Weak holonomicity for equivariant $\mathcal{D}$-modules on rigid analytic flag varieties

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Outline

$\mathcal{D}$-modules and representation theory

Theorem (Beilinson-Bernstein, 1981)

Let G be a connected complex reductive algebraic group with Lie algebra $\mathfrak{g}$ and $X := G/B$ be its flag variety ($B \subset G$ be a fixed Borel subgroup). The global section functor

$$\Gamma : Qcoh(\mathcal{D}_X) \longrightarrow Mod(U(\mathfrak{g}))$$
$$\mathcal{M} \longrightarrow \Gamma(X, \mathcal{M})$$

induces an equivalence of categories between the category of quasi-coherent (res. coherent) $\mathcal{D}_X$-modules and the category of $U(\mathfrak{g})/Z(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{g}$-modules (resp. finitely generated modules).

p-adic representations

Notations: K be a complete discrete valuation field (ex: finite extensions of $\mathbb{Q}_p$).
$R := \{a \in K, \mid a \mid \leq 1\}$, π uniformiser.

G: a locally analytic group over K.

Aim: Study locally analytic representations of G.

Definition

A representation $G \longrightarrow GL(V)$ is called locally analytic if

- V is a Hausdorf locally convex K-vector space.
- For each $v \in V$, the orbit map

$$G \longrightarrow V, \ \ g \longmapsto gv$$

is locally analytic with values in V.

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Distribution algebras

Definition

The strong dual of the locally convex K-vector space $C^{la}(G,K) := \{f : G \longrightarrow K, \text{ locally analytic}\}$

$$D(G,K) := (C^{la}(G,K))'_b$$

is called the distribution algebra of G.

Some remarks:

1. $G \subset D(G,K)$ via $g \longmapsto \delta_g(f) := f(g)$ (Dirac distribution). Therefore $K[G] \subset D(G,K)$
2. Let $\mathfrak{g} := Lie(G)$, then $U(\mathfrak{g}) \subset D(G,K)$ via

$$x \in \mathfrak{g} \longmapsto f \longmapsto \frac{d}{dt}\,|_{t=0}\,(f \circ \exp(-tx))(1)$$

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Important result: (Schneider-Teitelbaum) When G is compact, there is an (anti-) equivalence of categories between the category of admissible locally analytic representations and that of coadmissible $D(G,K)$-modules.

Geometric tools: Establish a p-adic version of Beilinson-Bernstein correspondence (Schmidt-Strauch-Huyghe for arithmetic $\mathcal{D}$-modules (Berthelot), Ardakov for $\widehat{D}$-modules on rigid analytic spaces)

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Group algebras

Recall that if R be a unital ring and G is an abstract group which acts on R via $G \longrightarrow Aut(R)$, then

$$R \rtimes G := \{r_1 g_1 + r_2 g_2 + ... + r_n g_n,\ r_i \in R, g_i \in G, n \in \mathbb{N}\}$$

with $(rg).(r'g') = r(g.r')(gg')$ is an associative ring.

$\Longrightarrow$ Each R-module with compatible action of G can be considered as a $R \rtimes G$-module.

Theorem (Equivariant $\mathcal{D}$-modules)

Let $\mathbf{X}$ be a rigid analytic space equipped with an action of a p-adic Lie group G and $\mathcal{M}$ be a G-equivariant $\mathcal{D}$-module. Then for any admissible open subset $\mathbf{U} \subset \mathbf{X}$ and $H \leq G$ which stablizes $\mathbf{U}$, $\mathcal{M}(\mathbf{U})$ is a $\mathcal{D}(\mathbf{U}) \rtimes H$-module.

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The completed kew-group algebra

Let **X** be a smooth rigid analytic variety, G a p-adic Lie group acting continuously on **X**.

$\rightsquigarrow$ For **U** an affinoid H-stable ($H \leq G$) s.t there is a free Lie lattice $\mathcal{L}$ of $Der(\mathcal{O}(\mathbf{U}))$ which is H-stable (H is then called **U**-small).
We define:

$$\widehat{\mathcal{D}}(\mathbf{U}, H) \simeq \varprojlim_{n} \widehat{U(\pi^n \mathcal{L})_K} \rtimes_{J_n} H,$$

where $...J_{n+1} \leq J_n ... \leq H$ open compact such that there exists $\beta : J_n \longrightarrow \widehat{U(\pi^n \mathcal{L})_K}^{\times}$ H-equivariant and

$$\widehat{U(\pi^n \mathcal{L})_K} \rtimes_{J_n} H = \widehat{U(\pi^n \mathcal{L})_K} \rtimes H / < \beta(g)^{-1} g - 1,\ g \in J_n > .$$

Theorem (K. Ardakov)

The K-algebra $\widehat{D}(\mathbf{U}, H)$ is Fréchet-Stein : each K-algebra $\widehat{U(\pi^n \mathcal{L})_K} \rtimes_{J_n} H$ is Banach Noetherian, transition maps are flat.

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Coadmissible equivariant modules

Definition

A module over a Fréchet-Stein K-algebra $U = \varprojlim_n U_n$ is coadmissible (denoted by $\mathcal{C}_{\widehat{D}(\mathbf{X},G)}$) if $M \cong \varprojlim_n M_n$ with $M_n \in Mod_c(U_n)$ such that

$$U_n \otimes_{U_{n+1}} M_{n+1} \xrightarrow{\sim} M_n.$$

Theorem (K.Ardakov)

Suppose that $\mathbf{X}$ is affinoid and $(\mathbf{X}, G)$ is small. Let M be a coadmissible $\widehat{D}(\mathbf{X}, G)$-module

(i) *If $(\mathbf{U}, H)$ is small, the $\widehat{D}(\mathbf{U}, H)$-module*

$$M(\mathbf{U}, H) := \widehat{D}(\mathbf{U}, H)\widehat{\otimes}_{\widehat{D}(\mathbf{X},H)} M$$

is coadmissible

(ii) *The correspondence $\mathbf{U} \longmapsto \varprojlim_H M(\mathbf{U}, H)$ induces a G-equivariant sheaf $Loc^{\mathbf{X}}_{\widehat{D}(\mathbf{X},G)}(M)$ of $\mathcal{D}$-modules on $\mathbf{X}$.*

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Definition

A G-equivariant $\mathcal{D}_{\mathbf{X}}$-module $\mathcal{M}$ is called coadmissible (denoted by $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$) if

(i) For each $\mathbf{U}$ affinoid, $\mathcal{M}(\mathbf{U})$ is equipped with a Fréchet topology such that the G-action is continuous.

(ii) $\mathcal{M}|_{\mathbf{U}} \cong Loc^{\mathbf{U}}_{\widehat{D}(\mathbf{U},H)}(M)$ for some coadmissible $\widehat{D}(\mathbf{U}, H)$-module M.

Main results
Weak holonomicity

Dimension theory

Theorem (V.)

(i) *The K-algebra $\widehat{D}(\mathbf{X}, G)$ is coadmissibly Auslander-Gorenstein, meaning that there exists a free Lie lattice $\mathcal{L}$ for $Der_K(\mathcal{O}(\mathbf{X}))$ such that each $D_n := \widehat{U(\pi^n \mathcal{L})}_K \rtimes_{J_n} G$ is Auslander-Gorenstein and*

$$\widehat{D}(\mathbf{X}, G) \cong \varprojlim_n D_n$$

(ii) *There is a dimension function on the category of coadmissible $\widehat{D}(\mathbf{X}, G)$ modules:* $d : \mathcal{C}_{\widehat{D}(\mathbf{X},G)} \longrightarrow \mathbb{Z}_{\geq 0}$ *given by*

$$d(M) := 2\dim \mathbf{X} - \min\{i \geq 0 : Ext^i(M, \widehat{D}(X, G)) \neq 0\}$$

Example

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If $P \in \mathcal{D}(\mathbf{X})$ is a regular differential operator (P is not a zero divisor of $\mathcal{D}(\mathbf{X})$). Then the coadmissible left $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module

$$M = \widehat{\mathcal{D}}(\mathbf{X}, G)/\widehat{\mathcal{D}}(\mathbf{X}, G)P$$

is of dimension $d(M) \leq 2d - 1$.

Proof.

Let $D := \mathcal{D}(X)$. We have $M \cong \varprojlim_n D_n/D_nP$ and $D_n/D_nP \cong D_n \otimes_D D/DP$. Now, D_n is flat over D, hence

$$Ext^i_D(D/DP, D) \otimes_D D_n \cong Ext^i_{D_n}(D_n \otimes_D D/DP, D_n).$$

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□

Dimension of coadmissible equivariant $\mathcal{D}$-modules

Definition

Let $\mathcal{U}$ be an admissible covering of $\mathbf{X}$ by affinoid subdomains and $\mathcal{M}$ be a coadmissible G-equivariant left $\mathcal{D}_{\mathbf{X}}$-module on $\mathbf{X}$. Then the dimension of $\mathcal{M}$ with respect to $\mathcal{U}$ is defined as follows:

$$d_{\mathcal{U}}(\mathcal{M}) := \sup \{d(\mathcal{M}(\mathbf{U}))|\mathbf{U} \in \mathcal{U}\},$$

where $d(\mathcal{M}(\mathbf{U}))$ is the dimension of the coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module $\mathcal{M}(\mathbf{U})$ for some open compact subgroup H of G which stablizes $\mathbf{U}$.

Notations

From now on:

- $\mathbb{G}$: connected, simply connected, split semi-simple algebraic group over K.
- $\mathbb{B} \subset \mathbb{G}$ a Borel subgroup.
- $G := \mathbb{G}(K)$ locally K-analytic group.
- $X = (\mathbb{G}/\mathbb{B})^{an}$ the rigid analytic flag variety of $\mathbb{G}$.

Weak holonomicity

Theorem (V.)

Let $0 \neq \mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$. *Then the dimension* $d(\mathcal{M})$ *is at least* $\dim \mathbf{X}$.

Definition

A coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-module $\mathcal{M}$ is called weakly holonomic if it is zero or $\dim \mathcal{M} = \dim \mathbf{X}$.

Remark: Weakly holonomic equivariant modules form an abelian subcategory $\mathcal{C}^{wh}_{\mathbf{X}/G}$ of $\mathcal{C}_{\mathbf{X}/G}$ and stable under extensions (the dimension function is exact!).

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Duality

Theorem (V.)

For each $i \geq 0$, there exists a functor $\mathcal{E}^i : \mathcal{C}_{\mathbf{X}/G} \longrightarrow \mathcal{C}^r_{\mathbf{X}/G}$ from coadmissible G-equivariant left $\mathcal{D}_X$-modules to coadmissible G-equivariant right $\mathcal{D}_X$-modules such that

$$\mathcal{E}^i(\mathcal{M})(\mathbf{U}) = Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U},H))$$

for each $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$ and $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, $H \leq G$ which is $\mathbf{U}$-small.

Theorem (V.)

The endofunctor $\mathbb{D} := \mathcal{H}om_{\mathcal{O}_{\mathbf{X}}}(\Omega_{\mathbf{X}}, \mathcal{E}^{\dim \mathbf{X}}(-))$ preserves weak holonomicity and satisfies $\mathbb{D}^2 = Id$.

Main results
Examples

p-adic Beilinson Bernstein correspondence

Remark: (K. Ardakov): $D(G,K) \cong \widehat{U}(\mathfrak{g},G) \supset U(\mathfrak{g}) \rtimes G$.

Theorem (Huyghe-Strauch-Schmidt, Ardakov)

The composition of two functors

$$\Gamma : \mathcal{C}_{\mathbf{X}/G} \longrightarrow \mathcal{C}_{\widehat{U}(\mathfrak{g},G)}, \quad \mathcal{M} \longmapsto \Gamma(\mathbf{X},\mathcal{M})$$

and

$$\mathcal{C}_{D(G,K)} \longrightarrow Rep_K^{adm}(G), \quad M \longmapsto (M)'_b$$

induces an anti-equivalence of category between the category of coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules and the category of admissible locally analytic representations of G with trivial central character.

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The cateogory $\mathcal{O}^{\mathfrak{p}}$

Let $\mathbb{P} \subset \mathbb{G}$ be a parabolic subgroup. Fix a maximal torus $\mathbb{T} \subset \mathbb{P}$ of $\mathbb{G}$ and write $\mathfrak{g} := Lie(\mathbb{G}), \mathfrak{p} := Lie(\mathbb{P}), \mathfrak{t} := Lie(\mathbb{T})$.

Definition

An $U(\mathfrak{g})$-module M is in the category $\mathcal{O}^{\mathfrak{p}}$ if:

(1) M is a finitely generated $U(\mathfrak{g})$-module.

(2) M is $\mathfrak{t}$ -semi-simple, i.e $M = \oplus_{\lambda \in \mathfrak{t}^*} M_\lambda$

(3) The action of $\mathfrak{p}$ on M is locally finite, which means that for every $m \in M$, the subspace $U(\mathfrak{p}).m \subset M$ is a finite-dimensional K-vector space.

Let $\mathfrak{m}_0 := Z(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{g}$. Define $\mathcal{O}^{\mathfrak{p}} := \{M \in \mathcal{O}^{\mathfrak{p}} : \ \mathfrak{m}_0 M = 0\}$.

Theorem (V.)

(i) *Let $\mathcal{N}$ be a P-equivariant weakly holonomic $\mathcal{D}_{\mathbf{X}}$-module. Then $ind_P^G(\mathcal{N})$ is a G-equivariant weakly holonomic $\mathcal{D}_{\mathbf{X}}$-module.*

(ii) *Let $\underline{M} \in \mathcal{O}_0^{\mathfrak{g}}$. Then $ind_P^G(Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},P)}(\widehat{U}(\mathfrak{g}, P) \otimes_{D(\mathfrak{g},P)} M))$ is a G-equivariant weakly holonomic $\mathcal{D}_{\mathbf{X}}$-module.*

The End