Degenerate double affine Hecke algebras and perverse sheaves over cyclically graded Lie algebras

Wille Liu, MPIM Bonn

Colloque Tournant 2021, Rennes

25 March 2021

Springer correspondence

- G : connected reductive algebraic group / $\mathbf{C}$
  $T \subset B \subset G$: max torus & Borel ; $U \subset B$: unip rad
  $W = N_G(T)/T$: Weyl group
- $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{b} = \mathrm{Lie}\, B$, $\mathfrak{u} = \mathrm{Lie}\, U$
  $\mathfrak{g}^{\mathrm{nil}} \subset \mathfrak{g}$: nilpotent cone
  $\dot{\mathfrak{g}} = G \times^B \mathfrak{u} \xrightarrow{\pi} \mathfrak{g}^{\mathrm{nil}}$, $\pi(g, x) = \mathrm{Ad}_g(x)$: Springer resolution

Theorem [Springer, Borho–MacPherson]

1. There is a bijection between
   (1) $\mathrm{Irrep}\, \mathbf{C}W$
   (2) $\{(z, \chi)\ ;\ z \in \mathfrak{g}^{\mathrm{nil}},\ \chi \in \mathrm{IrrLocSys}_G(Gz),\ \mathrm{H}^{\mathrm{top}}(\pi^{-1}(z))^{\chi} \neq 0\}/\sim.$
2. $\mathbf{I} = \pi_* \underline{\mathbf{C}}_{\dot{\mathfrak{g}}}[\dim \dot{\mathfrak{g}}]$ is a semisimple perverse sheaf and

$$\mathbf{I} = \bigoplus_{(z,\chi)/\sim} \mathrm{IC}(\chi) \otimes L_{z,\chi}, \quad (z, \chi) \leftrightarrow L_{z,\chi} \text{ via above bij}$$

Example : SL_2

- $G = \mathrm{SL}_2,\ B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix},\ T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$
- $\mathfrak{g} = \mathbf{C}e \oplus \mathbf{C}h \oplus \mathbf{C}f, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
- $\mathfrak{g}^{\mathrm{nil}} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \ ;\ a^2 + bc = 0 \right\} \cong A_1$-singularity
- Two G-orbits : $\mathfrak{g}^{\mathrm{nil}} = \mathrm{O}^0 \cup \mathrm{O}^{\mathrm{reg}}$
  $\mathrm{O}^0 = \{0\}, \quad \mathrm{O}^{\mathrm{reg}} = \{(2 \times 2)$-nilpotent matrices of rank $1\}$
- $\pi : \dot{\mathfrak{g}} \to \mathfrak{g}^{\mathrm{nil}}$: blow-up at 0, $\quad \dot{\mathfrak{g}} \cong \mathrm{T}^*\mathbf{P}^1$
- $W \cong \mathfrak{S}_2, \quad \mathrm{Irrep}(\mathbf{C}W) = \{V_{\mathrm{triv}}, V_{\mathrm{sgn}}\}$
- Springer correspondence :

$$\pi_* \underline{\mathbf{C}}_{\dot{\mathfrak{g}}}[2] \cong \left(\mathrm{IC}(\mathrm{O}^0) \otimes \chi_{\mathrm{sgn}}\right) \oplus \left(\mathrm{IC}(\mathrm{O}^{\mathrm{reg}}) \otimes \chi_{\mathrm{triv}}\right)$$

Affine Hecke algebras (AHAs)

- $P = \mathrm{Hom}(T, \mathbf{C}^\times) \circlearrowleft W$: weight lattice
- $R = R(G, T) \subset P$: root system ; $\Delta = \Delta(G, B, T) \subset R$: base
- $\{s_\alpha\}_{\alpha \in \Delta} \subset W$: simple reflections
- (extended) affine Hecke algebra $\mathbb{K}$ is the $\mathbf{C}[q^{\pm 1}]$-algebra
  - generated by the sets $\{X^\mu\}_{\mu \in P}$ and $\{T_\alpha\}_{\alpha \in \Delta}$
  - subject to the following relations for $\mu, \nu \in P$ and $\alpha \neq \beta \in \Delta$

$$\begin{aligned} X^0 &= 1, & X^\mu X^\nu &= X^{\mu+\nu} \\ (T_\alpha - q)(T_\alpha + 1) &= 0, & T_\alpha T_\beta T_\alpha \cdots &= T_\beta T_\alpha T_\beta \cdots \end{aligned}$$

$$T_\alpha X^\mu - X^{s_\alpha(\mu)} T_\alpha = (q-1) \frac{X^\mu - X^{s_\alpha(\mu)}}{1 - X^{-\alpha}}$$

About AHAs

- $\mathbb{K}$ is a q-deformation of $W \ltimes \mathbf{C}P = \mathbf{C}\widetilde{W_{R^\vee}}$ (=extended affine Weyl group of $R^\vee$).
- As vector space, there is a decomposition $\mathbb{K} = \mathcal{H}_{W,q} \otimes \mathbf{C}P$, where $\mathcal{H}_W$ is the Iwahori–Hecke algebra for W.
- We can decompose f.d. $\mathbb{K}$-modules according to the $\mathbf{C}[q^{\pm 1}] \otimes \mathbf{C}P$-spectrum.
- Centre : $Z(\mathbb{K}) = \mathbf{C}[q^{\pm 1}] \otimes (\mathbf{C}P)^W$.
- For $s = (\ell, q) \in T \times \mathbf{C}_q^\times$, completion : $\mathbb{K}_s^\wedge := \mathbb{K}^\wedge \otimes_{Z(\mathbb{K})} Z(\mathbb{K})_s^\wedge$
- f.d. $\mathbb{K}_s^\wedge$-modules $\leftrightarrow$ f.d. $\mathbb{K}$-modules on which $\mathbf{C}[q^{\pm 1}, X^\mu]_\mu$ acts with eigenvalues $(W \cdot \ell, q) \subset T \times \mathbf{C}_q^\times$.

K-theoretic construction of AHAs

- Let $\mathbf{C}_q^\times = \mathbf{C}^\times$ act on $\mathfrak{g}$ by weight -1 and trivially on G
  $\rightsquigarrow$ induced $\mathbf{C}_q^\times$-action on $\dot{\mathfrak{g}}$
  $\rightsquigarrow$ $\pi : \dot{\mathfrak{g}} \to \mathfrak{g}^{\mathrm{nil}}$ is $G \times \mathbf{C}_q^\times$-equivariant
- $\ddot{\mathfrak{g}} = \dot{\mathfrak{g}} \times_{\mathfrak{g}} \dot{\mathfrak{g}}$: Steinberg variety
- $\mathrm{K}^{G\times\mathbf{C}_q^\times}(\cdot)$: equivariant algebraic K-homology $\otimes\mathbf{C}$

Theorem [Ginzburg]

1. $\exists$ convolution product on $\mathrm{K}^{G\times\mathbf{C}_q^\times}(\ddot{\mathfrak{g}})$
2. $\mathbb{K} \cong \mathrm{K}^{G\times\mathbf{C}_q^\times}(\ddot{\mathfrak{g}})$
3. Let $s \in T \times \mathbf{C}_q^\times$. Then

$$\mathbb{K}_s^\wedge \cong \mathrm{K}^{G^s\times\mathbf{C}_q^\times}(\ddot{\mathfrak{g}}^s)^\wedge$$

Deligne–Langlands correspondence

Let $s = (\ell, q) \in T \times \mathbf{C}_q^\times$.
Let $\pi_s : \dot{\mathfrak{g}}^s \to (\mathfrak{g}^{\mathrm{nil}})^s$ be the restriction of $\pi : \dot{\mathfrak{g}} \to \mathfrak{g}^{\mathrm{nil}}$.

Theorem [Kazhdan–Lusztig]

Suppose q is not $\sqrt{1}$. There is a bijection between

(1) $\{$f.d. simple $\mathbb{K}_s^\wedge$-modules$\}/\sim$

(2) $\{(z, \chi)\ ;\ z \in (\mathfrak{g}^{\mathrm{nil}})^s,\ \chi \in \mathrm{IrrLocSys}_{G^s}(G^s z),\ \mathrm{H}^*(\pi_s^{-1}(z))^\chi \neq 0\}/\sim.$

However, when q is $\sqrt{1}$, we only have $(1) \hookrightarrow (2)$.

Q : In this case, remplacement for (1) ?

A : Degenerate double affine Hecke algebras (dDAHAs)

Example : SL_2

$\mathbb{K} = \mathbf{C}[q^{\pm 1}]\langle T, X\rangle/$(relations)

$$(T-q)(T+1)=0, \quad TX - X^{-1}T = (q-1)X$$

$\mathbb{K} \xrightarrow{\cong} \mathrm{K}^{G\times\mathbf{C}_q^\times}(\ddot{\mathfrak{g}})$ given by :

- $q \mapsto$ defining character of $\mathbf{C}_q^\times$
- $X \mapsto [\mathcal{O}_{\mathbf{P}^1}(-1)]$, $\quad T \mapsto -(q[\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(0,-2)] + [\mathcal{O}_{\mathbf{P}^1}])$, where

$$\begin{tikzcd}
\mathbf{P}^1 \arrow[r, hook, "\Delta"] & \mathbf{P}^1 \times \mathbf{P}^1 \arrow[r, "\subset"] \arrow[d] & \ddot{\mathfrak{g}} \arrow[d, "q"] \\
& \{0\} \arrow[r, "\subset"] & \mathfrak{g}.
\end{tikzcd}$$

Example : $\mathrm{SL}_2, \quad q \notin \sqrt{1}$

- Take $v \in \mathbf{R}_{>1}$ and $s = \left(\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, v^2 \right) \in T \times \mathbf{C}_q^\times$
- $G^s = T, \quad (\mathfrak{g}^{\mathrm{nil}})^s = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} = \mathbf{C}\, e$
- s-fixed points :

$$\begin{tikzcd}
\dot{\mathfrak{g}}^s \arrow[r, "\pi_s"] \arrow[d, "\cong"] & (\mathfrak{g}^{\mathrm{nil}})^s \arrow[r, equal] \arrow[d, "\cong"] & \mathrm{O}^0 \cup \mathrm{O}^e \arrow[d, "\cong"] \\
\mathbf{A}^1 \sqcup \{0\} \arrow[r] & \mathbf{A}^1 \arrow[r, equal] & \{0\} \cup \mathbf{C}^\times
\end{tikzcd}$$

- Deligne–Langlands corresp. :

$$(\mathrm{O}^e, \mathrm{triv}) \leftrightarrow L_{\mathrm{sph}} = \mathbb{K}/\, \mathbb{K}(q - v^2, T - v^2, X - v)$$
$$(\mathrm{O}^0, \mathrm{triv}) \leftrightarrow L_{\mathrm{asph}} = \mathbb{K}/\, \mathbb{K}(q - v^2, T + 1, X - v^{-1})$$

Example : $\mathrm{SL}_2, \quad q = -1$

- Take $v = \sqrt{-1}$ and $s = \left(\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, v^2 \right) \in T \times \mathbf{C}_q^\times$
- $G^s = T, \quad (\mathfrak{g}^{\mathrm{nil}})^s = \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \; ; \; ab = 0 \right\} = \mathbf{C}\, e \cup \mathbf{C}\, f \subset \mathbf{A}^2$
- s-fixed points :

$$\begin{tikzcd}
\dot{\mathfrak{g}}^s \arrow[r, "\pi_s"] \arrow[d, "\cong"] & (\mathfrak{g}^{\mathrm{nil}})^s \arrow[r, equal] \arrow[d, "\cong"] & \mathrm{O}^0 \cup \mathrm{O}^e \cup \mathrm{O}^f \arrow[d, "\cong"] \\
\mathbf{C}\, e \sqcup \mathbf{C}\, f \arrow[r] & \mathbf{C}\, e \cup \mathbf{C}\, f \arrow[r, equal] & \{0\} \cup \mathbf{C}^\times e \cup \mathbf{C}^\times f
\end{tikzcd}$$

- Deligne–Langlands corresp. fails for AHA at $\sqrt{1}$:

$$(\mathrm{O}^e, \mathrm{triv}) \leftrightarrow L_e = \mathbb{K} / \, \mathbb{K}(q+1, \, T+1, X - \sqrt{-1})$$

$$(\mathrm{O}^f, \mathrm{triv}) \leftrightarrow L_f = \mathbb{K} / \, \mathbb{K}(q+1, \, T+1, X + \sqrt{-1})$$

$$(\mathrm{O}^0, \mathrm{triv}) \leftrightarrow$$

Example : $\mathrm{SL}_2, \quad q=-1$

- Take $v=\sqrt{-1}$ and $s=\left(\begin{pmatrix} v & 0 \\ 0 & v^{-1}\end{pmatrix}, v^2\right) \in T \times \mathbf{C}_q^\times$
- $G^s = T, \quad (\mathfrak{g}^{\mathrm{nil}})^s = \left\{\begin{pmatrix} 0 & a \\ b & 0\end{pmatrix} \;;\; ab=0\right\} = \mathbf{C}\, e \cup \mathbf{C}\, f \subset \mathbf{A}^2$
- s-fixed points :

$$\begin{tikzcd}
\dot{\mathfrak{g}}^s \arrow[r, "\pi_s"] \arrow[d, "\cong"] & (\mathfrak{g}^{\mathrm{nil}})^s \arrow[r, equal] \arrow[d, "\cong"] & \mathrm{O}^0 \cup \mathrm{O}^e \cup \mathrm{O}^f \arrow[d, "\cong"] \\
\mathbf{C}\, e \sqcup \mathbf{C}\, f \arrow[r] & \mathbf{C}\, e \cup \mathbf{C}\, f \arrow[r, equal] & \{0\} \cup \mathbf{C}^\times e \cup \mathbf{C}^\times f
\end{tikzcd}$$

- Deligne–Langlands corresp. holds for dDAHA :

$$(\mathrm{O}^e, \mathrm{triv}) \leftrightarrow \tilde{L}_e = \mathbb{H}/\mathbb{H}(h+1/2, s_0-1, x-3/4)$$
$$(\mathrm{O}^f, \mathrm{triv}) \leftrightarrow \tilde{L}_f = \mathbb{H}/\mathbb{H}(h+1/2, s_1-1, x+1/4)$$
$$(\mathrm{O}^0, \mathrm{triv}) \leftrightarrow \tilde{L}_0 = \mathbb{H}/\mathbb{H}(h+1/2, s_1+1, s_0+1, x-1/4)$$

Degenerate double affine Hecke algebras

- G, B, T, as before
  Assume G is simple and $\pi_1(G) = 1$
- $R_{\mathrm{af}} = R \times \mathbf{Z}\delta$: real affine roots $\subset P \times \mathbf{Z}\delta$, δ : imaginary root
  $\Delta_{\mathrm{af}} = \Delta \cup \{\delta - \theta\}$: affine base, θ : highest root
  $W_{\mathrm{af}} = \langle s_\pi \rangle_{\pi \in \Delta_{\mathrm{af}}}$ affine Weyl group $\circlearrowright P \times \mathbf{Z}$
- The dDAHA $\mathbb{H}$ is the associative algebra over $\mathbf{C}[h]$
  - generated by the sets $\{x^\mu\}_{\mu \in P \times \mathbf{Z}\delta}$ and $\{s_\alpha\}_{\alpha \in \Delta_{\mathrm{af}}}$
  - subject to the relations for $\mu, \nu \in P \times \mathbf{Z}$ and $\alpha \neq \beta \in \Delta_{\mathrm{af}}$

$$x^\mu + x^\nu = x^{\mu+\nu}, \qquad x^\delta = 1$$

$$s_\alpha^2 = 1, \qquad s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \cdots$$

$$s_\alpha x^\mu - x^{s_\alpha(\mu)} s_\alpha = h \frac{x^\mu - x^{s_\alpha(\mu)}}{x^\alpha}$$

  - $\mathbb{H}$ is an h-deformation of $W_{\mathrm{af}} \ltimes \mathrm{Sym}(\mathbf{C}P)$

About dDAHAs

- The Double affine Hecke algebra (DAHA) was introduced by I. Cherednik in the 90s
- It organises the Macdonald polynomials ((q,t)-deformations of symmetric polynomials) and the Dunkl operators.
- Related to a lot of things : 2d CFT, $\mathrm{Hilb}^n(\mathbf{C}^2)$, knot invariants of planar-curve singularities, cluster algebras
- The degenerate form, dDAHA $\mathbb{H}$, captures a part of the DAHA.
- There is a triangular decomposition

$$\mathbb{H} = \mathbf{C}[h] \otimes (\mathbf{C}[T^\vee] \otimes \mathbf{C}W \otimes \mathbf{C}[\mathfrak{t}]), \quad T^\vee \text{ : dual torus}$$

- Set $\delta = 1$. There is an embedding $\mathbb{H} \hookrightarrow \mathbf{C}[h] \otimes \mathcal{D}(T^{\vee,\mathrm{reg}}) \rtimes W$. The subalgebra $\mathbf{C}[\mathfrak{t}]$ is sent to the Dunkl operators. $\rightsquigarrow$ affine Knizhnik–Zamolodchikov equations

Example : dDAHA for SL_2

- $R = \{\pm\alpha_1\}$, fundamental weight $x = \alpha_1/2$.
- Affine root system of SL_2 :

$$R_{\mathrm{af}} = \{\pm\alpha_1 + n\delta \ ; \ n \in \mathbf{Z}\}\,, \quad \delta : \ \text{imaginary root.}$$

  Base $\Delta_{\mathrm{af}} = \{\alpha_1, \alpha_0 = \delta - \alpha_1\}$.
- $\mathbb{H} = \mathbf{C}[h]\langle x, s_0, s_1\rangle/$(relations)

$$s_1^2 = 1 = s_0^2, \quad s_1 x + x s_1 = h, \quad s_0 x - (1 - x) s_0 = -h.$$

- Set $Y = s_0 s_1$. Then we have

$$\begin{aligned} \mathbb{H} &\hookrightarrow \mathbf{C}[h] \otimes \mathcal{D}(\mathbf{C}_Y^\times \backslash \{1\}) \rtimes \mathfrak{S}_2 \\ Y &\mapsto Y \\ s_1 &\mapsto s_1 \\ x &\mapsto Y\partial_Y - h(1 - Y^{-1})(1 - s_1) + h \end{aligned}$$

- We will only consider those $\mathbb{H}$-modules on which x acts locally finitely.

Cohomlogical construction of dDAHAs

- $G_{\mathrm{af}} = G((\varpi))$ loop group, $\mathbf{C}_t^\times = \mathbf{C}^\times \circlearrowright G_{\mathrm{af}}$ loop rotation
- $B_{\mathrm{af}} \subset G_{\mathrm{af}}$ standard Iwahori
  $T \subset G_{\mathrm{af}}$ max torus ; $U_{\mathrm{af}} \subset B_{\mathrm{af}}$ pro-unipotent radical
- $\dot{\mathfrak{g}}_{\mathrm{af}} = G_{\mathrm{af}} \times^{B_{\mathrm{af}}} \mathfrak{u}_{\mathrm{af}} \xrightarrow{\pi} \mathfrak{g}_{\mathrm{af}}, \quad \pi(g, x) = \mathrm{Ad}_g(x)$ Springer resolution
  $\ddot{\mathfrak{g}}_{\mathrm{af}} = \dot{\mathfrak{g}}_{\mathrm{af}} \times_{\mathfrak{g}_{\mathrm{af}}} \dot{\mathfrak{g}}_{\mathrm{af}}$ Steinberg variety

Theorem [Vasserot, Braverman–Finkelberg–Nakajima]

$\exists$ convolution on $\mathrm{H}_*^{G_{\mathrm{af}} \rtimes \mathbf{C}_t^\times \times \mathbf{C}_q^\times}(\ddot{\mathfrak{g}}_{\mathrm{af}})$ s.t.

$$\mathbb{H}_\delta \cong \mathrm{H}_*^{G_{\mathrm{af}} \rtimes \mathbf{C}_t^\times \times \mathbf{C}_q^\times}(\ddot{\mathfrak{g}}_{\mathrm{af}}),$$

where $\mathbb{H}_\delta$ is defined as $\mathbb{H}$ without the relation $x^\delta = 1$.

Springer correspondence for dDAHA

- Let $\sigma = (\lambda, m, d) \in \mathrm{Hom}(\mathbf{C}^\times, T \times \mathbf{C}_t^\times \times \mathbf{C}_q^\times)$. Suppose $m > 0, \ d < 0$.
- Let $\pi_\sigma : (\dot{\mathfrak{g}}_{\mathrm{af}})^\sigma \to (\mathfrak{g}_{\mathrm{af}}^{\mathrm{nil}})^\sigma$ be the restriction of $\pi : \dot{\mathfrak{g}}_{\mathrm{af}} \to \mathfrak{g}_{\mathrm{af}}$.
- Let $\mathbb{H}_\sigma^\wedge$ denote the completion of $\mathbb{H}$ at the ideal gen by $\{x^\mu - \langle \mu, \lambda/m \rangle\}_{\mu \in P} \cup \{h - d/m\}$.

Theorem [Vasserot]

$\exists$ bijection between

1. {simple smooth $\mathbb{H}_\sigma^\wedge$-modules}$/\sim$
2. $\{(z, \chi) \ ; \ z \in (\mathfrak{g}_{\mathrm{af}}^{\mathrm{nil}})^\sigma, \ \chi \in \mathrm{IrrLocSys}_{G_{\mathrm{af}}^\sigma}(G_{\mathrm{af}}^\sigma z), \ \mathrm{H}^*(\pi_\sigma^{-1}(z))^\chi \neq 0\}/\sim$.

Remark

- $m \neq 0 \implies \dot{\mathfrak{g}}_{\mathrm{af}}^\sigma = \bigsqcup$ (varieties of finite type)
- $m \cdot d < 0 \implies \pi_\sigma(\dot{\mathfrak{g}}_{\mathrm{af}}^\sigma) \subset \mathfrak{g}_{\mathrm{af}}^{\mathrm{nil}}$

Cyclically graded Lie algebras

Principle : $\mathbf{Z}$-grading on $\mathfrak{g}_{\mathrm{af}} \longleftrightarrow \mathbf{Z}/m$-grading on $\mathfrak{g}$

(a.k.a. Moy–Prasad = Vinberg)

Recall that $\sigma = (\lambda, m, d) : \mathbf{C}^\times \to T \times \mathbf{C}_t^\times \times \mathbf{C}_q^\times$

Assume $m \cdot d < 0$

- $\mathfrak{g}_{\mathrm{af}}^\sigma = \bigoplus_{k \in \mathbf{Z}} \left\{ \varpi^k \mathfrak{g}_\alpha \ ; \ \alpha \in R(G, T) \cup \{0\} \, , \ \langle \lambda, \alpha \rangle + mk = 2d \right\}.$

Set $\zeta = \exp(\frac{2\pi\sqrt{-1}}{m})$, $\ell = \lambda(\zeta)$ and $q = \zeta^d$.

Put $s = (\ell, q) \in T \times \mathbf{C}_q^\times$. Then

- $\mathfrak{g}^s = \{\mathfrak{g}_\alpha \ ; \ \alpha \in R(G, T) \cup \{0\} \, , \ \langle \lambda, \alpha \rangle \equiv 2d \pmod{m}\}$
- Thus $\mathfrak{g}_{\mathrm{af}}^\sigma \cong \mathfrak{g}^s$ by $\varpi \mapsto 1$, similarly $G_{\mathrm{af}}^\sigma \cong G^s$

$$\rightsquigarrow \quad [(\mathfrak{g}_{\mathrm{af}}^{\mathrm{nil}})^\sigma / G_{\mathrm{af}}^\sigma] \cong [(\mathfrak{g}^{\mathrm{nil}})^s / G^s]$$

Summary

$$\{\text{ simple smooth } \mathbb{H}_\sigma^\wedge\text{-modules}\}/\sim\ \longleftrightarrow \{(z,\chi)\ ;\ z \in (\mathfrak{g}_{\mathrm{af}}^{\mathrm{nil}})^\sigma,\ \mathrm{H}^*(\pi_\sigma^{-1}(z))^\chi \neq 0\}/\sim$$

$$\{\text{f.d. simple } \mathbb{K}_s^\wedge\text{-modules}\}/\sim\ \hookrightarrow \{(z,\chi)\ ;\ z \in (\mathfrak{g}^{\mathrm{nil}})^s,\ \mathrm{H}^*(\pi_s^{-1}(z))^\chi \neq 0\}/\sim$$

Summary

$$\begin{tikzcd}
\{\text{ simple smooth } \mathbb{H}^{\wedge}_{\sigma}\text{-modules}\}/\sim \arrow[r, leftrightarrow] & \{(z,\chi)\ ;\ z \in (\mathfrak{g}^{\mathrm{nil}}_{\mathrm{af}})^{\sigma},\ \mathrm{H}^*(\pi_{\sigma}^{-1}(z))^{\chi} \neq 0\}/\sim \arrow[d, leftrightarrow, "\text{Lusztig–Yun}"] \\
\{\text{f.d. simple } \mathbb{K}^{\wedge}_{s}\text{-modules}\}/\sim \arrow[r, hook] & \{(z,\chi)\ ;\ z \in (\mathfrak{g}^{\mathrm{nil}})^{s},\ \mathrm{H}^*(\pi_{s}^{-1}(z))^{\chi} \neq 0\}/\sim
\end{tikzcd}$$

Summary

$$\begin{tikzcd}
\{\text{ simple smooth } \mathbb{H}_\sigma^\wedge\text{-modules}\}/\sim \arrow[r, leftrightarrow] & \{(z,\chi)\ ;\ z \in (\mathfrak{g}_{\mathrm{af}}^{\mathrm{nil}})^\sigma,\ \mathrm{H}^*(\pi_\sigma^{-1}(z))^\chi \neq 0\}/\sim \arrow[d, leftrightarrow, "\text{Lusztig–Yun}"] \\
\{\text{f.d. simple } \mathbb{K}_s^\wedge\text{-modules}\}/\sim \arrow[u, hook, "???"] \arrow[r, hook] & \{(z,\chi)\ ;\ z \in (\mathfrak{g}^{\mathrm{nil}})^s,\ \mathrm{H}^*(\pi_s^{-1}(z))^\chi \neq 0\}/\sim
\end{tikzcd}$$

Summary

$$\begin{tikzcd}
\{\text{ simple smooth } \mathbb{H}^{\wedge}_{\sigma}\text{-modules}\}/\sim \arrow[r, leftrightarrow] & \{(z,\chi)\ ;\ z \in (\mathfrak{g}^{\mathrm{nil}}_{\mathrm{af}})^{\sigma},\ \mathrm{H}^*(\pi_{\sigma}^{-1}(z))^{\chi} \neq 0\}/\sim \\
\{\text{f.d. simple } \mathbb{K}^{\wedge}_{s}\text{-modules}\}/\sim \arrow[r, hook] \arrow[u, hook, "\text{algebraic KZ-functor}"] & \{(z,\chi)\ ;\ z \in (\mathfrak{g}^{\mathrm{nil}})^{s},\ \mathrm{H}^*(\pi_{s}^{-1}(z))^{\chi} \neq 0\}/\sim \arrow[u, leftrightarrow, "\text{Lusztig–Yun}"']
\end{tikzcd}$$

Algebraic Knizhnik–Zamolodchikov functor

We have

$$\mathbb{H}_\sigma^\wedge \cong \mathrm{H}_*^{G_{\mathrm{af}}^\sigma \times \mathbf{C}_q^\times}((\ddot{\mathfrak{g}}_{\mathrm{af}})^\sigma)_0^\wedge$$

$$\mathbb{K}_s^\wedge \cong \mathrm{K}^{G^s \times \mathbf{C}_q^\times}(\ddot{\mathfrak{g}}^s)_1^\wedge \xrightarrow{\mathrm{RR}} \mathrm{H}_*^{G^s \times \mathbf{C}_q^\times}(\ddot{\mathfrak{g}}^s)_0^\wedge$$

$$\ddot{\mathfrak{g}}_{\mathrm{af}}^\sigma = \dot{\mathfrak{g}}_{\mathrm{af}}^\sigma \times_{\mathfrak{g}_{\mathrm{af}}^\sigma} \dot{\mathfrak{g}}_{\mathrm{af}}^\sigma, \qquad \ddot{\mathfrak{g}}^s = \dot{\mathfrak{g}}^s \times_{\mathfrak{g}^s} \dot{\mathfrak{g}}^s$$

$$\begin{tikzcd}
\dot{\mathfrak{g}}_{\mathrm{af}}^\sigma \arrow[d, "\pi_\sigma"] & \dot{\mathfrak{g}}^s \arrow[d, "\pi_s"] \\
\mathfrak{g}_{\mathrm{af}}^\sigma \arrow[r, "\sim"] & \mathfrak{g}^s
\end{tikzcd}$$

Set

$$\mathcal{V} = \mathrm{H}_*^{G^s \times \mathbf{C}_q^\times}(\dot{\mathfrak{g}}_{\mathrm{af}}^\sigma \times_{\mathfrak{g}^s} \dot{\mathfrak{g}}^s)_0^\wedge, \qquad \mathbb{H}_\sigma^\wedge \circlearrowright \mathcal{V} \circlearrowleft \mathbb{K}_s^\wedge$$

Definition

The algebraic KZ-functor is

$$\mathbb{V} = \mathrm{Hom}_{\mathbb{H}_\sigma^\wedge}(\mathcal{V}, -) : \mathbb{H}_\sigma^\wedge\text{-mod}^{\mathrm{sm}} \to \mathbb{K}_s^\wedge\text{-mod}^{\mathrm{fd}}$$

Example : SL_2

Recall the Deligne–Langlands correspondence :

$$\mathrm{O}^0 = \{0\}\,, \quad \mathrm{O}^e = \begin{pmatrix} 0 & \mathbf{C}^\times \\ 0 & 0 \end{pmatrix}, \quad \mathrm{O}^f = \begin{pmatrix} 0 & 0 \\ \mathbf{C}^\times & 0 \end{pmatrix}$$

$$(\mathrm{O}^e, \mathrm{triv}) \leftrightarrow \tilde{L}_e = \mathbb{H}/\mathbb{H}(h+1/2, s_0-1, x-3/4)$$

$$(\mathrm{O}^f, \mathrm{triv}) \leftrightarrow \tilde{L}_f = \mathbb{H}/\mathbb{H}(h+1/2, s_1-1, x+1/4)$$

$$(\mathrm{O}^0, \mathrm{triv}) \leftrightarrow \tilde{L}_0 = \mathbb{H}/\mathbb{H}(h+1/2, s_1+1, s_0+1, x-1/4)$$

and

$$(\mathrm{O}^e, \mathrm{triv}) \leftrightarrow L_e = \mathbb{K}/\,\mathbb{K}(q+1,\, T+1,\, X-\sqrt{-1})$$

$$(\mathrm{O}^f, \mathrm{triv}) \leftrightarrow L_f = \mathbb{K}/\,\mathbb{K}(q+1,\, T+1,\, X+\sqrt{-1})$$

$$(\mathrm{O}^0, \mathrm{triv}) \leftrightarrow \emptyset$$

We have $\mathbb{V}(\tilde{L}_e) = L_e, \quad \mathbb{V}(\tilde{L}_f) = L_f, \quad \mathbb{V}(\tilde{L}_0) = 0.$

Properties of $\mathbb{V}$

Remark

$\mathbb{H}_\sigma^\wedge\text{-mod}^{\mathrm{sm}}$ and $\mathbb{K}_s^\wedge\text{-mod}^{\mathrm{fd}}$ are abelian category of finite length with finitely many simple objects.

Theorem [L]

The following holds for $\mathbb{V} : \mathbb{H}_\sigma^\wedge\text{-mod}^{\mathrm{sm}} \to \mathbb{K}_s^\wedge\text{-mod}^{\mathrm{fd}}$

1. $\mathbb{V}$ is a quotient functor of abelian category
2. $\ker \mathbb{V} = \{M \in \mathbb{H}_\sigma^\wedge\text{-mod}^{\mathrm{sm}} \ ; \ \dim_{\mathrm{GK}} M < r(G)\}$
3. $L \in \mathbb{H}_\sigma^\wedge\text{-mod}^{\mathrm{sm}}$ simple module. Then $\mathbb{V}(L) \neq 0$ iff P_L (proj cover of L) is relatively injective over $Z(\mathbb{H}_\sigma^\wedge)$.
4. $\mathbb{V}$ satisfies the "bicommutant property"

$$\mathrm{End}_{\mathbb{H}_\sigma^\wedge}(\mathcal{V}) \cong (\mathbb{K}_s^\wedge)^{\mathrm{op}}, \quad \mathrm{End}_{(\mathbb{K}_s^\wedge)^{\mathrm{op}}}(\mathcal{V}) \cong \mathbb{H}_\sigma^\wedge$$

Shifting the slope d/m

Given parameters $\sigma = (\lambda, m, d)$ and $\sigma' = (\lambda, m, d')$ such that $\frac{d-d'}{m} \in \mathbf{Z}$.

- Suppose that $d \cdot d' > 0$. Not hard to show :

$$\mathbb{H}_{\sigma}^{\wedge}\text{-mod}^{\mathrm{sm}} \cong \mathbb{H}_{\sigma'}^{\wedge}\text{-mod}^{\mathrm{sm}}$$

- Suppose that $d \cdot d' < 0$. Then

$$\mathrm{im}(\pi_{\sigma}) \subseteq (\mathfrak{g}^{\mathrm{nil}})^{s} \quad \text{but} \quad \mathrm{im}(\pi_{\sigma'}) \nsubseteq (\mathfrak{g}^{\mathrm{nil}})^{s}.$$

There are algebraic KZ-functors $\mathbb{V}$:

$$\mathbb{H}_{\sigma}^{\wedge}\text{-mod}^{\mathrm{sm}} \xrightarrow{\mathbb{V}} \mathbb{K}_{s}^{\wedge}\text{-mod}^{\mathrm{fd}} \xleftarrow{\mathbb{V}} \mathbb{H}_{\sigma'}^{\wedge}\text{-mod}^{\mathrm{sm}}$$

Shifting the slope d/m

Given parameters $\sigma = (\lambda, m, d)$ and $\sigma' = (\lambda, m, d')$ such that $\frac{d-d'}{m} \in \mathbf{Z}$.

- Suppose that $d \cdot d' > 0$. Not hard to show :

$$\mathbb{H}^{\wedge}_{\sigma}\text{-mod}^{\mathrm{sm}} \cong \mathbb{H}^{\wedge}_{\sigma'}\text{-mod}^{\mathrm{sm}}$$

- Suppose that $d \cdot d' < 0$. Then

$$\mathrm{im}(\pi_\sigma) \subseteq (\mathfrak{g}^{\mathrm{nil}})^s \quad \text{but} \quad \mathrm{im}(\pi_{\sigma'}) \nsubseteq (\mathfrak{g}^{\mathrm{nil}})^s.$$

There are algebraic KZ-functors $\mathbb{V}$:

$$\begin{tikzcd}
\mathrm{D}^b(\mathbb{H}^{\wedge}_{\sigma}\text{-mod}^{\mathrm{sm}}) \arrow[r, "\mathbb{V}"] \arrow[rr, dashed, bend right, "\cong \text{ (shift functor)}"'] & \mathrm{D}^b(\mathbb{K}^{\wedge}_{s}\text{-mod}^{\mathrm{fd}}) & \mathrm{D}^b(\mathbb{H}^{\wedge}_{\sigma'}\text{-mod}^{\mathrm{sm}}) \arrow[l, "\mathbb{V}"']
\end{tikzcd}$$

Derived equivalence between $\mathbb{H}_{\sigma}^{\wedge}$ and $\mathbb{H}_{\sigma'}^{\wedge}$

- Recall $T \subset B_{\mathrm{af}} \subset G_{\mathrm{af}}$ standard Iwahori, $\mathfrak{u}_{\mathrm{af}} \subset \mathfrak{b}_{\mathrm{af}}$, pro-unipotent radicals.
- $\dot{\mathfrak{g}}_{\mathrm{af}} = G_{\mathrm{af}} \times^{B_{\mathrm{af}}} \mathfrak{u}_{\mathrm{af}} \xrightarrow{\pi} \mathfrak{g}_{\mathrm{af}}, \quad \pi(g, x) = \mathrm{Ad}_g(x)$ Springer resolution
- Set

$$\mathcal{T} = \mathrm{H}_*^{G_{\mathrm{af}}^{\sigma} \times \mathbf{C}_q^{\times}} (\dot{\mathfrak{g}}_{\mathrm{af}}^{\sigma'} \times_{\mathfrak{g}^s} \dot{\mathfrak{g}}_{\mathrm{af}}^{\sigma})_0^{\wedge}, \quad \mathbb{H}_{\sigma}^{\wedge} \circlearrowright \mathcal{T} \circlearrowleft \mathbb{H}_{\sigma'}^{\wedge}$$

Theorem (in progress)

The following holds :

- Let $A = Z(\mathbb{H}_{\sigma}^{\wedge})$. Then $\mathcal{T} \cong \mathrm{Hom}_A(\mathbb{H}_{\sigma}^{\wedge}, A)$ as $\mathbb{H}_{\sigma}^{\wedge}$-module.
- $\mathcal{T}$ is a tilting complex $(\mathrm{Ext}^{>0}_{\mathbb{H}_{\sigma}^{\wedge}}(\mathcal{T}, \mathcal{T}) = 0,\ \langle \mathcal{T} \rangle = \mathrm{D}^b(\mathbb{H}_{\sigma}^{\wedge}\text{-mod}^{\mathrm{sm}}))$.
- $\mathcal{T}$ satisfies the bicommutant property
- $\mathrm{RHom}_{\mathbb{H}_{\sigma}^{\wedge}}(\mathcal{T}, -)$ yields an equivalence of derived categories

$$\mathrm{D}^b(\mathbb{H}_{\sigma}^{\wedge}\text{-mod}^{\mathrm{sm}}) \cong \mathrm{D}^b(\mathbb{H}_{\sigma'}^{\wedge}\text{-mod}^{\mathrm{sm}}).$$

Remarks

- The derived equivalence

$$\mathrm{D}^b(\mathbb{H}^\wedge_\sigma\text{-mod}^{\mathrm{sm}}) \cong \mathrm{D}^b(\mathbb{H}^\wedge_{\sigma'}\text{-mod}^{\mathrm{sm}}).$$

  is an analogue of the Ringel duality for *rational Cherednik algebras*.
- We conjecture the same derived equivalence for $\mathbb{H}$ of *unequal parameters*, which will be an analogue of a theorem of I. Losev for rational Cherednik algebras.
- The proof of Losev makes use of an analogue for symplectic reflection singularity of Bezrukavnikov–Kaledin's theorem of derived McKay correspondence and quantisation in characteristic $p > 0$.

Tool : Braden's hyperbolic restriction theorem

- X : a complex algebraic variety
- Let $\mathbf{C}^\times$ act on X by $\alpha : \mathbf{C}^\times \to \mathrm{Aut}(X)$
- $$\begin{tikzcd} & X & \\ X^+ \arrow[ur, hook, "j^+"] \arrow[dr, "p^+"'] & & X^- \arrow[ul, hook', "j^-"'] \arrow[dl, "p^-"] \\ & X^\alpha & \end{tikzcd}$$

$$X^+ = \left\{ x \in X \, ; \, \lim_{t \to 0} \alpha(t)x \in X \right\}$$

$$X^- = \left\{ x \in X \, ; \, \lim_{t \to \infty} \alpha(t)x \in X \right\}$$

X^α : α-fixed points.

- $p^+(x) = \lim_{t \to 0} \alpha(t)x, \quad p^-(x) = \lim_{t \to \infty} \alpha(t)x$

Theorem (Braden).

There is an isomorphism of functors $\mathrm{D}^b(X) \to \mathrm{D}^b(X^\alpha)$

$$(p^+)_! \circ (j^+)^* \cong (p^-)_* \circ (j^-)^!.$$

- Let $\tau = (\mu, n) : \mathbf{C}^\times \to T \times \mathbf{C}_t^\times$ be a regular cocharacter (i.e. $(G_{\mathrm{af}})^\tau = T$).
- Suppose $n > 0$. It defines two subgroups :

$$B^+ = \left\{ g \in G_{\mathrm{af}} \; ; \; \lim_{t \to 0} \tau(t) g \in G_{\mathrm{af}} \right\} \subset G_{\mathrm{af}}.$$

$$B^- = \left\{ g \in G_{\mathrm{af}} \; ; \; \lim_{t \to \infty} \tau(t) g \in G_{\mathrm{af}} \right\} \subset G_{\mathrm{af}}.$$

- B^+ is an Iwahori subgroup of G_{af}
  B^- is an integral $\mathbf{C}[\varpi^{-1}]$-model for G_{af}
  $B^+ \cap B^- = (G_{\mathrm{af}})^\tau = T$
- Two "affine Springer resolutions"

$$\dot{\mathfrak{g}}_{\mathrm{af}} = G_{\mathrm{af}} \times^{B^+} \mathfrak{u}^+ \to \mathfrak{g}_{\mathrm{af}}$$

$$\bar{\mathfrak{g}}_{\mathrm{af}} = G_{\mathrm{af}} \times^{B^-} \mathfrak{u}^- \to \mathfrak{g}_{\mathrm{af}}$$

$$\begin{tikzcd}
 & \mathfrak{g}^{\sigma}_{\mathrm{af}} & \\
(\mathfrak{b}^{+})^{\sigma} \arrow[ur, hook, "j^{+}"] \arrow[dr, "p^{+}"'] & & (\mathfrak{b}^{-})^{\sigma} \arrow[ul, hook', "j^{-}"'] \arrow[dl, "p^{-}"] \\
 & \mathfrak{t} &
\end{tikzcd}$$

- Thus by Braden's theorem,

$$(p^{+})_{!} \circ (j^{+})^{*} \cong (p^{-})_{*} \circ (j^{-})^{!}.$$

- By some easy Ext-calculus, it gives

$$\mathrm{H}_{*}^{G^{s}\times\mathbf{C}_{q}^{\times}}(\dot{\mathfrak{g}}^{\sigma}_{\mathrm{af}} \times_{\mathfrak{g}^{\sigma}_{\mathrm{af}}} \bar{\mathfrak{g}}^{\sigma}_{\mathrm{af}}) \cong \mathrm{Hom}_{A}(\mathrm{H}_{*}^{G^{s}\times\mathbf{C}_{q}^{\times}}(\dot{\mathfrak{g}}^{\sigma}_{\mathrm{af}} \times_{\mathfrak{g}^{\sigma}_{\mathrm{af}}} \dot{\mathfrak{g}}^{\sigma}_{\mathrm{af}}), A)$$

- Not hard to show

$$\mathrm{H}_{*}^{G^{s}\times\mathbf{C}_{q}^{\times}}(\bar{\mathfrak{g}}^{\sigma}_{\mathrm{af}} \times_{\mathfrak{g}^{\sigma}_{\mathrm{af}}} \bar{\mathfrak{g}}^{\sigma}_{\mathrm{af}}) \cong_{\mathrm{Morita}} \mathrm{H}_{*}^{G^{s}\times\mathbf{C}_{q}^{\times}}(\dot{\mathfrak{g}}^{\sigma'}_{\mathrm{af}} \times_{\mathfrak{g}^{\sigma}_{\mathrm{af}}} \dot{\mathfrak{g}}^{\sigma'}_{\mathrm{af}})$$

Fin