Equivariant multiplicities via representations of quantum affine algebras

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1. Cluster structures on coordinate rings
2. Equivariant multiplicities and Mirković-Vilonen bases
3. Categorification of $\mathbb{C}[N]$ via representations of quantum affine algebras
4. Main results

Coordinate rings

We consider a simple finite-dimensional Lie algebra $\mathfrak{g}$ and we denote by $\mathfrak{n}$ the nilpotent subalgebra arising from a triangular decomposition of $\mathfrak{g}$. Let $\mathbf{N}$ denote the corresponding Lie group and let us consider the ring $\mathbb{C}[\mathbf{N}]$ of regular functions on $\mathbf{N}$.

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Good bases of $\mathbb{C}[\mathbf{N}]$:

- Dual canonical basis (Lusztig) / Upper global basis (Kashiwara)
- Dual semicanonical basis (Lusztig)
- Mirković-Vilonen basis

Remarkable multiplicative properties of these bases ?

Example

Assume $\mathfrak{g}$ of type A_2 i.e. $\mathfrak{g} = \mathfrak{sl}_3$. Then

$$\mathbf{N} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad a, b, c \in \mathbb{C} \right\}.$$

It is natural to consider functions sending a matrix in $\mathbf{N}$ onto certain of its minors :

$$x : M \mapsto \Delta_{\{1\},\{2\}}(M) \qquad x' : M \mapsto \Delta_{\{1,2\},\{1,3\}}(M)$$
$$y : M \mapsto \Delta_{\{1,2\},\{2,3\}}(M) \qquad z : M \mapsto \Delta_{\{1\},\{3\}}(M).$$

Example

The respective images under x, x', y and z of an element of $\mathbf{N}$ of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

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In this case the dual canonical and dual semicanonical bases coincide and can be written as follows :

$$\left\{ x^i y^j z^k, i, j, k \in \mathbb{N} \right\} \bigcup \left\{ x'^i y^j z^k, i, j, k \in \mathbb{N} \right\}.$$

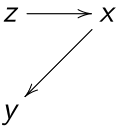
Cluster algebras

Initial data :

- $N \geqslant 1$ algebraically independant variables $x_1, \ldots, x_N$,
- a quiver Q with N vertices, without any loops or two cycles.

Such a data is called a **seed**.

Example :



Mutations of seeds

Mutation in the direction k :

- the variables $x_i, i \neq k$ are left unchanged,
- one remplaces x_k by a new variable x'_k given by :

$$x'_k := \frac{1}{x_k}\left(\prod_{j\to k} x_j + \prod_{k\to l} x_l\right)$$

- one remplaces Q by a new quiver Q', uniquely determined by Q and k.

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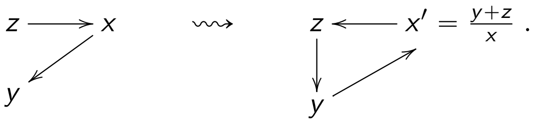

Definition

Let $\mathbb{T}$ be the tree whose nodes correspond to seeds and edges to mutations. The variables $x_1^{\mathcal{S}}, \dots, x_N^{\mathcal{S}}, \mathcal{S} \in \mathbb{T}$ are called **cluster variables**.

Definition : cluster algebra (Fomin-Zelevinsky, 2000)

The cluster algebra $\mathcal{A}$ associated to the initial seed $((x_1, \dots, x_N), Q)$ is the $\mathbb{Q}$-sub-algebra of $\mathbb{Q}(x_1, \dots, x_N)$ generated by the cluster variables.

Cluster structure on $\mathbb{C}[\mathbf{N}]$ and flag minors

Theorem (Berenstein-Fomin-Zelevinsky, Geiss-Leclerc-Schröer)

1. The algebra $\mathbb{C}[\mathbf{N}]$ has a cluster algebra structure of rank equal to the length of w_0 (= number of positive roots).
2. Construction of a (finite) family of seeds in $\mathbb{C}[\mathbf{N}]$ called *standard seeds* indexed by the set of reduced expressions of w_0.

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2. Construction of a (finite) family of seeds in $\mathbb{C}[\mathbf{N}]$ called *standard seeds* indexed by the set of reduced expressions of w_0.

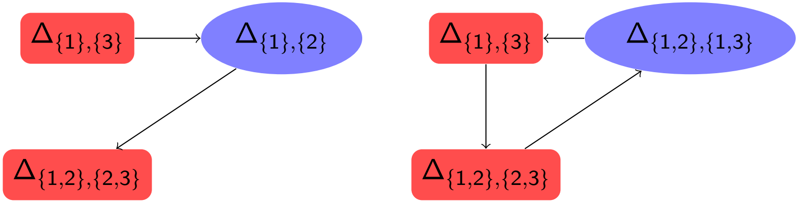
For every reduced expression $\mathbf{i}$ of w_0, $\mathcal{S}^{\mathbf{i}} = ((x_1^{\mathbf{i}}, \ldots, x_N^{\mathbf{i}}), Q^{\mathbf{i}})$. For each $1 \leqslant k \leqslant N$ the cluster variable $x_k^{\mathbf{i}}$ is called **the kth flag minor associated to $\mathbf{i}$**.

Example

Let us go back to the example $\mathfrak{g} = \mathfrak{sl}_3$. One has

$$w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$$

and the cluster structure of $\mathbb{C}[N]$ is composed of two distinct seeds :



Equivariant multiplicities

Let X be a closed projective scheme, endowed with the action of a torus T. Let X^T denote the set of fixed points in X for this action.

Proposition (Brion)

The set of classes of points in X^T forms a basis of $H_\bullet^T(X)$.

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The set of classes of points in X^T forms a basis of $H_\bullet^T(X)$.

In particular, for any closed T-invariant subvariety $Y \subset X$ one has :

$$[Y] = \sum_{p \in X^T} \epsilon_p^T(Y)[\{p\}].$$

Definition (Joseph, Rossmann, Brion)

For every $p \in X^T$, and for each closed T-invariant subvariety $Y \subset X$, the coefficient $\epsilon_p^T(Y) \in Frac(\mathbb{C}[T])$ is the (T)-equivariant multiplicity of Y at the point p.

Geometric Satake Correspondence and MV bases

Let $\mathbf{G}$ be the Lie goup of $\mathfrak{g}$ and let $\mathbf{G}^\vee$ be its Langlands dual. Fix a maximal torus $\mathbf{T}^\vee$ in $\mathbf{G}^\vee$ and a Borel subgroup $\mathbf{B}^\vee$ of $\mathbf{G}^\vee$ containing $\mathbf{T}^\vee$.

We set

$$\mathcal{O} := \mathbb{C}[[t]] \quad \text{and} \quad \mathcal{K} := \mathbb{C}((t)).$$

and we define the affine Grassmannian of $G^\vee$ as :

$$Gr_{\mathbf{G}^\vee} := \mathbf{G}^\vee(\mathcal{K})/\mathbf{G}^\vee(\mathcal{O}).$$

Geometric Satake Correspondence and MV bases

For every $\lambda \in P^+$ we let $L(\lambda)$ denote the finite-dimensional irreducible representation of $\mathbf{G}$ of highest weight λ.

Geometric Satake Correspondence (Mirković-Vilonen, 2000)

For each $(\lambda, \mu) \in P^+ \times P$, there is a closed subvariety $\mathcal{MV}^{\lambda,\mu}$ of $Gr_{\mathbf{G}^\vee}$ such that there is an isomorphism

$$H_\bullet(\mathcal{MV}^{\lambda,\mu}) \simeq L(\lambda)_\mu.$$

The irreducible components of $\mathcal{MV}^{\lambda,\mu}$ are called the **MV cycles** of type λ and of weight μ.

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Consequence

One can construct a basis of $\mathbb{C}[\mathbf{N}]$, whose elements are indexed by the set of (stable) Mirković-Vilonen cycles.

Equivariant multiplicities of MV cycles

In a recent work, Baumann-Kamnitzer-Knutson investigated the equivariant multiplicities of Mirković-Vilonen cycles.

The torus $\mathbf{T}^\vee(\mathbb{C})$ acts on $Gr_{\mathbf{G}^\vee}$, with

$$Gr_{\mathbf{G}^\vee}^{\mathbf{T}^\vee} := \{L_\mu, \mu \in P\}.$$

where P denotes the weight lattice of $\mathbf{G}$.
For any MV cycle $Z \subset Gr_{\mathbf{G}^\vee}$ and for each $\mu \in P$, let us consider the equivariant multiplicity

$$\epsilon_{L_\mu}^{\mathbf{T}^\vee}(Z) \in \mathbb{C}(\alpha_1, \ldots, \alpha_n).$$

Baumann-Kamnitzer-Knutson's morphism $\bar{D}$

For every $i \in I$, choose $e_i \in \mathfrak{n}_{\alpha_i}$ and for every sequence $\mathbf{j} = (j_1, \ldots, j_d)$ set $e_{\mathbf{j}} := e_{j_1} \cdots e_{j_d} \in U(\mathfrak{n})$. Then consider the map

$$\begin{array}{cccc} \bar{D}: & \mathbb{C}[N] & \longrightarrow & \mathbb{C}(\alpha_1, \ldots, \alpha_n) \\ & f & \longmapsto & \sum_{\mathbf{j}} (f, e_{\mathbf{j}}) \frac{1}{\alpha_{j_1}(\alpha_{j_1}+\alpha_{j_2})\cdots(\alpha_{j_1}+\cdots+\alpha_{j_d})}. \end{array}$$

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Theorem (Baumann-Kamnitzer-Knutson, 2020)

1. The map $\bar{D}$ is an algebra morphism.
2. If Z is a stable MV cycle of weight μ and if b_Z is the corresponding element of the MV basis of $\mathbb{C}[\mathbf{N}]$, then one has

$$\bar{D}(b_Z) = \epsilon_{L_\mu}^{\mathbf{T}^\vee}(Z).$$

Evaluation of $\bar{D}$ on the flag minors of $\mathbb{C}[\mathbf{N}]$

Conjecture (C, 2020)

Let $\mathfrak{g}$ be a simple Lie algebra of arbitrary simply-laced type. Then for any reduced expressions $\mathbf{i}$ of w_0, the flag minors of the standard seed $\mathcal{S}^{\mathbf{i}}$ satisfy the following properties :

1. For every $1 \leqslant j \leqslant N$, $\bar{D}(x_j^{\mathbf{i}}) = 1/P_j$ where P_j is a product of positive roots.
2. $\forall 1 \leqslant j \leqslant N, P_j P_{j_-(\mathbf{i})} = \beta_j \prod_{l<j<l_+, i_l \sim i_j} P_l$.
3. $\forall 1 \leqslant j \leqslant N, \forall 1 \leqslant i \leqslant N, \quad P_{j_+(\mathbf{i})}/P_j \underset{\beta_i \to 0}{=} \mathcal{O}(1/\beta_i)$.

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Theorem 1 (C, 2020)

Assume $\mathfrak{g}$ is of type $A_n, n \geqslant 1$ or D_4. Then the Conjecture holds.

Motivations

1. The morphism $\bar{D}$ admits natural expressions in terms of various categorifications of $\mathbb{C}[\mathbf{N}]$, such as quiver Hecke (Khovanov-Lauda-Rouquier) algebras or representations of preprojective algebras (Geiss-Leclerc-Schröer).
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2. The proof of Theorem 1 relies on certain monoidal categorification results via quiver Hecke algebras, due to Kang-Kashiwara-Kim-Oh and Kashiwara-Kim. The same techniques seem unlikely to be appropriate for the cases $D_n, n \geqslant 5$ and $E_n, n = 6, 7, 8$.

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3. Meaning of these properties, especially the polynomial identities 2) ?

Representations of quantum affine algebras

- $U_q(\widehat{\mathfrak{g}})$: the quantum affine algebra associated to a simply-laced type Lie algebra $\mathfrak{g}$.
- $\mathcal{C}$: the monoidal category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$.
- $\mathcal{Y}$: the torus $\mathbb{Z}[Y_{i,a}^{\pm 1}, i \in I, a \in \mathbb{C}^{\times}]$ where the $Y_{i,a}$ are some indeterminates.
- There is an injective ring morphism (Frenkel-Reshetikhin, 1999)

$$\chi_q : K_0(\mathcal{C}) \longrightarrow \mathcal{Y}$$

inducing a bijection $\mathfrak{m} \longmapsto L(\mathfrak{m})$ between the set of honest monomials in the $Y_{i,a}$ (called dominant monomials) and the simple objects in $\mathcal{C}$ up to isomorphism.
The simples $L(Y_{i,a})$ are called fundamental representations.

Hernandez-Leclerc's category $\mathcal{C}_Q$

Fix an orientation Q of the Dynkin diagram of $\mathfrak{g}$ and let ξ be a height function adapted to Q i.e.

$$\xi(j) = \xi(i) - 1 \qquad \text{if there exists an arrow } i \to j \text{ in } Q.$$

Definition (Hernandez-Leclerc)

Let $\mathcal{C}_Q$ denote the monoidal subcategory generated by the fundamental representations $L(Y_{i,q^s}), i \in I, s \in S_i$ where S_i is a finite set contained in $\xi(i) + 2\mathbb{Z}$.

For simplicity we write $Y_{i,s} := Y_{i,q^s}$. The modules

$$X_{i,k}^{(s)} := L(Y_{i,s} Y_{i,s+2} \cdots Y_{i,s+2k-2}) \quad i \in I, s \in \mathbb{Z}, k \geqslant 1$$

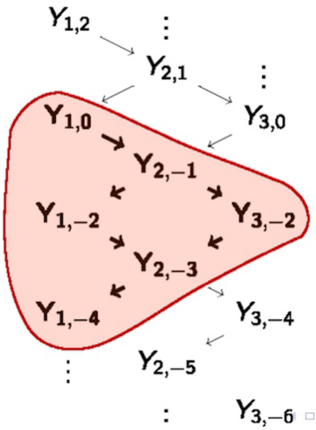
are called Kirillov-Reshetikhin modules.

Theorem (Hernandez-Leclerc, 2015)

1. There is an algebra isomorphism $\mathbb{C} \otimes K_0(\mathcal{C}_Q) \simeq \mathbb{C}[\mathbf{N}]$ inducing a bijection from the set of isomorphism classes of simple objects in $\mathcal{C}_Q$ to the elements of the dual canonical basis of $\mathbb{C}[\mathbf{N}]$.
2. Furthermore, if $\mathbf{i}_Q$ denotes a reduced expression of w_0 adapted to Q, then the cluster variables of the standard seed $\mathcal{S}^{\mathbf{i}_Q}$ are identified with the classes of the Kirillov-Reshetikhin modules of the form $X_{i,s} := L(Y_{i,s} Y_{i,s+2} \cdots Y_{i,\xi(i)})$.

Example in type A_3

$$\mathfrak{g} = \mathfrak{sl}_4,\ Q = 1 \to 2 \to 3,\ \xi(1) = 0, \xi(2) = -1, \xi(3) = -2.$$



Truncated q-characters

- Let $\mathcal{Y}_Q$ be the subtorus $\mathcal{Y}$ given by

$$\mathcal{Y}_Q := \mathbb{Z}[Y_{i,s}^{\pm 1}, i \in I, s \in S_i].$$

- There is an injective ring homomorphism

$$\widetilde{\chi}_q : K_0(\mathcal{C}_Q) \longrightarrow \mathcal{Y}_Q$$

  called **truncated q-character**, defined as follows : for every module V in $\mathcal{C}_Q$, kill all the terms of $\chi_q(V)$ that do not belong to $\mathcal{Y}_Q$.

- For example $\widetilde{\chi}_q(X_{i,s}) = Y_{i,s} \cdots Y_{i,\xi(i)}$.

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Inverses of quantum Cartan matrices

Let $\mathfrak{g}$ be of simply-laced type and let $C(z)$ be the corresponding quantum Cartan matrix :

$$C_{i,j}(z) := \begin{cases} z + z^{-1} & \text{if } i = j, \\ -1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is invertible :

$$C(z)^{-1} = \tilde{C}(z) = \left(\tilde{C}_{i,j}(z)\right)_{i,j}.$$

For every $i, j \in I$ we write :

$$\tilde{C}_{i,j}(z) = \sum_{m \geqslant 1} \tilde{C}_{i,j}(m) z^m.$$

Inverses of quantum Cartan matrices

By convention we extend this definition to all integers by setting $\tilde{C}_{i,j}(m) := 0$ if $m \leqslant 0$.

Lemma (consequence of the definition)

For every $i, j \in I$ we have

$$\begin{cases} \tilde{C}_{i,j}(m+1) + \tilde{C}_{i,j}(m-1) - \sum_{i \sim k} \tilde{C}_{i,k}(m) = 0 & \text{for any } m \neq 0 \\ \tilde{C}_{i,j}(1) = \delta_{i,j}. \end{cases}$$

Hernandez-Leclerc show that these coefficients can be understood in terms of the Euler form associated to a chosen orientation of the Dynkin diagram of $\mathfrak{g}$.

Auslander-Reiten-theoretic notations

- $\mathbf{i}_Q$: an arbitrary reduced expression of w_0 adapted to Q.
- $n_Q(i)$: the number of occurrences of the letter i in the reduced word $\mathbf{i}_Q$.
- τ_Q : the Coxeter transformation associated to Q.
- $\gamma_i := \sum_{j \in B(i)} \alpha_j$ where $B(i)$ is the set of indices j such that there is a path from j to i in Q.

Then we have

$$\Phi_+ = \{\tau_Q^{r-1}(\gamma_i), i \in I, 1 \leqslant r \leqslant n_Q(i)\}.$$

Definition of $\widetilde{D}_Q$

We define an algebraic morphism $\widetilde{D}_Q : \mathbb{C} \otimes \mathcal{Y}_Q \longrightarrow \mathbb{C}(\alpha_1, \ldots, \alpha_n)$ as follows :

$$\forall i \in I, s \in S_i, \quad \widetilde{D}_Q(Y_{i,s}) := \prod_{\substack{j \in I \\ p \in S_j}} \left(\tau_Q^{(\xi(j)-p)/2}(\gamma_j) \right)^{\tilde{C}_{i,j}(p-s-1) - \tilde{C}_{i,j}(p-s+1)}$$

Lemma (C-Li)

For each $i \in I$ and $s \in S_i$, one has

$$\widetilde{D}_Q(\widetilde{\chi}_q(X_{i,s})) = \prod_{\substack{j \in I \\ 1 \leqslant l \leqslant n_Q(j)}} \frac{1}{\left(\tau^{(\xi(j)-p)/2}(\gamma_j)\right)^{\tilde{C}_{i,j}(p-s+1)}}.$$

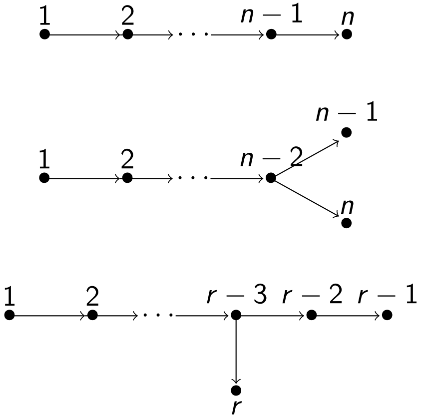
Main result 1

Theorem 2 (C-Li)

Let $\mathfrak{g}$ be a simply-laced type Lie algebra and let Q be an arbitrary orientation of the Dynkin diagram of $\mathfrak{g}$. Then the following diagram commutes :

$$\begin{array}{ccccc} \mathbb{C}[\mathbf{N}] & \xrightarrow{\simeq} & \mathbb{C} \otimes K_0(\mathcal{C}_Q) & \xhookrightarrow{\tilde{\chi}_q} & \mathbb{C} \otimes \mathcal{Y}_Q \\ & \searrow^{\bar{D}} & & \swarrow_{\tilde{D}_Q} & \\ & & \mathbb{C}(\alpha_i, i \in I) & & \end{array}$$

Choice of a particular orientation Q_0.



Type A_n

Type D_n

Types $E_r, r = 6, 7, 8$

Proof in the case $Q = Q_0$

- **Types** $A_n, n \geqslant 1$ **and** $D_n, n \geqslant 4$. Check that $\widetilde{D}_{Q_0}$ and $\bar{D}$ agree on the classes of the fundamentals modules of $\mathcal{C}_Q$.

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  1. Evaluation of $\widetilde{D}_{Q_0}$: the classes of Kirillov-Reshetikhin modules in $\mathcal{C}_Q$ are cluster variables, related to each other by explicit sequences of cluster mutations in $\mathbb{C}[\mathbf{N}]$ (T-systems) :

$$[X_{i,k}^{(s)}][X_{i,k}^{(s-2)}] = [X_{i,k+1}^{(s-2)}][X_{i,k-1}^{(s)}] + \prod_{j \sim i} [X_{j,k}^{(s-1)}].$$

  The previous Lemma provides the initial step.

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- **Types** $E_n, n = 6, 7, 8$. We use a computer software.

Main result 2

Theorem 3 (C-Li)

Let $\mathfrak{g}$ be a simple Lie algebra of arbitrary simply-laced type. Then for any reduced expressions $\mathbf{i}$ of w_0, the flag minors of the standard seed $\mathcal{S}^{\mathbf{i}}$ satisfy the following properties :

1. For every $1 \leqslant j \leqslant N$, $\bar{D}(x_j^{\mathbf{i}}) = 1/P_j$ where P_j is a product of positive roots.
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3. $\forall 1 \leqslant j \leqslant N, \forall 1 \leqslant i \leqslant N, \quad P_{j_+(\mathbf{i})}/P_j \underset{\beta_i \to 0}{=} \mathcal{O}\left(1/\beta_i\right)$.

Sketch of proof

It suffices to prove the desired statement for the standard seed $\mathcal{S}^{\mathbf{i}_{Q_0}}$. We set $\widetilde{D} := \widetilde{D}_{Q_0}$.

$$\widetilde{D}(\widetilde{\chi}_q(X_{i,s})) = \prod_{j,l} \frac{1}{\left(\tau_{Q_0}^{(\xi_0(j)-p)/2}(\gamma_j)\right)^{\tilde{C}_{i,j}(p-s+1)}} \Longrightarrow \text{Property 1)}$$

recursive relations between the inverse quantum Cartan matrix coefficients $\tilde{C}_{i,j}(m)$ ⟹ Property 2)

Hernandez-Leclerc's expression of $\tilde{C}_{i,j}(m)$ in terms of the Euler form of Q ⟹ Property 3)

Main result 1 : the general case

Let Q be an arbitrary orientation of the Dynkin diagram of $\mathfrak{g}$.

- Use Theorem 4 for the standard seed $\mathcal{S}^{\mathbf{i}_Q}$. In particular, one can use the Property 2) to check that $\bar{D}$ and $\widetilde{D}_Q$ coincide on the flag minors $x_1^{\mathbf{i}_Q}, \ldots, x_N^{\mathbf{i}_Q}$.
- Conclude : the morphisms $\bar{D}$ and $\widetilde{D}_Q$ agree on the cluster variables of one seed, hence they coincide on the whole cluster algebra $\mathbb{C}[\mathbf{N}]$.

Perspectives and possible future questions

- Extend the definition of $\widetilde{D}_Q$ to the Grothendieck ring of Hernandez-Leclerc's category $\mathcal{C}^-$, which contains $\mathcal{C}_Q$ as a monoidal subcategory.
  $\rightsquigarrow$ Use the cluster structure of $K_0(\mathcal{C}^-)$ to obtain new rational fractions, not belonging to the image of $\bar{D}$.
- (jw. A. Dranowski and J. Kamnitzer) Investigate possible deformations of $\bar{D}$ using the theory of (q, t)-characters (Nakajima, Hernandez, Hernandez-Leclerc).