The Cox ring of a complexity one variety
(Colloque Tournant 2021 du GDR Théorie de Lie Algébrique et Géométrique)

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Conventions

In this talk, we work over an algebraically closed field k of characteristic zero. An **algebraic group** is an **affine** k-group scheme of finite type.

What is a Cox ring ?

Fact : Any finitely generated integral k-algebra A defines an affine variety $X := \operatorname{Spec} A$. Morevover, A is recovered by taking the algebra $\mathcal{O}(X)$ of regular functions on X.

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This closed immersion is determined by the very ample invertible sheaf $\mathcal{L} := \iota^*\mathcal{O}(1)$ and the pullbacks $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ of homogeneous coordinates on $\mathbb{P}^n_k$.

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The image S_ι of the natural graded morphism

$$k[x_0, \dots, x_n] \to \bigoplus_{m \geqslant 0} \Gamma(X, \mathcal{L}^{\otimes m}), x_i \mapsto s_i$$

is the **homogeneous coordinate ring** of $\iota(X)$.

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Problem : This algebra depends on ι !
Idea : Define a **total coordinate ring** (or **Cox ring**) containing all the sections of (isomorphism classes of) invertible sheaves.

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$$\mathrm{WDiv}(X)/\mathrm{PDiv}(X) \to \mathrm{Cl}(X),\ [D] \mapsto [\mathcal{O}_X(D)],$$

where $\Gamma(U, \mathcal{O}_X(D)) := \{f \in k(X)^*; (\mathrm{div}(f) + D)_{|U} \geqslant 0\} \cup \{0\}$.

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Example : $\mathrm{Cl}(X) \simeq \mathbb{Z}/n\mathbb{Z} \simeq \langle [\mathcal{L}] \rangle$, $\mathrm{Cox}(X) = \bigoplus_{k=0}^{n-1} \Gamma(\mathcal{L}^{\otimes k}, X)$

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... must choose an isomorphism $\mathcal{L}^{\otimes n} \simeq \mathcal{O}_X$.

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A **rigidified divisorial sheaf** is a pair $(\mathcal{F}, f)$ where $\mathcal{F}$ is a divisorial sheaf on X, and $f : i^*\mathcal{F} \to k$ is an isomorphism of k-vector spaces.

A morphism $v : (\mathcal{F}, f) \to (\mathcal{G}, g)$ of rigidified divisorial sheaves is an $\mathcal{O}_X$-module morphism such that the diagram

$$\begin{array}{ccc} i^*\mathcal{F} & \xrightarrow{i^*v} & i^*\mathcal{G} \\ \downarrow f & \swarrow g & \\ k & & \end{array}$$

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Let E be the abelian group of isomorphism classes of rigidified divisorial sheaves.

The Cox ring of a normal variety

Important features :

- a rigidified divisorial sheaf has no non-trivial automorphism,
- the canonical morphism $E \to \mathrm{Cl}(X), [(\mathcal{F}, f)] \mapsto [\mathcal{F}]$ is an isomorphism.

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It follows that the Cox sheaf of the pointed variety (X, x)

$$\mathcal{R}_X = \bigoplus_{[\mathcal{F}] \in \mathrm{Cl}(X)} \mathcal{F}^x$$

has a canonical multiplication law via

$$\Gamma(U, \mathcal{F}^x) \otimes_{\mathcal{O}_X(U)} \Gamma(U, \mathcal{G}^x) \to \Gamma(U, (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{\vee\vee})^x).$$

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This doesn't depend on the choice of the point x up to graded isomorphism.

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The gradings translate in an action of the **diagonalizable group** $\Gamma_{\mathrm{Cl}(X)}$ on $\hat{X}$ and $\tilde{X}$.

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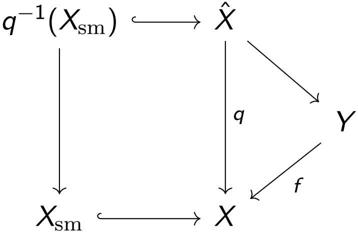
The structural morphism q is an **almost principal $\Gamma_{\mathrm{Cl}(X)}$-bundle**. Moreover, it is universal in the sense that it factors through any almost principal bundle $f : Y \to X$ under a diagonalizable group.



Examples

If X is a smooth complete intersection of dimension $\geqslant 3$ in $\mathbb{P}_k^n$, then the restriction $\mathrm{Cl}(\mathbb{P}_k^n) \to \mathrm{Cl}(X)$ is an isomorphism, and the Cox ring coincide with the homogeneous coordinate ring of the embedding $X \subset \mathbb{P}_k^n$.

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The simplest example is the projective space :

$$\begin{array}{ccc} \mathbb{A}_k^{n+1} \setminus \{0\} & \xrightarrow{j} & \mathbb{A}_k^{n+1} \\ \big\downarrow /\mathbb{G}_m & & \\ \mathbb{P}_k^n & & \end{array}$$

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Given a projective $\mathbb{Q}$-factorial MDS X, this chamber decomposition also describes all the projective varieties sharing the same Cox ring as X. Of course, they are all birationally equivalent.

Examples of Mori Dream Spaces

- The open subsets of the projective line are the only one dimensional MDS,
- a $K3$ surface is a MDS if and only if its automorphism group is finite,
- spherical varieties,
- Fano varieties.

Equivariant Cox ring

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Idea : Define an equivariant analogue of the Cox ring and relate it to the ordinary one.

Equivariant Cox ring

Let G be an algebraic group. A G-**linearized divisorial sheaf** $\mathcal{F}$ on a normal G-variety $G \times X \xrightarrow{\sigma} X$ is the given of an isomorphism $\sigma^*\mathcal{F} \to p_X^*\mathcal{F}$ satisfying a "cocycle condition".

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The isomorphism translates in a commutative square

$$\begin{array}{ccc} G \times L & \xrightarrow{\alpha} & L \\ \downarrow & & \downarrow \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

where $L := \mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}(X)}(\mathcal{F}))$. The cocycle condition guarantees that α is an action.

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The abelian group of isomorphism classes of G-linearized divisorial sheaves is the **equivariant class group** $\mathrm{Cl}^G(X)$.

Equivariant Cox ring

Similarly as before, under the assumption that $\mathcal{O}(X)^{*G} \simeq k^*$ we can define the **equivariant Cox sheaf and ring** :

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$$\begin{array}{ccccc} G \times \Gamma_{\mathrm{Cl}^G(X)} \times \hat{X}^G & \longrightarrow & \hat{X}^G & \hookrightarrow & \tilde{X}^G \\ \downarrow & & \downarrow & & \\ G \times X & \longrightarrow & X & & \end{array}$$

Relation between $\mathrm{Cox}^G(X)$ and $\mathrm{Cox}(X)$

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The morphism φ is finite in general, and surjective if ϕ is surjective (e.g. when $\mathrm{Pic}(G) = 0$).

Description of $\ker \varphi$

$\mathrm{Cox}^G(X)$ has a natural structure of $k[\hat{G}]$-algebra :
$\forall [\mathcal{F}] \in \mathrm{Cl}^G(X), \forall s \in \mathrm{Cox}^G(X)_{[\mathcal{F}]}, \forall \lambda \in \hat{G}$, define $\lambda.s$ as the unique element of $\mathrm{Cox}^G(X)_{[(\mathcal{F} \otimes \mathcal{O}(\lambda))^{\vee\vee}]}$ such that $\varphi(\lambda.s) = \varphi(s)$.

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Suppose that ϕ is surjective and $\mathcal{O}(X)^* \simeq k^*$. Then $\ker \varphi = ((1-\lambda_i.1)_i)$, where $(\lambda_i)_i$ is any $\mathbb{Z}$-basis of $\hat{G}$.

Proposition

Under the above assumptions, we have

$$\mathrm{Cox}(X) \simeq \mathrm{Cox}^G(X)/((1-\lambda_i.1)_i).$$

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A normal variety of complexity one is an MDS if and only if it is a rational variety (Brion, Knop, Luna, Vust).

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Definition

The finite set of B-stable prime divisors that doesn't lie in the image of π^* is the set of *exceptional divisors*. For an exceptional divisor E, the image of $\pi_{|E}$ is either dense or a point in $\mathbb{P}^1_k$. In the former case, we say that E is *central*, in the latter case, the image point is called *exceptional*.

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- the set of exceptional points is denoted $\mathcal{S}_X$.
- $\pi^*(x) = \sum_j n_{j,x} E_j^x$, $x = [\alpha_x : \beta_x] \in \mathcal{S}_X \subset \mathbb{P}^1_k$
- $(E_k)_k$ are the (exceptional) central divisors,
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Theorem

$\mathrm{Cox}^G(X)^U$ is generated over $k[\hat{G}]$ by $a, b, (s_{j,x})_{j,x}, (s_k)_k$. The ideal of relations contains

$$\beta_x a - \alpha_x b = \lambda_i \prod_j s_{j,x}^{n_{j,x}}, \text{ for } x \in \mathcal{S}_X \ (\lambda_i \in \hat{G}).$$

If moreover the common degree of a and b has no $\mathbb{Z}$-torsion, then these relations generate the whole ideal.

Applications

When $G = \mathbb{T}$ is a torus and $\mathcal{O}(X)^{\mathbb{T}} \simeq k$, this provides a description of $\mathrm{Cox}^{\mathbb{T}}(X)$ (hence of $\mathrm{Cox}(X)$) by generators and relations.

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The description of $\mathrm{Cox}^G(X)^U$ allows to characterize the log terminality in the total coordinate space of an **almost homogeneous variety** of complexity one by reducing to the case of varieties with torus action.

The result on U-invariants provides homogeneous generators for the equivariant Cox ring (as a $k[\hat{G}]$-algebra) : the elements of the bases of the simple G-modules spanned by the elements $a, b, (s_{j,x})_{j,x}, (s_k)_k$.