G_0, K) & \xrightarrow{\varphi_r} & D_r(G_{L/\mathbb{Q}_p}, K). \end{array}$$

Let $\mathcal{R}$ be a system of coset representatives for G/H. Choose a system of coset representatives $\mathcal{R}'$ in G^0 for coset representatives of the cokernel of the inclusion $G/H \xrightarrow{\Delta} \prod_\sigma G_\sigma/H_\sigma = G_{L/\mathbb{Q}_p}/H_{L/\mathbb{Q}_p}$. Then $\varphi_r(\mathcal{R})\mathcal{R}'$ equals a system of coset representatives for $G_{L/\mathbb{Q}_p}/H_{L/\mathbb{Q}_p}$ and so the vertical arrows in the above diagram are finite free ring extensions on $\mathcal{R}$ resp. $\varphi_r(\mathcal{R})\mathcal{R}'$. Consider the map of left $D_r(G_0, K)$-modules

$$\oplus_{g' \in \mathcal{R}'}\ D_r(G_0, K) \otimes_{D_r(H_0,K)} D_r(H_{L/\mathbb{Q}_p}, K) \longrightarrow D_r(G_{L/\mathbb{Q}_p}, K) \qquad (6)$$

induced by $(\lambda \otimes \mu)_{g' \in \mathcal{R}'} \mapsto \sum_{g' \in \mathcal{R}'} \varphi_r(\lambda)\mu\, g'$. On the level of vector spaces this map factores through

$$\oplus_{g \in \mathcal{R}, g' \in \mathcal{R}'}\ gD_r(H_{L/\mathbb{Q}_p}, K) \xrightarrow{\sim} \sum_{g,g'} \varphi_r(g)D_r(H_{L/\mathbb{Q}_p}, K)g' = D_r(G_{L/\mathbb{Q}_p}, K)$$

and hence, is bijective (using that $g' D_r(H_{L/\mathbb{Q}_p}, K) = D_r(H_{L/\mathbb{Q}_p}, K) g'$ by normality of $H_{L/\mathbb{Q}_p}$ in $G_{L/\mathbb{Q}_p}$). It therefore suffices to establish the claim for H. In other words, we may assume in the following that G is uniform*. By the construction of uniform* subgroups (cf. [Sch], Cor. 4.4) we may assume that the associated $\mathbb{Z}_p$-basis $v_1 = 1, ..., v_n$ is as described in Lem. 4.3. This lemma together with Lem. 4.2 then yields a commutative diagram (+)

$$\begin{array}{ccc} U_r(\mathfrak{g}_0, K) & \longrightarrow & U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \\ \downarrow & & \downarrow \\ D_r(G_0, K) & \xrightarrow{\varphi_r} & D_r(G_{L/\mathbb{Q}_p}, K) \end{array}$$

where the upper vertical arrow is an isometry with dense image. Let $\mathfrak{X} := \{\mathfrak{x}_1, ..., \mathfrak{x}_d\}$ be the associated L-basis of $\mathfrak{g}_L$ for G. Then the nd elements $\exp(v_i \mathfrak{x}_j)$ are a minimal set S of topological generators for G. Hence, according to Thm. 2.3, the left vertical arrow is finite free on a basis $\mathcal{R}$ in $\mathbb{Z}[G]$. Putting $\mathbf{b}^\alpha = b_{11}^{\alpha_{11}} \cdots b_{nd}^{\alpha_{nd}}, b_{ij} = \exp(v_i \mathfrak{x}_j) - 1 \in \mathbb{Z}[G]$ one has $\mathcal{R} = \{\mathbf{b}^\alpha,\ \alpha_{ij} < \epsilon(r,p) \text{ for all } ij\}$ with $\epsilon(r,p)$ depending only on r and p. Apply this to the group $G_{L/\mathbb{Q}_p}$ as well. More precisely, $G_{L/\mathbb{Q}_p}$ is uniform* with $v_1, ..., v_n$ and $n = |Gal(L/\mathbb{Q}_p)|$ copies of $\mathfrak{X}$ as bases. By a previous argument the set $\varphi_r(S) \subseteq G_{L/\mathbb{Q}_p}$ may be completed to a minimal set S' of generators for the uniform group $\prod_\sigma G_0$. Choose an ordering $h'_1, ..., h'_{ndn}$ of S' such that $\varphi_r(S) = \{h'_1, ..., h'_{nd}\}$. Put $b'_k = h'_k - 1$ and form the set $\mathcal{R}' := \{\mathbf{b}'^\alpha,\ \alpha_k < \epsilon(r,p) \text{ for all } k\}$. Again by Thm. 2.3 the right vertical arrow is finite free on the basis $\mathcal{R}'$. Let $\mathcal{R}'_< = \{\mathbf{b}'^\alpha \in \mathcal{R}', \alpha_k = 0 \text{ for all } k > nd\}$ and $\mathcal{R}'_> = \{\mathbf{b}'^\alpha \in \mathcal{R}', \alpha_k = 0 \text{ for all } k \leq nd\}$. The chosen ordering of S' implies that $\varphi_r(\mathcal{R}) = \mathcal{R}'_<$. Now recall that $D_r(G_0, K)$ and $D_r(G_{L/\mathbb{Q}_p}, K)$ are Zariski rings (cf. section 2) with respect to the norm filtrations and φ_r is norm-preserving (Lem. 4.2) whence a filtered morphism. Thus, it suffices to prove faithful flatness of the graded map $gr^{\cdot}_r\, \varphi_r : gr^{\cdot}_r\, D_r(G_0, K) \to gr^{\cdot}_r\, D_r(G_{L/\mathbb{Q}_p}, K)$ ([LVO], II.1.2.2). To do this consider the graded version of (+) (where the upper objects are formed with respect to the induced filtrations via the vertical inclusions of (+))

$$\begin{array}{ccc} gr^{\cdot}_r\, U_r(\mathfrak{g}_0, K) & \xrightarrow{\sim} & gr^{\cdot}_r\, U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \\ \downarrow & & \downarrow \\ gr^{\cdot}_r\, D_r(G_0, K) & \xrightarrow{gr^{\cdot}_r\, \varphi_r} & gr^{\cdot}_r\, D_r(G_{L/\mathbb{Q}_p}, K). \end{array}$$

Recall that all rings occuring are commutative. For the rest of this proof $\sigma(.)$ always denotes the principal symbol map. By Prop. 2.4 the vertical arrows are finite free on the basis $\sigma(\mathcal{R})$ resp. $\sigma(\mathcal{R}')$ and $\sigma(\mathcal{R}') = \{st,\ s \in \sigma(\mathcal{R}'_<),\ t \in \sigma(\mathcal{R}'_>)\}$. Since $gr^{\cdot}_r\, \varphi_r \neq 0$ one has $gr^{\cdot}_r\, \varphi_r(\sigma(b_{ij})) \neq 0$ for all ij whence $gr^{\cdot}_r\, \varphi_r(\sigma(\mathcal{R})) = \sigma(\varphi_r(\mathcal{R})) = \sigma(\mathcal{R}'_<)$. Then the map of $gr^{\cdot}_r\, D_r(G_0, K)$-modules

$$\oplus_{x' \in \sigma(\mathcal{R}'_>)}\ gr^{\cdot}_r\, D_r(G_0, K) \otimes_{gr^{\cdot}_r\, U_r(\mathfrak{g}_0, K)} gr^{\cdot}_r\, U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \longrightarrow gr^{\cdot}_r\, D_r(G_{L/\mathbb{Q}_p}, K)$$

induced by $(\lambda \otimes \mu)_{x' \in \sigma(\mathcal{R}'_>)} \mapsto \sum_{x' \in \sigma(\mathcal{R}'_>)} gr^{\cdot}_r\, \varphi_r(\lambda)\, \mu\, x'$ is bijective. It follows that $gr^{\cdot}_r\, D_r(G_{L/\mathbb{Q}_p}, K)$ is a finite free $gr^{\cdot}_r\, D_r(G_0, K)$-module (of rank $\#\mathcal{R}'_>$). $\square$

Corollary 4.5 *The map φ is injective.*

Proof: Since all φ_r are injective (for r sufficiently close to 1) and compatible with transition with respect to $r' \leq r$ the map φ is injective by left-exactness of the projective limit. $\square$

For the rest of this section we assume $K/\mathbb{Q}_p$ to be finite. Consider the faithfully flat algebra map $D^c(G,K) \to D(G_0,K)$ stated in (5). We compose it with φ and show our first main result.

Theorem 4.6 *The map $D^c(G,K) \longrightarrow D(G_{L/\mathbb{Q}_p},K)$ is faithfully flat.*

Proof: We show only left faithful flatness. For flatness we are reduced, by the usual argument, to show that the map $D(G_{L/\mathbb{Q}_p},K)\otimes_{D^c(G,K)}J \longrightarrow D(G_{L/\mathbb{Q}_p},K)$ is injective for any left ideal $J \subseteq D^c(G,K)$. The ring $D^c(G,K)$ being noetherian the left hand side is a coadmissible $D(G_{L/\mathbb{Q}_p},K)$-module. By left exactness of the projective limit we are thus reduced to show that $D_r(G_{L/\mathbb{Q}_p},K) \otimes_{D^c(G,K)} J \to D_r(G_{L/\mathbb{Q}_p},K)$ is injective for all r (sufficiently close to 1). This is clear since

$$D^c(G,K) \to D(G_0,K) \to D_r(G_0,K) \xrightarrow{\varphi_r} D_r(G_{L/\mathbb{Q}_p},K)$$

is flat (the second map by [ST5], remark 3.2).

For faithful flatness we have to show $D(G_{L/\mathbb{Q}_p},K) \otimes_{D^c(G,K)} M \neq 0$ for any nonzero left $D^c(G,K)$-module M. By the first step we may assume that M is finitely generated. Then $D(G_{L/\mathbb{Q}_p},K) \otimes_{D^c(G,K)} M$ is coadmissible whence we are reduced, by the equivalence of categories between coadmissible modules and coherent sheafs ([ST5], Cor. 3.3), to find an index r such that

$$D_r(G_{L/\mathbb{Q}_p},K) \otimes_{D^c(G,K)} M \neq 0.$$

Put $N := D(G_0,K) \otimes_{D^c(G,K)} M \in \mathcal{C}_{G_0}$. Then $N \neq 0$ whence $N_r \neq 0$ for some r (sufficiently close to 1). It follows that $D_r(G_{L/\mathbb{Q}_p},K) \otimes_{D^c(G,K)} M$ equals

$$D_r(G_{L/\mathbb{Q}_p},K) \otimes_{D(G_0,K)} N = D_r(G_{L/\mathbb{Q}_p},K) \otimes_{D_r(G_0,K)} N_r$$

and the right-hand side is nonzero by faithful flatness of φ_r. $\square$

Each choice $\sigma \in Gal(L/\mathbb{Q}_p)$ gives rise to the locally L-analytic manifold K_σ arising from restriction of scalars via σ. The space $C^{an}(G,K_\sigma) = C^{an}(G_{\sigma^{-1}},K)$ is called the space of *locally σ-analytic* functions ([B-VAR], 5.14.3). This motivates the following definition: given $V \in \mathrm{Rep}_K(G)$ an element $v \in V$ is called a *locally σ-analytic vector* if $o_v \in C^{an}(G_{\sigma^{-1}},V)$. Let $V_{\sigma-an}$ denote the subspace of all these vectors in V.

Consider the Lie algebra map $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0 \simeq \prod_\sigma \mathfrak{g}_\sigma \to \mathfrak{g}_{\sigma^{-1}}$. The kernel $\mathfrak{g}_\sigma^0$ acts on V whence one deduces as in case $\sigma = id$ that $V_{\sigma-an} = V^{\mathfrak{g}_\sigma^0}$, functorial in V and that passage to σ-analytic vectors is a left exact functor $\mathrm{Rep}_K^a(G_0) \to \mathrm{Rep}_K^a(G_{\sigma^{-1}})$. Note also that given $\sigma \neq \tau$ one has $V_{\sigma-an} \cap V_{\tau-an} = V^{\mathfrak{g}_0} = V^\infty$, the smooth vectors in V.

We consider the base extension functor $M \mapsto M \otimes_{D^c(G,K)} D(G_{L/\mathbb{Q}_p},K)$ on finitely generated $D^c(G,K)$-modules and pull back to representations. This yields a functor

$$F : \mathrm{Ban}_G^{adm}(K) \longrightarrow \mathrm{Rep}_K^a(G_{L/\mathbb{Q}_p})$$

which is exact and faithful according to Thm. 4.6. Given $V \in \mathrm{Ban}_G^{adm}(K)$ note that, by exactness and since $D^c(G,K)$ is noetherian, the coadmissible module associated to $F(V)$ is even finitely presented. We deduce the second main result.

Theorem 4.7 *The functor F is exact and faithful. Given $V \in \mathrm{Ban}_G^{adm}(K)$ the representation $F(V)$ is strongly admissible. Viewed as a $G_{\sigma^{-1}}$-representation, $\sigma \in Gal(L/\mathbb{Q}_p)$, it contains $V_{\sigma-an}$ as a closed subrepresentation and functorial in V. In case $L = \mathbb{Q}_p$ the functor coincides with $F_{\mathbb{Q}_p}$.*

Proof: It remains to see the latter statements. The projection $pr_\sigma : \prod_\sigma G_\sigma \to G_\sigma$ induces a continuous inclusion $pr_\sigma^* : C^{an}(G_\sigma, K) \to C^{an}(G_{L/\mathbb{Q}_p}, K)$. By definition of the locally convex topologies on both sides it is a compact locally convex inductive limit of isometries and so, according to [E1], Prop. 1.1.41, a topological embedding with closed image. Dualizing we obtain a continuous algebra surjection $(pr_\sigma)_* : D(G_{L/\mathbb{Q}_p}, K) \to D(G_\sigma, K)$ exhibiting $D(G_\sigma, K)$ as coadmissible $D(G_{L/\mathbb{Q}_p}, K)$-module. It follows from [ST5], Lem. 3.8 and its proof that $V_b' \otimes_{D^c(G,K)} D(G_\sigma) \in \mathcal{C}_{G_\sigma}$ lies in $\mathcal{C}_{G_{L/\mathbb{Q}_p}}$ and that the two canonical topologies coincide. The $D(G_{L/\mathbb{Q}_p}, K)$-linear surjection

$$\mathrm{id} \otimes (pr_\sigma)_* : V_b' \otimes_{D^c(G,K)} D(G_{L/\mathbb{Q}_p}, K) \longrightarrow V_b' \otimes_{D^c(G,K)} D(G_\sigma, K)$$

then lies in $\mathcal{C}_{G_{L/\mathbb{Q}_p}}$ and is therefore continuous and strict. It is also $D(G_\sigma, K)$-linear and the right-hand side equals $(V_{\sigma^{-1}-an})_b'$ according to the σ^{-1}-analytic version of Prop. 3.4 (cf. remarks above). Passing to strong duals yields a closed G_σ-equivariant topological embedding $V_{\sigma^{-1}-an} \to F(V)$. It is natural in V and clearly onto in case $L = \mathbb{Q}_p$. $\square$

Remark: Let $H \subseteq G$ be a compact subgroup and denote the versions of the functor F relative to H resp. G by F_H resp. F_G. The natural map

$$D^c(G,K) \otimes_{D^c(H,K)} D(H_{L/\mathbb{Q}_p}, K) \to D(G_{L/\mathbb{Q}_p}, K)$$

being an isomorphism of bimodules is equivalent to $L = \mathbb{Q}_p$ (cf. (6)). It follows that in case $L \neq \mathbb{Q}_p$ the functors F_H and F_G do not commute with the restriction functors induced by $H \subseteq G$ resp. $H_{L/\mathbb{Q}_p} \subseteq G_{L/\mathbb{Q}_p}$. Therefore there is no naive generalization of the functor F to noncompact groups in the case $L \neq \mathbb{Q}_p$ and we leave this matter as an open problem.

5 Standard resolutions

Turning back to a general extension $L/\mathbb{Q}_p$ (not necessarily Galois) the functors $F_{\mathbb{Q}_p}^L$ and F_L defined previously remain interesting in themselves. We begin their study with some general analysis of certain base extension functors between coadmissible modules.

Let G be a locally L-analytic group. For the rest of this section we *fix* an ideal $\mathfrak{h}$ of $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0$ stable under $L \otimes \mathrm{Ad}(g)$ for all $g \in G$ where Ad refers to the adjoint action of G. Denote by $(\mathfrak{h})$ the two-sided ideal generated by $\mathfrak{h}$ in $L \otimes_{\mathbb{Q}_p} U(\mathfrak{g}_0)$ as well as in $D(G_0, K)$. Put $D := D(G_0, K)/(\mathfrak{h})$, $\mathcal{C}_D := \mathcal{C}_{G_0} \cap \mathcal{M}(D)$. Then $\mathcal{C}_D$ (with D-linear maps) is an abelian category. If G is compact then since $(\mathfrak{h})$ is closed D is a Fréchet-Stein algebra and if $(\mathfrak{h})_r$ denotes the closure $(\mathfrak{h}) \subseteq D_r(G_0, K)$, the coherent sheaf associated to D equals $D_r := D_r(G_0, K)/(\mathfrak{h})_r$

([ST5], Prop. 3.7). If we base extend the standard complex $U(\mathfrak{h}) \otimes_L \dot{\bigwedge} \mathfrak{h}$ via $U(\mathfrak{h}) \subseteq D(G_0, K)$ then the complex

$$D(G_0, K) \otimes_L \bigwedge^{l} \mathfrak{h} \to ... \to D(G_0, K) \otimes_L \bigwedge^{0} \mathfrak{h} \to D, \tag{7}$$

$l := \dim_L \mathfrak{h}$, consists of finite free left $D(G_0, K)$-modules.

Lemma 5.1 *The complex (7) is a free resolution of the left $D(G_0, K)$-module D by $D(G_0, K)$-bimodules.*

Proof: Let us prove that $D(G_0, K) \otimes_L \dot{\bigwedge} \mathfrak{h}$ is acyclic. Choosing $H \subseteq G$ compact open and using $\mathfrak{h}$-invariant decompositions

$$D(G_0, K) = \oplus_{g \in G/H} g D(H_0, K), \quad D = \oplus_{g \in G/H} g(D(H_0, K)/(\mathfrak{h}))$$

we are reduced to G compact. The complex (7) consists then of coadmissible left $D(G_0, K)$-modules whence acyclicity may be tested on coherent sheafs. It thus suffices to see that $D_r(G_0, K) \otimes_L \dot{\bigwedge} \mathfrak{h}$ is exact for a fixed radius r. The maps $U(\mathfrak{g}_0, K) \to U_r(\mathfrak{g}_0, K) \to D_r(G_0, K)$ are flat, the first by [ST5], remark 3.2 and the second by Thm. 2.3. We are thus reduced to show that base extending the standard complex via $U(\mathfrak{h}) \subseteq U(\mathfrak{h}, K) \subseteq U(L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0, K) = U(\mathfrak{g}_0, K)$ is an exact operation. By Lem. 2.8 this holds for the extension $U(\mathfrak{h}) \subseteq U(\mathfrak{h}, K)$ and $U(\mathfrak{h}, K) \subseteq U(L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0, K) = U(\mathfrak{g}_0, K)$ is clearly a free ring extension. This proves acyclicity. Using that $\mathfrak{h}$ is Ad-stable we may endow the complex (7) with a right $D(G_0, K)$-module structure as follows. The adjoint action of G_0 on $\mathfrak{h}$ is locally analytic and extends functorially to a continuous right action on $\bigwedge^q \mathfrak{h}$ given explicitly via $(\mathfrak{x}_1 \wedge ... \wedge \mathfrak{x}_q)g = \mathrm{Ad}(g^{-1})\mathfrak{x}_1 \wedge ... \wedge \mathrm{Ad}(g^{-1})\mathfrak{x}_q$. Letting G_0 act on $D(G_0, K)$ by right multiplication we give $D(G_0, K) \otimes \bigwedge^q \mathfrak{h}$ the right diagonal G_0-action which extends to a separately continuous right $D(G_0, K)$-module structure (cf. [ST6], Appendix). The identity $g\mathfrak{x}g^{-1} = \mathrm{Ad}(g)\mathfrak{x}$ in $D(G_0, K)$ implies that the differential ∂ of (7) respects the diagonal right G_0-action. Since $K[G_0] \subseteq D(G_0, K)$ is dense ([ST2], Lem. 3.1) ∂ respects the right $D(G_0, K)$-module structure by continuity. □

Proposition 5.2 *Given $X \in \mathcal{C}_{G_0}$ one has $\mathrm{Tor}_*^{D(G_0,K)}(X, D) \in \mathcal{C}_D$. Furthermore, $\mathrm{Tor}_*^{D(G_0,K)}(X, D) \simeq H_*(\mathfrak{h}, X)$ in Vec_K natural in X which is topological with respect to the canonical topology on the left hand side. In particular, $\mathrm{Tor}_*^{D(G_0,K)}(., D) = 0$ in degrees $* > \dim_L \mathfrak{h}$.*

Proof: Let $X \in \mathcal{C}_{G_0}$ be given. By the above lemma $\mathrm{Tor}_*^{D(G_0,K)}(X, D) \simeq h_*(X \otimes_L \dot{\bigwedge} \mathfrak{h})$ in Vec_K. By the usual argument with double complexes, the right module structure on $X \otimes_L \dot{\bigwedge} \mathfrak{h}$ makes the isomorphism right $D(G_0, K)$-equivariant. Now X is coadmissible and $\bigwedge^q \mathfrak{h}$ is finite dimensional. Hence dualising [E1], Prop. 6.1.5 yields that each right module $X \otimes_L \bigwedge^q \mathfrak{h}$ is coadmissible and its canonical topology coincides with the tensor product topology. Since $\mathcal{C}_{G_0}$ is abelian we obtain $\mathrm{Tor}_*^{D(G_0,K)}(X, D) \in \mathcal{C}_{G_0} \cap \mathcal{M}(D) = \mathcal{C}_D$. The remaining statements are now clear. □

In the compact case the associated coherent sheafs are easily computed.

Corollary 5.3 *Let G be compact. Given $X \in \mathcal{C}_{G_0}$ then* $\mathrm{Tor}_*^{D(G_0,K)}(X,D)_r = \mathrm{Tor}_*^{D_r(G_0,K)}(X_r,D_r)$. *Also,* $\mathrm{Tor}_*^{D_r(G_0,K)}(X_r,D_r) \simeq H_*(\mathfrak{h},X_r)$ *in* Vec_K.

Proof: Let $P. \to X$ be a projective resolution in $\mathcal{M}(D(G_0,K))$. By flatness of $D(G_0,K) \to D_r(G_0,K)$ the complex $P. \otimes_{D(G_0,K)} D_r(G_0,K) \to X_r$ is a projective resolution of the $D_r(G_0,K)$-module X_r. Since $D_r \simeq D \otimes_{D(G_0,K)} D_r(G_0,K))$ as $(D(G_0,K),D_r(G_0,K))$-bimodules ([ST5], Cor. 3.1) we have as right $D_r(G_0,K)$-modules

$$\begin{aligned}\mathrm{Tor}_*^{D_r(G_0,K)}(X_r,D_r) &\simeq h_*(P. \otimes_{D(G_0,K)} D_r)\\ &\simeq h_*(P. \otimes_{D(G_0,K)} D) \otimes_{D(G_0,K)} D_r(G_0,K)\\ &\simeq \mathrm{Tor}_*^{D(G_0,K)}(X,D) \otimes_{D(G_0,K)} D_r(G_0,K).\end{aligned}$$

□

Since $\mathcal{C}_{G_0} \subseteq \mathcal{M}(D(G_0,K))$ is a full embedding (and similarly for $\mathcal{C}_D$) we have

Corollary 5.4 *The functors* $\mathrm{Tor}_*^{D(G_0,K)}(.,D)$ *form a homological δ-functor between $\mathcal{C}_{G_0}$ and $\mathcal{C}_D$.*

Proposition 5.5 *There is a commutative (up to natural isomorphism) diagram of functors*

$$\begin{array}{ccc}\mathrm{Rep}_K^a(G_0) & \xrightarrow{V\mapsto V^{\mathfrak{h}}} & \mathrm{Rep}_K^a(G_0)\\ {\scriptstyle V\mapsto V_b'}\downarrow & & \downarrow{\scriptstyle V\mapsto V_b'}\\ \mathcal{C}_{G_0} & \xrightarrow{M\mapsto M\otimes_{D(G_0,K)}D} & \mathcal{C}_{G_0}\end{array}$$

Proof: Suppose first that G is compact. By continuity of the Lie action and since $\mathfrak{h}$ is Ad-stable $V \mapsto V^{\mathfrak{h}}$ is an auto-functor of $\mathrm{Rep}_K^a(G_0)$. The complex $V_b' \otimes_L \dot{\bigwedge}\mathfrak{h}$ consists of coadmissible right modules whence the differential is strict. Hence Lem. 2.7 for $* = 0$ implies that restriction of functionals yields a G_0-isomorphism $V_b'/V_b'\mathfrak{h} \simeq (V^{\mathfrak{h}})_b'$ of topological vector spaces, functorial in V. By local analyticity this extends to an isomorphism of right $D(G_0,K)$-modules $V_b' \otimes_{D(G_0,K)} D \simeq (V^{\mathfrak{h}})_b'$ natural in V. This settles the compact case. If G is arbitrary the result follows easily from the compact case by choosing a compact open subgroup. □

6 Higher analytic vectors

We are mainly interested in the choice $\mathfrak{h} := \mathfrak{g}^0$ where, as before, $\mathfrak{g}^0 = \ker(L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0 \to \mathfrak{g})$. Hence $D = D(G_0,K)/(\mathfrak{g}^0) \simeq D(G,K)$ and $\mathcal{C}_D = \mathcal{C}_{G_0} \cap \mathcal{M}(D(G,K)) = \mathcal{C}_G$.

Theorem 6.1 *Passage to analytic vectors $F_{\mathbb{Q}_p}^L$ extends to a cohomological δ-functor $(R^iF_{\mathbb{Q}_p}^L)_{i\geq 0}$ with $R^iF_{\mathbb{Q}_p}^L = 0$ for $i > ([L:\mathbb{Q}_p]-1)\dim_L G$.*

Proof: We may clearly replace the right upper resp. lower corner in Prop. 5.5 by $\mathrm{Rep}_K^a(G)$ resp. $\mathcal{C}_G$ without changing the statement. Both vertical arrows are anti-equivalences between abelian categories and therefore exact functors. By direct calculation pulling back the functors $\mathrm{Tor}_*^{D(G_0,K)}(.,D(G,K)):\mathcal{C}_{G_0}\rightarrow\mathcal{C}_G$ (cf. Cor. 5.4) yields a cohomological δ-functor extending $F_{\mathbb{Q}_p}^L$. Finally, $\dim_L\mathfrak{g}^0=([L:\mathbb{Q}_p]-1)\dim_L G$. $\Box$

The functors $R^iF_{\mathbb{Q}_p}^L$ can be expressed without referring to coadmissible modules. Endowing $V\in\mathrm{Rep}_K^a(G_0)$ with the uniquely determined separately continuous left $D(G_0,K)$-module structure we may consider the G-representation (*)

$$\mathrm{Ext}^i_{D(G_0,K)}(D(G,K),V)$$

where the left G-action comes from right multiplication on $D(G,K)$.

Corollary 6.2 *The G-representation $(*)$ lies in $\mathrm{Rep}_K^a(G)$ and there is a natural isomorphism*

$$R^iF_{\mathbb{Q}_p}^L(V)\simeq\mathrm{Ext}^i_{D(G_0,K)}(D(G,K),V)$$

of admissible G-representations.

Proof: Taking cohomology on the complex $\mathrm{Hom}_{D(G_0,K)}(D(G_0,K)\otimes_L\dot{\bigwedge}\mathfrak{g}^0,V)=\mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}^0,V)$ yields an isomorphism $\mathrm{Ext}^*_{D(G_0,K)}(D(G,K),V)\simeq H^*(\mathfrak{g}^0,V)$ in Vec_K natural in V. We endow the Ext group with the locally convex topology of compact type of the right-hand side. Using Lem. 2.7 as well as Prop. 5.2 we obtain a natural isomorphism in Vec_K

$$\mathrm{Tor}_*^{D(G_0,K)}(V_b',D(G,K))_b'\simeq\mathrm{Ext}^*_{D(G_0,K)}(D(G,K),V)$$

which is topological. To check that it is G-equivariant amounts to check the G-equivariance of the isomorphisms of complexes $(h_*(V'\otimes_L\dot{\bigwedge}\mathfrak{g}^0))'\simeq h^*((V'\otimes_L\dot{\bigwedge}\mathfrak{g}^0)')$ appearing in the proof of Lem. 2.7 and $(V'\otimes_L\dot{\bigwedge}\mathfrak{g}^0)'\simeq\mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}^0,V)$ (remark before [loc.cit.]). These are direct computations. $\Box$

Let G be compact and $K/\mathbb{Q}_p$ be finite. Using the notation of section 3 recall that $F_{\mathbb{Q}_p}$ is exact and preserves injective objects (Thm. 3.1). Hence we have the following direct application of our results to Banach space representations.

Proposition 6.3 *Let G be compact and $K/\mathbb{Q}_p$ be finite. Then*

$$R^iF_L=R^iF_{\mathbb{Q}_p}^L\circ F_{\mathbb{Q}_p}\text{ and }R^iF_L=0,\ i>([L:\mathbb{Q}_p]-1)\dim_L G$$

for the right-derived functors of F_L.

We conclude with two further applications when varying $\mathfrak{h}$.

1. σ-analytic representations. Assume $L/\mathbb{Q}_p$ is Galois. Letting $\mathfrak{h}:=\mathfrak{g}_\sigma^0$ in Cor. 5.4 and pulling back to representations yields: for each $\sigma\in Gal(L/\mathbb{Q}_p)$ the left-exact functor $\mathrm{Rep}_K^a(G_0)\rightarrow\mathrm{Rep}_K^a(G_{\sigma^{-1}})$, $V\mapsto V_{\sigma-an}$ extends to a δ-functor. The higher functors vanish in degrees $>([L:\mathbb{Q}_p]-1)\dim_L G$.

2. Smooth representations. Let $\mathrm{Rep}_K^{\infty,a}(G)\subseteq\mathrm{Rep}_K^a(G_0)$ denote the full abelian subcategory of smooth-admissible representations ([ST5], §6). The

equivalence $\mathrm{Rep}^a_K(G_0) \simeq \mathcal{C}_{G_0}$ induces an equivalence $\mathrm{Rep}^{\infty,a}_K(G) \simeq \mathcal{C}_\infty$ where $\mathcal{C}_\infty = \mathcal{C}_{G_0} \cap \mathcal{M}(D^\infty(G,K))$ and $D^\infty(G,K) = D(G_0,K)/(\mathfrak{g}_0)$ denotes the algebra of smooth distributions ([ST6], §1). Since $V^\infty = H^0(\mathfrak{g}_0, V)$ we may put $\mathfrak{h} := L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0$, apply the results of the preceding section and obtain: passage to smooth vectors $\mathrm{Rep}^a_K(G_0) \to \mathrm{Rep}^{\infty,a}_K(G)$, $V \mapsto V^\infty$ extends to a δ-functor vanishing in degrees $> [L:\mathbb{Q}_p]\,\mathrm{dim}_L G$.

7 Analytic vectors in induced representations

As an application we study the interaction of the functors $R^i F^L_{\mathbb{Q}_p}$ with locally analytic induction. This implies an explicit formula for the higher analytic vectors in principal series representations. As usual G denotes a locally L-analytic group.

We let $P \subseteq G$ be a closed subgroup with Lie algebra $\mathfrak{p}$. Let Ind^G_P denote the locally analytic induction viewed as a functor from admissible P-representations (W,ρ), finite dimensional over K, to admissible G-representations. Explicitly,

$$\mathrm{Ind}^G_P(W) := \{f \in C^{an}(G,W),\ f(gb) = \rho(b)^{-1} f(g) \text{ for all } g \in G, b \in P\}$$

and G acts by left translations. One has the isomorphism of right $D(G,K)$-modules

$$W'_b \otimes_{D(P,K)} D(G,K) \xrightarrow{\sim} (\mathrm{Ind}^G_P W)'_b \tag{8}$$

mapping $\phi \otimes \lambda$ to the functional $f \mapsto \lambda(g \mapsto \phi(f(g)))$ ([OS], 2.4).

Recall that $\mathfrak{g}^0 := \ker\,(L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0 \to \mathfrak{g})$. We **assume** for the rest of this section that there is a compact open subgroup $G' \subseteq G$ such that an "Iwasawa decomposition"

$$G = G'P \tag{9}$$

holds.

Lemma 7.1 *Given $X \in \mathcal{M}(D(G_0,K))$ there is a natural isomorphism*

$$\mathrm{Tor}^{D(G'_0,K)}_*(X, D(G',K)) \simeq \mathrm{Tor}^{D(G_0,K)}_*(X, D(G,K))$$

as right $D(G',K)$-modules.

Proof: Let $* = 0$. For $X = D(G_0,K)$ the claim follows from the fact that $G' \subseteq G$ is open whence $\mathfrak{g}^0 \subseteq D(G'_0,K)$ with $D(G',K) = D(G'_0,K)/(\mathfrak{g}^0)$. The case of arbitrary X follows from this. In general a projective resolution $P. \to X$ of X as $D(G_0,K)$-module remains a projective resolution of X as $D(G'_0,K)$-module since $D(G_0,K)$ is free over $D(G'_0,K)$. The claim then follows from the case $* = 0$. □

Now put $P' := P \cap G'$. Using (9) restriction of functions induces a topological G'-isomorphism

$$\mathrm{Ind}^G_P W \xrightarrow{\sim} \mathrm{Ind}^{G'}_{P'} W \tag{10}$$

where the right-hand side has the obvious meaning ([Fea], 4.1.4).

Lemma 7.2 (i) *For all $Y \in \mathcal{M}(D(P,K))$ the natural map*

$$Y \otimes_{D(P',K)} D(G',K) \to Y \otimes_{D(P,K)} D(G,K)$$

is an isomorphism of $D(G',K)$-modules.

(ii) *We have a commutative diagram*

$$\begin{array}{ccc} W' \otimes_{D(P',K)} D(G',K) & \longrightarrow & (\mathrm{Ind}_{P'}^{G'} W)' \\ \downarrow & & \downarrow \\ W' \otimes_{D(P,K)} D(G,K) & \longrightarrow & (\mathrm{Ind}_{P}^{G} W)' \end{array}$$

of right $D(G',K)$-modules in which all four maps are isomorphisms.

Proof: This is a straightforward generalization of [ST6], Lem. 6.1 using (9). □

Recall from last section that $F^L_{\mathbb{Q}_p}$ extends to a δ-functor $(R^i F^L_{\mathbb{Q}_p})_{i\geq 0}$ (Thm. 6.1). Assume that we are given a finite dimensional locally $\mathbb{Q}_p$-analytic P-representation W. We abbreviate in the following

$$V := \mathrm{Ind}_{P_0}^{G_0} W \in \mathrm{Rep}_K^a(G_0)$$

and study the admissible G-representations $R^i F^L_{\mathbb{Q}_p}(V)$. Let $Q. = D(P_0,K)\otimes_L \dot{\bigwedge}\mathfrak{p}^0$ resp. $\tilde{Q}. = D(G_0,K)\otimes_L \dot{\bigwedge}\mathfrak{g}^0$ denote the standard resolutions for the bimodules $D(P,K)$ resp. $D(G,K)$ as referred to in Lem. 5.1. We have the natural morphism of complexes of $D(P_0,K)$-bimodules $Q. \to \tilde{Q}.$ induced by $D(P_0,K) \to D(G_0,K)$ and $\mathfrak{p}^0 \to \mathfrak{g}^0$. Tensoring with the right $D(P_0,K)$-equivariant map $W' \to V', w \mapsto w\otimes 1$ arising from (8) gives rise to a morphism of complexes of right $D(P_0,K)$-modules

$$W' \otimes_{D(P_0,K)} Q. \to V' \otimes_{D(G_0,K)} \tilde{Q}.$$

Taking homology and extending scalars yields a map

$$f : \mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P)} D(G) \longrightarrow \mathrm{Tor}_*^{D(G_0)}(V', D(G))$$

of right $D(G)$-modules where we have abbreviated $D(G) := D(G,K), D(P) := D(P,K)$ etc.

Proposition 7.3 *The map f is an isomorphism.*

Proof: First note that the right-hand side is a priori coadmissible by Prop. 5.2. We now have bijective maps of right $D(G')$-modules

$$\mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P)} D(G) \xrightarrow{\sim} \mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P')} D(G')$$

(Lem. 7.2 (i) applied to $Y := \mathrm{Tor}_*^{D(P_0)}(W', D(P))$) and

$$\mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P')} D(G') \xrightarrow{\sim} \mathrm{Tor}_*^{D(P_0')}(W', D(P')) \otimes_{D(P')} D(G')$$

(Lem. 7.1 applied to $P' \subseteq P$ and $W' \in \mathcal{M}(D(P_0,K))$. Their composite fits into the diagram of right $D(G')$-modules

$$\begin{array}{ccc} \mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P)} D(G) & \xrightarrow{f} & \mathrm{Tor}_*^{D(G_0)}(V', D(G)) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_*^{D(P_0')}(W', D(P')) \otimes_{D(P')} D(G') & \longrightarrow & \mathrm{Tor}_*^{D(G_0')}(V', D(G')) \end{array}$$

where the right hand vertical arrow is due to Lem. 7.1 and bijective. The lower horizontal arrow is defined analogously to the upper one using Lem. 7.2 (ii). Tracing through the definitions of the maps involved this diagram commutes. We may thus assume that G is compact. Then both sides of our map are coadmissible: $\mathrm{Tor}_*^{D(P_0)}(W', D(P))$ is finite dimensional over K (Prop. 5.2) hence is a finitely presented $D(P)$-module. We introduce another map of right $D(G)$-modules

$$f' : \mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P)} D(G) \longrightarrow \mathrm{Tor}_*^{D(G_0)}(V', D(G)) \qquad (11)$$

as follows. Choose a projective resolution $P. \to W'$ by right $D(P_0)$-modules according to Lem. 7.4 below. Then $\tilde{P}. := P. \otimes_{D(P_0)} D(G_0)$ is a projective resolution for $W' \otimes_{D(P_0)} D(G_0) = V'$ whence the natural map

$$P. \otimes_{D(P_0)} D(P) \to \tilde{P}. \otimes_{D(G_0)} D(G)$$

induced by $m \otimes \lambda \mapsto m \otimes 1 \otimes \lambda$ for $m \in P_n,\ m \otimes 1 \in \tilde{P}_n,\ \lambda \in D(P) \subseteq D(G)$ gives our map f'. We claim that it is bijective: by coadmissibility this may be tested on coherent sheafs. Let us realize $D(G_0)$ and $D(G)$ as Fréchet-Stein algebras via the families of norms appearing in Prop. 2.6. In particular $D_r(P) \to D_r(G)$ is flat for all r. Denote by W'_r resp. $V'_r = W'_r \otimes_{D_r(P_0)} D_r(G_0)$ the coherent sheafs associated to W' resp. V'. Then the coherent sheafs associated to both sides of (11) are given by $\mathrm{Tor}_*^{D_r(P_0)}(W'_r, D_r(P)) \otimes_{D_r(P)} D_r(G)$ resp. $\mathrm{Tor}_*^{D_r(G_0)}(V'_r, D_r(G))$ according to Cor. 5.3. Put $P_r. := P. \otimes_{D(P_0)} D_r(P_0)$ resp. $\tilde{P}_r. := \tilde{P}. \otimes_{D(G_0)} D_r(G_0)$. By [ST5], remark 3.2 $P_r. \to W'_r$ resp. $\tilde{P}_r. \to V'_r$ are projective resolutions of W'_r resp. V'_r and the map

$$f' \otimes_{D(G)} D_r(G) : \mathrm{Tor}_*^{D_r(P_0)}(W'_r, D_r(P)) \otimes_{D_r(P)} D_r(G) \to \mathrm{Tor}_*^{D_r(G_0)}(V'_r, D_r(G))$$

coincides with the one induced by

$$P_r. \otimes_{D_r(P_0)} D_r(P) \to \tilde{P}_r. \otimes_{D_r(G_0)} D_r(G).$$

Since $D_r(P) \to D_r(G)$ is flat $f' \otimes_{D(G)} D_r(G)$ is bijective and since this holds for all r the map f' is bijective. Using a standard double complex argument now shows that $f = f'$ whence the proposition. $\square$

The following lemma was used in the preceding proof.

Lemma 7.4 *Assume G is compact. There is a projective resolution $P. \to W'$ by right $D(P_0)$-modules such $P. \otimes_{D(P_0)} D(G_0)$ is acyclic.*

Proof: By density of analytic vectors (Thm. 3.1) W' being finite dimensional over K implies that $W' \otimes_{D^c(P_0)} D(P_0) = W'$. Now choose a finite free resolution $P'.$ of the $D^c(P_0)$-module W'. By the flatness result (5) $P. := P'. \otimes_{D^c(P_0)} D(P_0)$ is a finite free resolution of the $D(P_0)$-module W' and it remains to see the last statement. Now every kernel $K_n \subseteq P_n$ of the differential in $P.$ is a finitely presented $D(P_0)$-module whence the morphism $K_n \otimes_{D(P_0)} D(G_0) \to P_n \otimes_{D(P_0)} D(G_0)$ lies in $\mathcal{C}_{G_0}$. Its injectivity follows therefore on coherent sheafs from flatness of $D_r(P_0) \to D_r(G_0)$ and left-exactness of the projective limit using the Fréchet-Stein structure of Prop. 2.6. $\square$

Theorem 7.5 *The functors $R^i F^L_{\mathbb{Q}_p}$ commute with induction: given a finite dimensional locally $\mathbb{Q}_p$-analytic P-representation W one has an isomorphism*

$$R^i F^L_{\mathbb{Q}_p} \circ \mathrm{Ind}^{G_0}_{P_0}(W) \simeq \mathrm{Ind}^G_P \circ R^i F^L_{\mathbb{Q}_p}(W)$$

as admissible G-representations functorial in W.

Proof: This follows from dualising the isomorphism in the preceding proposition which is, by construction, functorial in W. □

The functor Ind^G_P is nonzero on objects ([Fea], Satz. 4.3.1) whence

Corollary 7.6 *We have $R^i F^L_{\mathbb{Q}_p}(\mathrm{Ind}^{G_0}_{P_0} W) \neq 0$ if and only if $R^i F^L_{\mathbb{Q}_p}(W) \neq 0$. In particular $R^i F^L_{\mathbb{Q}_p}(\mathrm{Ind}^{G_0}_{P_0} W) = 0$ for all $i > ([L : \mathbb{Q}_p] - 1)\dim_L \mathfrak{p}$.*

The above results apply in particular when G equals the L-points of a connected reductive group over L and $P \subseteq G$ is a parabolic subgroup. If W is a one dimensional P-representation K_χ given by a locally $\mathbb{Q}_p$-analytic character $\chi : P \to K^\times$ we may determine the vector space $R^i F^L_{\mathbb{Q}_p}(K_\chi)$ completely. For simplicity we assume that G is quasi-split and let $P := B$ be a Borel subgroup with Lie algebra $\mathfrak{b}$. Then $\mathfrak{b} = \mathfrak{t}\mathfrak{u}$ (semidirect product) where $\mathfrak{t}$ is a maximal toral subalgebra and $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$. Let $\mathfrak{b}^0$ be the kernel of $K \otimes_{\mathbb{Q}_p} \mathfrak{b}_0 \to K \otimes_L \mathfrak{b}$ and define $\mathfrak{t}^0$ and $\mathfrak{u}^0$ analogously. Then K_χ is a $\mathfrak{b}^0$-module via $K \otimes_{\mathbb{Q}_p} \mathrm{d}\chi$ where $\mathrm{d}\chi : \mathfrak{b}_0 \to K$ denotes the differential of χ. We denote this module as well as the induced (note that $[\mathfrak{b}^0, \mathfrak{b}^0] = \mathfrak{u}^0$) $\mathfrak{t}^0$-module by $K_{\mathrm{d}\chi}$.

Corollary 7.7 *There is an isomorphism in Vec_K*

$$R^i F^L_{\mathbb{Q}_p}(K_\chi) \simeq \sum_{j+k=i} \bigwedge^j \mathfrak{t}^0 \otimes_K H^k(\mathfrak{u}^0, K_{\mathrm{d}\chi})^{\mathfrak{t}^0}.$$

Proof: By Prop. 5.2 we have that $R^i F^L_{\mathbb{Q}_p}(K_\chi) = H^i(\mathfrak{b}^0, K_{\mathrm{d}\chi})$ in Vec_K. The algebras $\mathfrak{b}^0$ resp. $\mathfrak{t}^0$ resp. $\mathfrak{u}^0$ are direct products of scalar extensions of $\mathfrak{b}$ resp. $\mathfrak{t}$ resp. $\mathfrak{u}$. In particular, $\mathfrak{t}^0 \subseteq \mathfrak{b}^0$ is a reductive subalgebra whence [HS], Thm. 12 implies that there is an isomorphism

$$H^i(\mathfrak{b}^0, K_{\mathrm{d}\chi}) \simeq \sum_{j+k=i} H^j(\mathfrak{t}^0, K) \otimes_K H^k(\mathfrak{b}^0, \mathfrak{t}^0, K_{\mathrm{d}\chi})$$

in Vec_K where $H^*(\mathfrak{b}^0, \mathfrak{t}^0, K_{\mathrm{d}\chi})$ is the relative Lie algebra cohomology with respect to $\mathfrak{t}^0$. Now $\mathfrak{t}^0$ is even toral whence the argument preceding [loc.cit.], Thm. 13 implies that $H^*(\mathfrak{b}^0, \mathfrak{t}^0, K_{\mathrm{d}\chi}) \simeq H^*(\mathfrak{u}^0, K_{\mathrm{d}\chi})^{\mathfrak{t}^0}$. Finally, since $\mathfrak{t}^0$ is abelian, the differential in $\mathrm{Hom}_K(\dot{\bigwedge}\mathfrak{t}^0, K)$ vanishes identically and so one obtains $H^j(\mathfrak{t}^0, K) = \mathrm{Hom}_K(\bigwedge^j \mathfrak{t}^0, K)$. □

Remark: One has $H^k(\mathfrak{u}^0, K_{\mathrm{d}\chi}) = H^k(\mathfrak{u}^0, K)$ in Vec_K and since $\mathfrak{u}^0$ is nilpotent [D], Thm. 2 implies that $\dim_K H^k(\mathfrak{u}^0, K) \geq 1$ for $0 \leq k \leq \dim_K \mathfrak{u}^0$.

Corollary 7.8 *If χ is smooth one has $R^1 F^L_{\mathbb{Q}_p}(\mathrm{Ind}^{G_0}_{P_0} K_\chi) \neq 0$.*

Proof: We have $\mathrm{d}\chi = 0$ whence $H^0(\mathfrak{u}^0, K_{\mathrm{d}\chi})^{\mathfrak{t}^0} = K$. By the corollary there is an injection $\bigwedge^1 \mathfrak{t}^0 \to R^1 F^L_{\mathbb{Q}_p}(K_\chi)$ whence the claim follows from Cor. 7.6. □

References

[B-TVS] Bourbaki, N.: Topological Vector Spaces. Chap. 1-5. Berlin Heidelberg: Springer 2003

[B-VAR] Bourbaki, N.: Variétés différentielles et analytiques. Fascicule de résultats. Paris: Hermann 1967

[BW] Borel, A., Wallach, N.: Continuous cohomology, Discrete subgroups, and Representations of Reductive groups. Princeton Univ. Press 1980

[D] Dixmier, J.: Cohomologie des algèbres de Lie nilpotent. Acta. Sci. Math. 16 (1995)

[DDMS] Dixon, J.D., du Sautoy, M.P.F., Mann, A., Segal, D.: Analytic Pro-p Groups. Cambridge Univ. Press 1999

[E1] Emerton, M.: Locally analytic vectors in representations of non-archimedean locally p-adic analytic groups. To appear in: Memoirs of the AMS

[E2] Emerton, M.: Locally analytic representation theory of p-adic reductive groups: A summary of some recent developments. To appear in: Proc. LMS Symp. Durham 2004

[F] Frommer, H.: The locally analytic principal series of split reductive groups. Münster: SFB-preprint 265 (2003)

[Fea] Féaux de Lacroix, C.T.: Einige Resultate über die topologischen Darstellungen p-adischer Liegruppen auf unendlich dimensionalen Vektorräumen über einem p-adischen Körper. Münster: Schriftenreihe Math. Inst. Univ. Münster, 3. Serie, Heft 23 (1999)

[HS] Hochschild, G., Serre, J.-P.: Cohomologie of Lie algebras. Ann. of Math. 57 (1953)

[Ko1] Kohlhaase, J.: Invariant distributions on p-adic analytic groups. Duke Math. J. 137 (2007)

[Ko2] Kohlhaase, J.: The cohomology of locally analytic representations. Münster: Preprint (2007)

[LVO] Li Huishi, van Oystaeyen, F.: Zariskian Filtrations. Dordrecht: Kluwer 1996

[NFA] Schneider, P.: Nonarchimedean Functional Analysis. Berlin Heidelberg New York: Springer 2002

[OS] Orlik, S., Strauch, M.: On the irreducibiliy of locally analytic principal series representations. Preprint (2006)

[P] Pirkovski, A.Y.: Stably flat completions of universal enveloping algebras. Diss. Math. 441 (2006)

[S] Schneider, P.: Continuous representation theory of p-adic Lie groups. Proc. ICM Madrid 2006, vol. II (2006)

[Sch] Schmidt, T.: Auslander Regularity of p-adic Distribution Algebras. Representation Theory 12 (2008)

[ST1] Schneider, P., Teitelbaum, J.: $U(\mathfrak{g})$-finite locally analytic representations. Representation Theory 5 (2001)

[ST2] Schneider, P., Teitelbaum, J.: Locally analytic distributions and p-adic representation theory, with applications to GL_2. J. Am. Math. Soc. 15 (2002)

[ST3] Schneider, P., Teitelbaum, J.: Banach space representations and Iwasawa theory. Israel J. Math. 127 (2002)

[ST4] Schneider, P., Teitelbaum, J.: p-adic Fourier theory. Documenta Math. 6 (2001)

[ST5] Schneider, P., Teitelbaum, J.: Algebras of p-adic distributions and admissible representations. Inv. Math. 153 (2003)

[ST6] Schneider, P., Teitelbaum, J.: Duality for admissible locally analytic representations. Representation Theory 9 (2005)

[T] Teitelbaum, J.: Admissible analytic representations. An introduction and three questions. Talk at Harvard Eigensemester 2006. Available at: http://www2.math.uic.edu/ jeremy/harvard.

Tobias Schmidt
Mathematisches Institut
Westfälische Wilhelms-Universität Münster
Einsteinstr. 62
D-48149 Münster, Germany
toschmid@math.uni-muenster.de