Positivity of an analog of Kazhdan-Lusztig polynomials for finite-dimensional representations of quantum loop algebras

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based on a joint work (arXiv:2101.07489) with
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Journées du GDR TLAG 2021
25/Mar/2021

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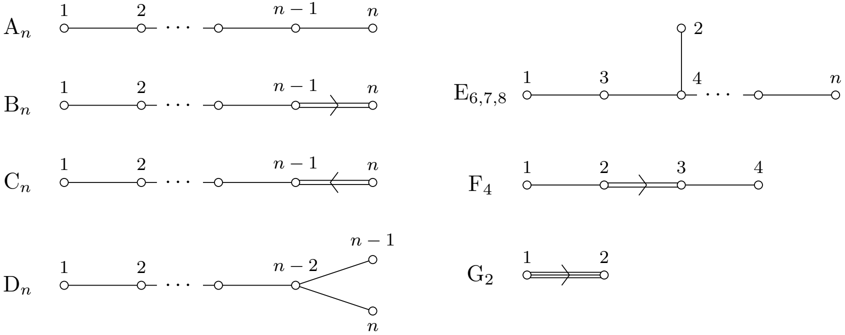
Notation

- $\mathfrak{g}$: simple Lie algebra, fin-dim/$\mathbb{C}$
- $\overset{1:1}{\longleftrightarrow}$ Dynkin diagram (type $\mathrm{A}_n, \mathrm{B}_n, \ldots, \mathrm{G}_2$)



- $I := \{1, \ldots, n\}$: set of vertices
- Fix a triangular decomp. $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$
- $\mathfrak{h} = \bigoplus_{i \in I} \mathbb{C} h_i$: Cartan subalg $\overset{\text{dual}}{\longleftrightarrow}$ $\mathfrak{h}^* = \bigoplus_{i \in I} \mathbb{C} \varpi_i$

Classical finite-dimensional representation theory

The category $\mathrm{Rep}(\mathfrak{g}) := \{$fin-dim $\mathfrak{g}$-rep's$\}$ is :

- *semisimple* as abelian cat.
- *symmetric* as monoidal cat. $(V \otimes V' \overset{\text{flip}}{=} V' \otimes V)$

From $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, one can consider :

- highest weight theory
  $\rightsquigarrow \mathrm{Irr}\,\mathrm{Rep}(\mathfrak{g}) = \{L(\lambda) \mid \lambda \in \bigoplus_{i \in I} \mathbb{N}\varpi_i\}$
- characters

$$\chi(V) = \sum_{\lambda \in \mathfrak{h}^*} \dim(V_\lambda) e^\lambda, \quad V_\lambda = \{v \in V \mid h{\cdot}v = \lambda(h)v, \forall h \in \mathfrak{h}\},$$

$\rightsquigarrow$ alg. emb. $\chi\colon K(\mathrm{Rep}(\mathfrak{g})) \hookrightarrow \mathbb{Z}[y_i^{\pm 1} \mid i \in I], \quad y_i := e^{\varpi_i}$

NB : $\chi(L(\lambda))$ is computed by Weyl's character formula.

General highest weight representations

A non-zero $U(\mathfrak{g})$-module V is called a highest weight module if $\exists v \in V, \lambda \in \mathfrak{h}^*$ s.t.

$$\mathfrak{n}_+ \cdot v = 0, \quad h \cdot v = \lambda(h)v \ (\forall h \in \mathfrak{h}), \quad V = U(\mathfrak{n}_-) \cdot v.$$

Then λ is called the highest weight of V.

For each $\lambda \in \mathfrak{h}^*$, Verma module $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda$ (universal h.w. module) has a unique simple quotient $L(\lambda)$:

$$M(\lambda) \twoheadrightarrow L(\lambda)$$

Then

$$\dim L(\lambda) < \infty \quad \Leftrightarrow \quad \lambda \text{ is dominant.}$$

A fundamental problem : $\chi(L(\lambda)) = ?$ for general λ

Kazhdan-Lusztig theory

W : Weyl group of $\mathfrak{g}$, $\rho := \sum_{i \in I} \varpi_i$

$$M_w := M(-w\rho - \rho), \quad L_w := L(-w\rho - \rho) \quad \text{for } w \in W.$$

Let $\mathcal{O}_0$ be the category of $\mathfrak{h}$-semisimple $U(\mathfrak{g})$-modules which are finite extensions of L_w for various $w \in W$. (*non-semisimple*)

We have two $\mathbb{Z}$-bases of $K(\mathcal{O}_0)$: $\{[L_w]\}_{w \in W}$ and $\{[M_w]\}_{w \in W}$.

$$[M_w] = [L_w] + \sum_{v < w} p_{w,v} [L_v].$$

To obtain $\chi(L_w)$, it suffices to compute $p_{w,v} = [M_w : L_v]$.

Theorem (Beilinson-Bernstein, Brylinski-Kashiwara 1981)

We have $p_{w,v} = p_{w,v}(1)$ with $p_{w,v}(t) \in \mathbb{Z}[t]$ being the Kazhdan-Lusztig polynomial, which is given by an algorithm.

NB : Proof uses the geometry of flag varieties.

Kazhdan-Lusztig polynomials

$\mathcal{H}_t(W) = \bigoplus_{w \in W} \mathbb{Z}[t^{\pm 1}] T_w$: Iwahori-Hecke algebra of W

- with the standard basis $\{T_w\}_{w \in W}$, and
- the bar-involution $\overline{T_w} = T_{w^{-1}}^{-1}, \bar{t} = t^{-1}$.

Kazhdan-Lusztig (1979) constructed the canonical basis (KL basis) $\{C_w\}_{w \in W}$ characterized by

1. $\overline{C_w} = C_w$, and
2. $T_w = C_w + \sum_{v<w} p_{w,v}(t) C_v$ for some $p_{w,v}(t) \in t\mathbb{Z}[t]$.

Here $p_{w,v}(t)$ is called Kazhdan-Lusztig polynomial.

Theorem (Kazhdan-Lusztig 1980)

The KL polynomial $p_{w,v}(t)$ coincides with the Poincaré polynomial of local intersection cohomology of a Schurbert variety.
Hence it is positive : $p_{w,v}(t) \in t\mathbb{N}[t]$.

NB : Elias-Williamson (2014) gave an algebraic proof of the positivity for general Coxeter group via Soergel's theory.

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Quantum loop algebras

$U_q(L\mathfrak{g})$: quantum loop algebra of $\mathfrak{g}$

- $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[z^{\pm 1}]$: loop algebra of $\mathfrak{g}$ (∞-dim/$\mathbb{C}$)
- $U_q(L\mathfrak{g})$ is a Hopf algebra over $\Bbbk := \overline{\mathbb{Q}(q)}$.

The category $\mathscr{C}_\mathfrak{g} := \{\text{fin-dim } U_q(L\mathfrak{g})\text{-modules (of type } \mathbf{1})\}$ is

- *non-semisimple* as abelian cat.
- *non-symmetric* as *rigid* $\otimes$-cat. (sometimes $V \otimes V' \not\simeq V' \otimes V$)

rigid = $\forall V \in \mathscr{C}_\mathfrak{g}$ has its right/left dual module $\mathcal{D}^{\pm 1}(V) \in \mathscr{C}_\mathfrak{g}$

$$V \otimes \mathcal{D}(V) \xrightarrow{can} \Bbbk \qquad \mathcal{D}^{-1}(V) \otimes V \xrightarrow{can} \Bbbk$$

NB : $\mathcal{D} \not\simeq \mathcal{D}^{-1}$, and $\{\mathcal{D}^k\}_{k \in \mathbb{Z}}$ defines a faithful action $\mathbb{Z} \curvearrowright \mathscr{C}_\mathfrak{g}$.

Simple modules

The decomp. $L\mathfrak{g} = L\mathfrak{n}_- \oplus L\mathfrak{h} \oplus L\mathfrak{n}_+$ induces

$$U_q(L\mathfrak{g}) = U_q(L\mathfrak{n}_-)U_q(L\mathfrak{h})U_q(L\mathfrak{n}_+),$$

which allows us to consider :

- ℓ-highest weight theory ($\ell =$ "loop").
- q-characters ($q =$ "quantum")

$$\chi_q(V) \in \mathcal{Y} := \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I, a \in \Bbbk^\times]$$

for $V \in \mathscr{C}_\mathfrak{g}$, where $Y_{i,a}$ is a quantum loop analog of $y_i = e^{\varpi_i}$.

Define the set of dominant monomials

$$\mathcal{M} := \{ \text{ monomials in } Y_{i,a} \text{ for various } (i,a) \in I \times \Bbbk^\times \ \} \subset \mathcal{Y}$$

Theorem (Chari-Pressley 1995)

$$\mathcal{M} \ni m \quad \overset{1:1}{\longleftrightarrow} \quad \exists! \ \ell\text{-h.w. module } L(m) \in \operatorname{Irr} \mathscr{C}_\mathfrak{g}$$

q-characters

Theorem (Frenkel-Reshetikhin 1999)

The q-character homomorphism yields an algebra embedding

$$\chi_q \colon K(\mathscr{C}_\mathfrak{g}) \hookrightarrow \mathcal{Y}, \quad [V] \mapsto \chi_q(V).$$

In particular, $K(\mathscr{C}_\mathfrak{g})$ is commutative.

Moreover, $K(\mathscr{C}_\mathfrak{g})$ is isomorphic to a polynomial ring :

$$K(\mathscr{C}_\mathfrak{g}) \simeq \mathbb{Z}[x_{i,a} \mid i \in I, a \in \mathbb{k}^\times], \quad [L(Y_{i,a})] \leftrightarrow x_{i,a}.$$

- $L(Y_{i,a})$ is called a fundamental module.
- $\chi_q(L(Y_{i,q}))$ is computable by an algorithm due to Frenkel-Mukhin (2001).

$\rightsquigarrow$ "standard basis" $\{$monomials in $[L(Y_{i,a})]\} \subset K(\mathscr{C}_\mathfrak{g})$

Standard modules

<u>Define</u> the standard module for each $m = \prod_{(i,a)} Y_{i,a}^{u_{i,a}(m)} \in \mathcal{M}$, as a suitably ordered tensor product :

$$M(m) := \overrightarrow{\bigotimes_{(i,a)\in I\times\mathbb{k}^\times}} L(Y_{i,a})^{\otimes u_{i,a}(m)} \stackrel{hd}{\twoheadrightarrow} L(m)$$

- $\chi_q(M(m)) = \prod_{(i,a)} \chi_q(L(Y_{i,q}))^{u_{i,a}(m)}$ is computable.

We have two bases of $K(\mathscr{C}_\mathfrak{g})$: $\{[L(m)]\}_{m\in\mathcal{M}}$ and $\{[M(m)]\}_{m\in\mathcal{M}}$.

$$[M(m)] = [L(m)] + \sum_{m'<m} P_{m,m'}[L(m')]$$

holds in $K(\mathscr{C}_\mathfrak{g})$ with a suitable ordering $<$ on $\mathcal{M}$.

<u>Want</u> to compute $\chi_q(L(m))$ or $P_{m,m'} = [M(m) : L(m')]$ with an analog of Kazhdan-Lusztig algorithm.

Quantum Grothendieck rings

QGR $\mathcal{K}_t(\mathscr{C}_\mathfrak{g})$ is a t-deformation of $K(\mathscr{C}_\mathfrak{g})$, introduced by

- Nakajima (2004) and Varagnolo-Vasserot (2003) for $\mathfrak{g}$ of simply-laced type (ADE type) using quiver varieties,
- Hernandez (2004) for general $\mathfrak{g}$ in a purely algebraic way.

$\mathcal{K}_t(\mathscr{C}_\mathfrak{g})$ is a certain $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of a quantum torus $\mathcal{Y}_t$

$$\mathcal{K}_t(\mathscr{C}_\mathfrak{g}) \quad \subset \quad \mathcal{Y}_t = (\mathbb{Z}[t^{\pm 1/2}] \otimes_\mathbb{Z} \mathcal{Y}, *)$$

Here the t-deformed product $*$ is defined by using the quantum Cartan matrix of $\mathfrak{g}$.

Define the (anti) bar-involution on $\mathcal{Y}_t$ by

$$\overline{f(t) \otimes y} := f(t^{-1}) \otimes y \quad \text{for } f(t) \in \mathbb{Z}[t^{\pm 1/2}], y \in \mathcal{Y}.$$

Standard (q,t)-characters

By a t-analog of Frenkel-Mukhin's algorithm, we can construct

$$F_t(Y_{i,a}) \in \mathcal{K}_t(\mathscr{C}_{\mathfrak{g}}) \quad \text{for each } (i,a) \in I \times \Bbbk^\times$$

such that $F_t(Y_{i,a})|_{t=1} = \chi_q(L(Y_{i,a}))$.

The standard (q,t)-character $M_t(m)$ for $m = \prod_{(i,a)} Y_{i,a}^{u_{i,a}(m)} \in \mathcal{M}$ is defined by

$$M_t(m) := t^\star \overset{\rightarrow}{\underset{(i,a)}{*}} F_t(Y_{i,a})^{u_{i,a}(m)} = 1 \otimes m + \cdots .$$

- $M_t(m)|_{t=1} = \chi_q(M(m))$,
- $\{M_t(m)\}_{m\in\mathcal{M}}$ forms a $\mathbb{Z}[t^{\pm 1/2}]$-basis of $\mathcal{K}_t(\mathscr{C}_{\mathfrak{g}})$.

Simple (q,t)-characters

Theorem (Hernandez 2004)

There exists a unique "canonical" $\mathbb{Z}[t^{\pm 1/2}]$-basis $\{L_t(m)\}_{m\in\mathcal{M}}$ of $\mathcal{K}_t(\mathscr{C}_{\mathfrak{g}})$, called the simple (q,t)-characters, characterized by :

1. $\overline{L_t(m)} = L_t(m)$, and
2. there are $P_{m,m'}(t) \in t\mathbb{Z}[t]$ such that

$$M_t(m) = L_t(m) + \sum_{m'<m} P_{m,m'}(t) L_t(m').$$

Here $P_{m,m'}(t)$ is *an analog of Kazhdan-Lusztig polynomial*.

Write

$$L_t(m_1) * L_t(m_2) = \sum_{m\in\mathcal{M}} c^{m}_{m_1,m_2}(t) L_t(m),$$

with $c^{m}_{m_1,m_2}(t) \in \mathbb{Z}[t^{\pm 1/2}]$: t*-analog of structure constants*.

Kazhdan-Lusztig type conjectures

Conjecture (An analog of KL conjecture)

(KL) $L_t(m)|_{t=1} = \chi_q(L(m))$ for all $m \in \mathcal{M}$.

$(\Leftrightarrow P_{m,m'}(1) = P_{m,m'}.)$

Conjecture (Positivity conjectures)

(P1) $P_{m,m'}(t) \in t\mathbb{N}[t]$;

(P2) $c^m_{m_1,m_2}(t) \in \mathbb{N}[t^{\pm 1/2}]$;

(P3) $L_t(m)$ has non-negative coefficient in $\mathcal{Y}_t$.

Theorem (Nakajima 2004, Varagnolo-Vasserot 2003)

When $\mathfrak{g}$ is of simple-laced type, all the above conjectures are true.

NB : Proof uses the geometry of quiver varieties.
In particular, $P_{m,m'}(t)$ coincides with a Poincaré polynomial of a local intersection cohomology of a (graded) quiver variety.

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Theorem (F.-Hernandez-Oh-Oya)

1. The conjectures (P1) and (P2) are true for all $\mathfrak{g}$.
2. The conjectures (KL) and (P3) are true for $\mathfrak{g}$ of type B.

NB : (P1) follows from (P2) as $M_t(m) = t^{\star} \overrightarrow{\ast} L_t(Y_{i,a})^{u_{i,a}(m)}$.

Ideas of the proof

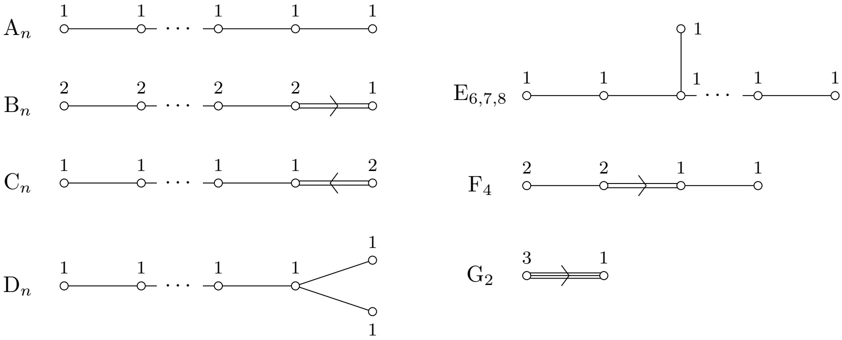
- Reduce the situation to a "skeleton" subcategory $\mathscr{C}_{\mathfrak{g},\mathbb{Z}} \subset \mathscr{C}_{\mathfrak{g}}$ introduced by Hernandez-Leclerc (2015, 2016).
- Proof of (P2) : Construct an isomorphism

$$\Psi \colon \mathcal{K}_t(\mathscr{C}_{\mathfrak{g},\mathbb{Z}}) \simeq \mathcal{K}_t(\mathscr{C}_{\tilde{\mathfrak{g}},\mathbb{Z}})$$

  where $\tilde{\mathfrak{g}}$ is of "unfolded" simply-laced type.
- Proof of (KL) in type B : Compare our isomorphism Ψ with the generalized quantum affine Schur-Weyl duality due to Kashiwara-Kim-Oh (2019).

Notation

- $r = \begin{cases} 1 & \text{if } \mathfrak{g} \text{ is of type ADE (simply-laced),} \\ 2 & \text{if } \mathfrak{g} \text{ is of type BCF (doubly-laced),} \\ 3 & \text{if } \mathfrak{g} \text{ is of type G (triply-laced).} \end{cases}$
- $d_i \in \{1, r\}$: the minimal symmetrizing numbers



Notation

Fix a "parity function" $\epsilon \colon I \to \{0,1\}$ satisfying

$$\epsilon_i \equiv \epsilon_j + \min(d_i, d_j) \pmod 2 \quad \text{if } i \text{ and } j \text{ are adjacent.}$$

Define $\quad \widehat{I} := \{(i,p) \in I \times \mathbb{Z} \mid p \equiv \epsilon_i \pmod 2\}$

Examples in type A_5 & B_3

$(i \setminus p)$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	•		•		•		•		•		•		•		•		•		•		•		•		•
2		•		•		•		•		•		•		•		•		•		•		•		•	
3	•		•		•		•		•		•		•		•		•		•		•		•		•
4		•		•		•		•		•		•		•		•		•		•		•		•	
5	•		•		•		•		•		•		•		•		•		•		•		•		•

$(i \setminus p)$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	•		•		•		•		•		•		•		•		•		•		•		•		•
2	•		•		•		•		•		•		•		•		•		•		•		•		•
3		•		•		•		•		•		•		•		•		•		•		•		•	

Hernandz-Leclerc's category $\mathscr{C}_{\mathfrak{g},\mathbb{Z}}$

Define

$$\widehat{I} := \{(i,p) \in I \times \mathbb{Z} \mid p \equiv \epsilon_i \pmod 2\}$$
$$\mathcal{M}_{\mathbb{Z}} := \{\text{monomials in } Y_{i,q^p} \text{ for various } (i,p) \in \widehat{I}\ \} \subset \mathcal{M}$$
$$\mathscr{C}_{\mathfrak{g},\mathbb{Z}} \subset \mathscr{C}_{\mathfrak{g}} : \text{ Serre subcat. gen'd by } \{L(m)\}_{m \in \mathcal{M}_{\mathbb{Z}}}$$

Proposition (Hernandez-Leclerc)

- $\mathscr{C}_{\mathfrak{g},\mathbb{Z}}$ is stable under taking $\otimes$ and $\mathcal{D}^{\pm 1}$ (rigid $\otimes$-subcat),
- $\mathscr{C}_{\mathfrak{g},\mathbb{Z}}$ is a "skeleton" of $\mathscr{C}_{\mathfrak{g}}$. In fact, we have

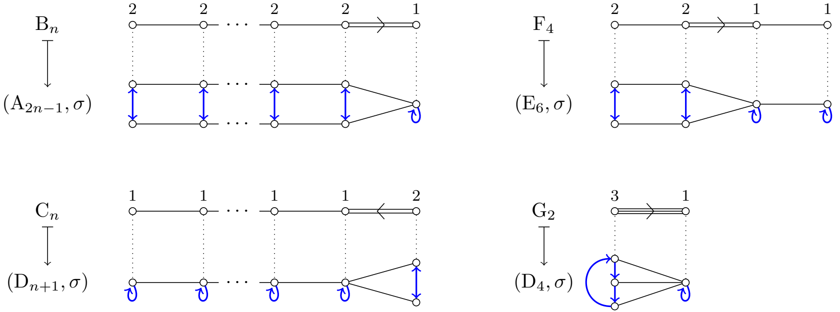
$$K(\mathscr{C}_{\mathfrak{g}}) \simeq \bigotimes_{c \in \Bbbk^\times / q^{2\mathbb{Z}}}' K(\tau_c^* \mathscr{C}_{\mathfrak{g},\mathbb{Z}})$$

where $\tau_c \in \mathrm{Aut}(U_q(L\mathfrak{g}))$ is a spectral param twist $z \mapsto cz$.

Unfoldings

Let $\mathfrak{g}$ be of non-simply-laced.

$$\mathfrak{g} \mapsto (\tilde{\mathfrak{g}}, \sigma) \text{ : simply-laced Lie algebra + diagram automorphism}$$



We can identify

- r = the order of σ.
- $I_{\mathfrak{g}} = I_{\tilde{\mathfrak{g}}}/\sigma$ and $d_i = \#$ of the corresponding σ-orbit.

QGR isomorphism between $\mathfrak{g} \leftrightarrow \tilde{\mathfrak{g}}$

Theorem (F.-Hernandez-Oh-Oya)

There is a (concrete) isomorphism of $\mathbb{Z}[t^{\pm 1/2}]$-algebras

$$\Psi\colon \mathcal{K}_t(\mathscr{C}_{\mathfrak{g},\mathbb{Z}}) \simeq \mathcal{K}_t(\mathscr{C}_{\tilde{\mathfrak{g}},\mathbb{Z}})$$

satisfying several nice properties :

- Ψ induces a bijection between the simple (q,t)-characters.
- Ψ is compatible with "t-analog of dualities" $\mathcal{D}_t^{\pm 1}$.

Thm proves (P2) and hence (P1). "*propagation of positivity*"

Remark

1. Ψ does not respect the standard (q,t)-characters.
2. Actually, we construct a collection of such isomorphisms Ψ labeled by pairs of Q-data : combinatorial generalization of a Dynkin quiver with a height function [F.-Oh, 2020].

Generalized quantum affine Schur-Weyl duality

Theorem (Kashiwara-Kim-Oh 2019)

When $(\mathfrak{g}, \tilde{\mathfrak{g}}) = (\mathrm{B}_n, \mathrm{A}_{2n-1})$, there are exact monoidal functors

$$\mathscr{C}_{\mathrm{B}_n,\mathbb{Z}} \leftarrow H_{\mathrm{A}_\infty}\text{-mod}_{\mathrm{fd}} \rightarrow \mathscr{C}_{\mathrm{A}_{2n-1},\mathbb{Z}}$$

where H_{A_∞} is the quiver Hecke (KLR) algebra of type A_∞.
It induces an ring isomorphism $F\colon K(\mathscr{C}_{\mathrm{B}_n,\mathbb{Z}}) \simeq K(\mathscr{C}_{\mathrm{A}_{2n-1},\mathbb{Z}})$ respecting the simple classes.

Theorem (F.-Hernandez-Oh-Oya)

When $\mathfrak{g}$ is of type B, the conjecture (KL) is true.

$$\begin{array}{ccc} \mathcal{K}_t(\mathscr{C}_{\mathrm{B}_n,\mathbb{Z}}) & \xrightarrow{\Psi} & \mathcal{K}_t(\mathscr{C}_{\mathrm{A}_{2n-1},\mathbb{Z}}) \\ \downarrow{\scriptstyle t=1} & & \downarrow{\scriptstyle t=1} \\ K(\mathscr{C}_{\mathrm{B}_n,\mathbb{Z}}) & \xrightarrow{F} & K(\mathscr{C}_{\mathrm{A}_{2n-1},\mathbb{Z}}) \end{array} \qquad \begin{array}{ccc} L_t^{\mathrm{B}_n}(m) & \longleftrightarrow & L_t^{\mathrm{A}_{2n-1}}(m') \\ \vdots{\scriptstyle t=1} & & \downarrow{\scriptstyle t=1} \\ [L^{\mathrm{B}_n}(m)] & \longleftrightarrow & [L^{\mathrm{A}_{2n-1}}(m')] \end{array}$$

Construction of Ψ

Eventually, it turns out that the *localized* QGR

$$\mathcal{K}_t(\mathscr{C}_{\mathbb{Z},\mathfrak{g}})_{loc} := \mathcal{K}_t(\mathscr{C}_{\mathbb{Z},\mathfrak{g}}) \otimes_{\mathbb{Z}[t^{\pm 1/2}]} \mathbb{Q}(t^{1/2})$$

shares the same presentation with $\mathcal{K}_t(\mathscr{C}_{\mathbb{Z},\tilde{\mathfrak{g}}})_{loc}$.

Theorem (Hernandez-Leclerc 2015, for simply-laced $\tilde{\mathfrak{g}}$)

For a Dynkin quiver Q of the same type as $\tilde{\mathfrak{g}}$, we have

$$\mathcal{K}_t(\mathscr{C}_{\mathbb{Z},\tilde{\mathfrak{g}}})_{loc} \simeq DH(Q)$$

where $DH(Q)$ is the derived Hall algebra of $D^b(\mathrm{Rep}(Q))$.

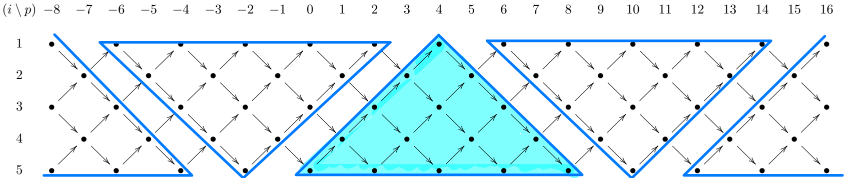
- The shift functor $[\pm 1]$ of $D^b(\mathrm{Rep}(Q))$ corresponds to the duality $\mathcal{D}^{\pm 1}$ of $\mathscr{C}_{\tilde{\mathfrak{g}},\mathbb{Z}}$.
- The heart $\mathrm{Rep}(Q) \subset D^b(\mathrm{Rep}(Q))$ corresponds to a certain monoidal subcategory $\mathscr{C}_Q \subset \mathscr{C}_{\tilde{\mathfrak{g}},\mathbb{Z}}$.

Roughly speaking ...

$$\begin{array}{ccc} \mathscr{C}_{\tilde{\mathfrak{g}},\mathbb{Z}} & \longleftrightarrow & D^b(\mathrm{Rep}(Q)) \\ \cup & & \cup \\ \mathscr{C}_Q & \longleftrightarrow & \mathrm{Rep}(Q) \end{array}$$

$$\begin{array}{ccc} \otimes & \longleftrightarrow & \text{extension} \\ \mathcal{D}^{\pm 1} & \longleftrightarrow & [\pm 1] \end{array}$$

Example in type A_5



HL isomorphism

Theorem (Hernandez-Leclerc 2015)

For any Dynkin quiver Q (with a choice of height function), we have an isomorphism

$$\begin{array}{cccc} \Phi_Q\colon & \mathcal{K}_t(\mathscr{C}_Q) & \simeq & \mathcal{A}_t[\tilde{N}_+], \\ & \cup & & \cup \\ & \{L_t(m)\} & \leftrightarrow & \text{dual canonical basis} \end{array}$$

where $\mathcal{A}_t[\tilde{N}_+]$ is the quantized coordinate ring of the unipotent group $\tilde{N}_+ = \exp(\tilde{\mathfrak{n}}_+)$ associated with $\tilde{\mathfrak{g}}$.

After localization, we have $\mathcal{K}_t(\mathscr{C}_Q)_{loc} \simeq U_t(\tilde{\mathfrak{n}}_+)$ and find

- $\mathcal{K}_t(\mathscr{C}_Q)_{loc}$ is presented by the q-Serre relations,
- Relations among $\mathcal{D}_t^k \mathcal{K}_t(\mathscr{C}_Q)_{loc}$ for various $k \in \mathbb{Z}$ are described by the q-Boson relations,

which yields the desired presentation and $\mathcal{K}_t(\mathscr{C}_{\tilde{\mathfrak{g}},\mathbb{Z}})_{loc} \simeq DH(Q)$.

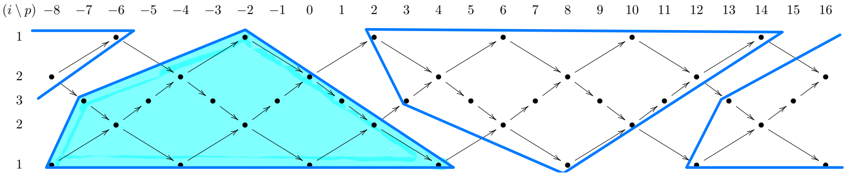
Generalization to non-simply-laced cases

For non-simply-laced $\mathfrak{g}$, we use the notion of Q-datum $\mathcal{Q}$ for $\mathfrak{g}$ to define a subcategory $\mathscr{C}_{\mathcal{Q}} \subset \mathscr{C}_{\mathfrak{g},\mathbb{Z}}$. [Oh-Suh 2019], [F.-Oh 2020]

A Q-datum for $\mathfrak{g}$ is a triple $\mathcal{Q} = (\tilde{\mathfrak{g}}, \sigma, \xi)$ where

- $(\tilde{\mathfrak{g}}, \sigma)$ is the unfolding of $\mathfrak{g}$ as before.
- $\xi \colon I_{\tilde{\mathfrak{g}}} \to \mathbb{Z}$ is a generalized height function, which subjects to certain axioms.

Example in type B_3



HLO isomorphism

Theorem (Hernandez-Oya in type B, FHOO in general)

For each Q-datum $\mathcal{Q}$ for $\mathfrak{g}$, we have an isomorphism

$$\begin{array}{cccc} \Phi_{\mathcal{Q}}\colon & \mathcal{K}_t(\mathscr{C}_{\mathcal{Q}}) & \simeq & \mathcal{A}_t[\tilde{N}_+] \\ & \cup & & \cup \\ & \{L_t(m)\} & \leftrightarrow & \text{dual canonical basis,} \end{array}$$

where $\mathcal{A}_t[\tilde{N}_+]$ is associated with $\tilde{\mathfrak{g}}$ (not $\mathfrak{g}$!!).

- After more efforts, we find that $\mathcal{K}_t(\mathscr{C}_{\mathbb{Z},\tilde{\mathfrak{g}}})_{loc}$ and $\mathcal{K}_t(\mathscr{C}_{\mathbb{Z},\mathfrak{g}})_{loc}$ share the same presentation and hence

$$\mathcal{K}_t(\mathscr{C}_{\mathbb{Z},\mathfrak{g}})_{loc} \simeq \mathcal{K}_t(\mathscr{C}_{\mathbb{Z},\tilde{\mathfrak{g}}})_{loc}.$$

- We prove that it preserves the simple (q,t)-characters.