A Beilinson-Bernstein Theorem for p-adic Analytic Quantum Groups

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GDR Théorie de Lie Algébrique et Géométrique
24th March 2021

Outline

1. The Beilinson-Bernstein Theorem: classical and p-adic
2. Quantum groups and quantum flag varieties
3. p-adic analytic quantum flag varieties
4. The Main Theorem

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Classical Beilinson-Bernstein

Theorem (Beilinson-Bernstein, '81)

Let G be a reductive algebraic group over an algebraically closed field of characteristic 0, with Borel subgroup B and Lie algebra $\mathfrak{g}$. Let λ be a regular dominant weight, $X = G/B$ the flag variety of G and $\mathcal{D}_X^\lambda$ the sheaf of λ-twisted differential operators on X. Then the global section functor $\mathcal{M} \mapsto \mathcal{M}(X)$ gives an equivalence of categories

$$\Gamma : \mathcal{D}_X^\lambda\text{-mod} \to U(\mathfrak{g})_\lambda\text{-mod}$$

between the category of (quasi-coherent) sheaves of $\mathcal{D}_X^\lambda$-modules and the category of $U(\mathfrak{g})$-modules with central character corresponding to λ.

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- Recently, there have been p-adic analytic analogues of this theorem (Ardakov-Wadsley, Huyghe-Patel-Schmidt-Strauch among others). The representation theoretic goal here is to use these to study geometrically representations of p-adic groups.

The p-adic setup

L complete discrete valuation field of mixed characteristic $(0, p)$
R its DVR, $\pi \in R$ uniformizer
$k = R/\pi R$ residue field.

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e.g. $L = \mathbb{Q}_p$, $R = \mathbb{Z}_p$, $\pi = p$, $k = \mathbb{F}_p$.

$\mathbf{G}$ a simply connected, split semisimple algebraic group over R
$\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$
$X = \mathbf{G}/\mathbf{B}$ flag scheme.

A p-adic B-B theorem

Ardakov-Wadsley, 2013:

- a family $(\widehat{\mathcal{U}_{n,L}})_{n\geq 0}$ of Banach completions of the envloping algebra $U(\mathfrak{g}_L)$ of the L-Lie algebra $\mathfrak{g}_L := \mathfrak{g} \otimes_R L$;

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- for a weight λ, a family $(\widehat{\mathcal{D}^{\lambda}_{n,L}})_{n\geq 0}$ of sheaves of completed deformed twisted crystalline differential operators on X.

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- for a weight λ, a family $(\widehat{\mathcal{D}^{\lambda}_{n,L}})_{n\geq 0}$ of sheaves of completed deformed twisted crystalline differential operators on X.

Theorem (Ardakov-Wadsley, 2013)

Suppose p is a very good prime for $\mathbf{G}$. *Then for any* $n \geq 0$ *and for* λ *regular and dominant, the global section functor gives an equivalence of categories between coherent sheaves of* $\widehat{\mathcal{D}^{\lambda}_{n,L}}$*-modules and finitely generated* $\widehat{\mathcal{U}_{n,L}}$*-modules with central character corresponding to* λ*.*

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- (Ardakov-Wadsley, Ardakov-Bode-Wadsley, 2014-now): Introduced a general theory of (coadmissible) $\widehat{\mathcal{D}}$-modules over rigid analytic spaces;
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- (Ardakov, 2018): Beilinson-Bernstein for coadmissible equivariant $\widehat{\mathcal{D}}$-modules on rigid analytic flag varieties;
- (Huyghe-Patel-Schmidt-Strauch, 2017): Beilinson-Bernstein for $\mathcal{D}^\dagger$-modules on formal models of flag varieties;
- (Ongoing research by people above): Applications to induced representations or line bundles on the Drinfeld upper half space.

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Quantized enveloping algebras

Setup: fix $q \in R^\times$ such that q is *not* a root of unity. Let G be the group of L-points of $\mathbf{G}$, and let $\mathfrak{g}$ be as before. Let P be the weight lattice of G, and $\alpha_1, \ldots, \alpha_n$ the simple roots. Write $q_i = q^{\frac{\langle \alpha_i, \alpha_i \rangle}{2}}$.

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Definition

The *quantized enveloping algebra* $U_q := U_q(\mathfrak{g})$ is the L-algebra generated by $E_{\alpha_1}, \ldots, E_{\alpha_n}$, $F_{\alpha_1}, \ldots, F_{\alpha_n}$ and K_λ ($\lambda \in P$), with relations

$$K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_0 = 1,$$
$$K_\lambda E_{\alpha_i} K_{-\lambda} = q^{\langle \lambda, \alpha_i \rangle} E_{\alpha_i}, \quad K_\lambda F_{\alpha_i} K_{-\lambda} = q^{-\langle \lambda, \alpha_i \rangle} F_{\alpha_i},$$
$$[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q_i - q_i^{-1}}$$

and some quantum Serre relations.

Quantized coordinate algebras

The quantized coordinate algebra $\mathcal{O}_q := \mathcal{O}_q(G)$ is constructed as the algebra of matrix coefficients of finite dimensional U_q-modules. Both U_q and $\mathcal{O}_q$ are L-Hopf algebras.

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Example

If $G = SL_2(L)$, then $\mathcal{O}_q$ is the L-algebra with generators a, b, c, d and relations

$$ab = qba, \quad ac = qca, \quad bc = cb, \quad bd = qdb$$
$$cd = qdc, \quad ad - da = (q - q^{-1})bc, \quad ad - qbc = 1$$

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There is a 'Borel subalgebra' $U_q^{\geq 0}$ which is the L-subalgebra of U_q generated by all the E's and the K's. Correspondingly, there is a quantized coordinate algebra $\mathcal{O}_q(B)$ which is a quotient of $\mathcal{O}_q$. Again these are L-Hopf algebras.

Quantum flag varieties

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Idea:

$$G/B \leftrightarrow \{\text{q.c. } \mathcal{O}_{G/B}\text{-modules}\} \leftrightarrow \{B\text{-equivariant q.c. } \mathcal{O}_G\text{-modules}\}.$$

and as B and G are affine, the latter can be characterised at the level of global sections. We can then quantize the corresponding category!

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and as B and G are affine, the latter can be characterised at the level of global sections. We can then quantize the corresponding category!

Definition (Backelin-Kremnitzer, 2006)

The *quantum flag variety* is the category $\mathcal{M}_{B_q}(G_q)$ whose objects are $\mathcal{O}_q$-modules M also equipped with a right $\mathcal{O}_q(B)$-comodule structure, such that the action map $\mathcal{O}_q \otimes_L M \to M$ is a comodule homomorphism. The morphisms are the maps preserving both structures.

Quantum flag varieties (ctd)

- Giving a right $\mathcal{O}_q(B)$-comodule structure on M is equivalent to giving a $U_q^{\geq 0}$-module structure on M which make M *integrable*, i.e. M is the direct sum of its weight spaces and the action of E_{α_i} is locally nilpotent for all $1 \leq i \leq n$.

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- We can also define a *global section functor* $\Gamma : \mathcal{M}_{B_q}(G_q) \to L$-v.s. by

$$\Gamma(M) = \mathrm{Hom}_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_q, M),$$

  or equivalently $\Gamma(M)$ is the L-vector subspace of M of $\mathcal{O}_q(B)$-coinvariants.

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  or equivalently $\Gamma(M)$ is the L-vector subspace of M of $\mathcal{O}_q(B)$-coinvariants.
- Backelin-Kremnitzer proved that $\mathcal{M}_{B_q}(G_q)$ is a non-commutative projective scheme.

Quantum $\mathcal{D}$-modules

For $\mathcal{D}$-modules the idea is again:

$$\{\mathcal{D}_{G/B}\text{-mod}\} \leftrightarrow \{B\text{-equivariant } \mathcal{D}_G\text{-modules}\}.$$

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The quantized ring of differential operators on G is defined to be the smash product algebra $\mathcal{D}_q = \mathcal{O}_q \# U_q$. This is generated by $\mathcal{O}_q$ and U_q (which are subalgebras) with the relation $\text{ad}(u)(f) = u \cdot f$ for all $u \in U_q$ and $f \in \mathcal{O}_q$.

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Definition

Let $\lambda \in P$. The category $\mathcal{D}^{\lambda}_{B_q}(G_q)$ of quantum λ-twisted $\mathcal{D}$-modules has objects $\mathcal{D}_q$-modules M which are also $\mathcal{O}_q(B)$-comodules such that:

1. the $\mathcal{D}_q$-action $\mathcal{D}_q \otimes_L M \to M$ is $U_q^{\geq 0}$-linear; and
2. the $U_q^{\geq 0} \subset \mathcal{D}_q$-action and the comodule action 'differ by λ'.

Quantum Beilinson-Bernstein

The category $\mathcal{D}^{\lambda}_{B_q}(G_q)$ has a distinguished object $\mathcal{D}^{\lambda}_q := \mathcal{D}_q / I_\lambda$, where $I_\lambda = \mathcal{D}_q \cdot \{E_{\alpha_i}, K_\mu - q^{\langle \lambda, \mu \rangle}\}$.

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This represents global sections, i.e. for $M \in \mathcal{D}^{\lambda}_{B_q}(G_q)$,

$$\Gamma(M) \cong \mathrm{Hom}_{\mathcal{D}^{\lambda}_{B_q}(G_q)}(\mathcal{D}^{\lambda}_q, M).$$

In particular $\Gamma(\mathcal{D}^{\lambda}_q) = \mathrm{End}_{\mathcal{D}^{\lambda}_{B_q}(G_q)}(\mathcal{D}^{\lambda}_q)$ is a ring.

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Theorem (Backelin-Kremnitzer, 2006)

Suppose λ is regular and dominant. Then global sections gives an equivalence of categories

$$\Gamma : \mathcal{D}^{\lambda}_{B_q}(G_q) \to \Gamma(\mathcal{D}^{\lambda}_q)\text{-}mod.$$

One way of interpreting this is to say that $\mathcal{M}_{B_q}(G_q)$ is '$\mathcal{D}$-affine'.

Issues with global sections

To get a full quantum B-B we need to compute $\Gamma(\mathcal{D}_q^\lambda)$. For that, we need to introduce the 'ad-finite part' of U_q. A notable difference between U_q and $U(\mathfrak{g}_L)$ is that the adjoint action of U_q on itself is not locally finite dimensional. So define

$$U_q^{\text{fin}} := \{u \in U_q | \dim_L \text{ad}(U_q)(u) < \infty\}.$$

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The centre $Z(U_q)$ is contained in U_q^{fin}, and given $\lambda \in P$, we have a central character χ_λ and we may form the quotient $U_q^\lambda := U_q^{\text{fin}}/(\ker \chi_\lambda)$.

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There is a natural map $U_q^\lambda \to \Gamma(\mathcal{D}_q^\lambda)$, and Backelin-Kremnitzer claimed it is an isomorphism. Unfortunately their proof has issues that are yet unresolved.

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Some integral forms and p-adic completions

From now on, we assume $q \equiv 1 \pmod{\pi}$.

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- We let U denote the R-subalgebra of U_q generated the generators E_{α_i}, F_{α_i} $(1 \leq i \leq n)$ and K_μ for $\mu \in P$ ('De Concini-Kac integral form').
- The quantized coordinate algebra also contains an R-subalgebra $\mathcal{A}_q \subset \mathcal{O}_q$ (dual to Lusztig's integral form of U_q). There is also an R-subalgebra $\mathcal{B}_q \subset \mathcal{O}_q(B)$ which is a quotient of $\mathcal{A}_q$. These are the quantum analogues of functions on $\mathbf{G}(R)$ and $\mathbf{B}(R)$ respectively.

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- We may also form the smash product $\mathcal{D} = \mathcal{A}_q \# U$ which is an R-subalgebra of $\mathcal{D}_q$.
- We can then take π-adic completions of all the above to form the L-Banach Hopf algebras $\widehat{U_q} := \widehat{U} \otimes_R L$, $\widehat{\mathcal{O}_q} := \widehat{\mathcal{A}_q} \otimes_R L$ and $\widehat{\mathcal{O}_q(B)} := \widehat{\mathcal{B}_q} \otimes_R L$, and the L-Banach algebra $\widehat{\mathcal{D}_q} := \widehat{\mathcal{D}} \otimes_R L$. These are all Noetherian.

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Idea: $\widehat{\mathcal{O}_q}$ is the quantum analogue of the algebra of *analytic* functions on $\mathbf{G}(R)$.

Some integral forms and p-adic completions (ctd)

Example (The SL_2 case)

$\mathcal{A}_q$ is generated by the generators a, b, c, d of $\mathcal{O}_q$ and is defined by the same relations.

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$\mathcal{A}_q$ is generated by the generators a, b, c, d of $\mathcal{O}_q$ and is defined by the same relations. Then, explicitly,

$$\widehat{\mathcal{O}_q} = \left\{ \sum \lambda_{ijkl} a^i b^j c^k d^l : |\lambda_{ijkl}| \to 0 \text{ as } i+j+k+l \to \infty \right\}.$$

Similarly U_q has generators E, F and $K^{\pm 1}$, and

$$\widehat{U_q} = \left\{ \sum \lambda_{ijk} E^i K^j F^k : |\lambda_{ijkl}| \to 0 \text{ as } |i+j+k| \to \infty \right\}.$$

Quantum analytic flag varieties

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Definition (D., 2018)

The analytic quantum flag variety is the category $\widehat{\mathcal{M}_{B_q}(G_q)}$ whose objects are Banach $\widehat{\mathcal{O}_q}$-modules $\mathcal{M}$ which are also Banach right $\widehat{\mathcal{O}_q(B)}$-comodules such that the $\widehat{\mathcal{O}_q}$-action $\widehat{\mathcal{O}_q}\widehat{\otimes}_L\mathcal{M} \to \mathcal{M}$ is a comodule homomorphism.

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The global sections functor $\Gamma : \widehat{\mathcal{M}_{B_q}(G_q)} \to \mathbf{Ban}_L$ is defined by

$$\Gamma(\mathcal{M}) = \mathrm{Hom}_{\widehat{\mathcal{M}_{B_q}(G_q)}}(\widehat{\mathcal{O}_q}, \mathcal{M})$$

or, equivalently, by taking $\widehat{\mathcal{O}_q(B)}$-coinvariants.

Quantum analytic flag varieties (ctd)

Question 1

What are Banach $\widehat{\mathcal{O}_q(B)}$-comodules?

Proposition (D., 2018)

The category of right Banach $\widehat{\mathcal{O}_q(B)}$-comodules is canonically equivalent to the category of *topologically integrable* modules over a certain Banach completion of $U_q^{\geq 0}$.

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Here, $\mathcal{M}$ being topologically integrable means that every $m \in \mathcal{M}$ can be written as a converging series of weight vectors and, for each $1 \leq i \leq n$, the sequence $\frac{E_{\alpha_i}^k}{[k]_{q_i}!} \cdot m$ converges to 0.

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Here, $\mathcal{M}$ being topologically integrable means that every $m \in \mathcal{M}$ can be written as a converging series of weight vectors and, for each $1 \leq i \leq n$, the sequence $\frac{E_{\alpha_i}^k}{[k]_{q_i}!} \cdot m$ converges to 0.
At $q = 1$, this means we can exponentiate the action to get a (continuous) representation of $\mathbf{B}(R)$.

Quantum analytic flag varieties (ctd 2)

Question 2

How to compute the cohomology $R\Gamma$ in $\widehat{\mathcal{M}_{B_q}(G_q)}$?

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Warning: $\widehat{\mathcal{M}_{B_q}(G_q)}$ is *not* abelian but instead quasi-abelian! We can still however right derive left exact functors for the right notion of exactness.

Quantum analytic flag varieties (ctd 2)

Question 2

How to compute the cohomology $R\Gamma$ in $\widehat{\mathcal{M}_{B_q}(G_q)}$?

Warning: $\widehat{\mathcal{M}_{B_q}(G_q)}$ is *not* abelian but instead quasi-abelian! We can still however right derive left exact functors for the right notion of exactness.

Theorem (D., 2018)

For any $\mathcal{M} \in \widehat{\mathcal{M}_{B_q}(G_q)}$, there is a standard complex $\check{C}(\mathcal{M})$ which computes $R\Gamma(\mathcal{M})$.

This complex is a quantum analogue of the Čech complex obtained from the covering of the flag variety by Weyl translates of the big cell. It is based on constructions of Joseph and De Concini-Lyubashenko.

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Corollary

Γ has finite cohomological dimension.

Outline

1. The Beilinson-Bernstein Theorem: classical and p-adic

2. Quantum groups and quantum flag varieties

3. p-adic analytic quantum flag varieties

4. The Main Theorem

Quantum analytic $\mathcal{D}$-modules

We can now define λ-twisted analytic $\mathcal{D}$-modules in $\widehat{\mathcal{M}_{B_q}(G_q)}$:

Definition (D., 2018)

Let $\lambda \in P$. The category $\widehat{\mathcal{D}^{\lambda}_{B_q}(G_q)}$ has objects Banach $\widehat{\mathcal{D}_q}$-modules $\mathcal{M}$ which are also Banach right $\widehat{\mathcal{O}_q(B)}$-comodules such that:

1. the $\widehat{\mathcal{D}_q}$-action $\widehat{\mathcal{D}_q}\widehat{\otimes}_L\mathcal{M} \to \mathcal{M}$ is $U_q^{\geq 0}$-linear; and
2. The two $U_q^{\geq 0}$-actions on $\mathcal{M}$ 'differ by λ'.

Again there is a distinguished object $\widehat{\mathcal{D}^{\lambda}_q}$ which represents global sections.

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We say $\mathcal{M}$ is *coherent* if it is finitely generated as a $\widehat{\mathcal{D}_q}$-module. The coherent $\mathcal{D}$-modules form a full subcategory $\mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$ which is abelian.

Beilinson-Bernstein

We need to introduce an 'ad-finite part' in these Banach algebras. Let $U^{\text{fin}} = U \cap U_q^{\text{fin}}$, so that we can define $U^\lambda = U^{\text{fin}}/(\ker \chi_\lambda)$. Then let $\widehat{U_q^\lambda} = \widehat{U^\lambda} \otimes_R L$, a Noetherian Banach L-algebra.

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Theorem (D., 2020)

Let $\lambda \in P$ be regular and dominant. Suppose that $p = char(k)$ is bigger than the Coxeter number of the root system of $\mathbf{G}$. Then global sections give an equivalence of abelian categories

$$\Gamma : coh(\widehat{\mathcal{D}_{B_q}^{\lambda}(G_q)}) \to f.g.\ \widehat{U_q^{\lambda}}\text{-}mod.$$

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- The idea here is to show that this is true modulo π^n for all n. An inductive argument reduces to showing it modulo π. There the condition $q \equiv 1 \pmod{\pi}$ is crucial as the situation becomes non-quantum.
- The result modulo π follows by work of Bezrukavnikov-Mirkovic-Rumynin (this is where the condition on p appears).

Thank you!