Perverse Monodromic Sheaves

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Notations

G: connected reductive complex algebraic group

B: Borel subgroup, with unipotent radical U

T: maximal torus

$\Bbbk$: algebraically closed field of characteristic $\ell > 0$

W: Weyl group, with $\leq$ the Bruhat order, $S =$ subset of simple reflections, $w_\circ$ the longest element of (W, S).

$(\mathrm{X}^*(T), \Phi, \mathrm{X}_*(T), \Phi^\vee)$ root datum of G, Φ^+ subset of positive roots (B is positive)

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We have a natural stratification

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The category $\mathscr{O}_{\mathrm{geom}}$ has representation theoretic interpretations.

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2. $\mathscr{O}_{\mathrm{geom}}$ is a highest weight category with weight poset $(W, \leq)$.

Highest Weight Category

Cline–Parshall–Scott, Beilinson–Ginzburg–Soergel.
A $\Bbbk$-linear abelian category $\mathcal{A}$ is highest weight with (finite) weight poset $(\mathscr{S}, \leq)$ if we have families of *standard*, *simple*, *costandard* objects and morphisms

$$(\Delta_s \to L_s \to \nabla_s)_{s \in \mathscr{S}}$$

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satisfying:

1. for any $s \in \mathscr{S}$, we have $\mathrm{End}(L_s) = \mathbb{k}$,
2. if $\mathscr{T} \subseteq \mathscr{S}$ is an ideal in which s is maximal, then $\Delta_s \to L_s$ is a projective cover and $L_s \to \nabla_s$ is an injective envelope in $\langle L_t \mid t \in \mathscr{T} \rangle_{\mathrm{Serre}}$,
3. the cokernel of $\Delta_s \to L_s$ and kernel of $L_s \to \nabla_s$ are in $\langle L_t \mid t < s \rangle_{\mathrm{Serre}}$,
4. for any $s, t \in \mathscr{S}$

$$\mathrm{Ext}^2_{\mathcal{A}}(\Delta_s, \nabla_t) = 0.$$

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3. $$\mathrm{Ext}^i_{\mathcal{A}}(\Delta_s, \nabla_t) = \begin{cases} \Bbbk & \text{if} \quad s = t \text{ and } i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

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Tiltings can be thought of as intermediate between projectives and injectives, and are very convenient to work with.

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4. let P_w be the projective cover of IC_w, and T_w be the indecomposable tilting object associated to w.

Tiltings and Ringel duality

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Geometric Ringel Duality: there exists an equivalence

$$\mathrm{Tilt}\mathscr{O}_{\mathrm{geom}} \xrightarrow{\sim} \mathrm{Proj}\mathscr{O}_{\mathrm{geom}}$$

mapping T_w to $P_{ww_\circ}$, where $w_\circ$ is the longest element of W.

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3. $\mathrm{End}(T_{w_\circ}) \cong \Bbbk[X_*(T)]/\langle \Bbbk[X_*(T)]_+^W \rangle$,
4. explicit description of the essential image.

Soergel Theory

$\mathfrak{g}$ a semisimple complex Lie algebra, $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ a Cartan and Borel subalgebras. Soergel obtained the following description of the principal block $\mathscr{O}_0$ of the BGG category $\mathscr{O}$ for representation of $\mathfrak{g}$.
Let P be the projective cover of the unique simple in $\mathscr{O}_0$ with antidominant highest weight.

1. $S(\mathfrak{h})/\langle S(\mathfrak{h})_+^W \rangle \xrightarrow{\sim} \mathrm{End}(P)$,
2. $\mathbb{V} := \mathrm{Hom}_{\mathscr{O}_0}(P, -)$ is fully faithful on projective objects,
3. explicit description the essential image of $\mathbb{V}$ (restricted to projectives).

The isomorphism

$$\Bbbk[\mathsf{X}_*(T)]/\langle \Bbbk[\mathsf{X}_*(T)]_+^W \rangle \xrightarrow{\sim} \mathrm{End}(T_{w_\circ})$$

is induced by a *monodromy morphism*

$$\Bbbk[\mathsf{X}_*(T)] \to \mathrm{End}(T_{w_\circ}).$$

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For $\mathscr{F}$ on X, this is given by a group morphism

$$\mathsf{X}_*(T) \longrightarrow \mathrm{Aut}(\mathscr{F})$$

i.e. an action of $\mathsf{X}_*(T)$ on the objects of $D^b_{\mathcal{S}}(X, \Bbbk)$.

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We extend this to an algebra morphism

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G/U is a stratified right T-variety; we can then define a monodromy morphism.

Proposition

A perverse sheaf $\mathscr{F} \in P_{(B)}(G/U, \Bbbk)$ is in $\mathscr{O}_{\mathrm{geom}} \subseteq P_{(B)}(G/U, \Bbbk)$ iff the right monodromy morphism of $\mathscr{F}$ factors through

$$\Bbbk[X_*(T)] \to \Bbbk[X_*(T)]/\langle e^\lambda - 1\rangle = \Bbbk \to \mathrm{End}(\mathscr{F}).$$

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$$P_{[-,\underline{t}]}$$

the full subcategory of $P_{(B)}(G/U, \Bbbk)$ whose objects are those $\mathscr{F}$ such that the right monodromy morphism $\Bbbk[X_*(T)] \to \mathrm{End}(\mathscr{F})$ factors through

$$\Bbbk[X_*(T)] \longrightarrow \Bbbk[X_*(T)]/\langle e^{\lambda} - \lambda(t) \rangle \to \mathrm{End}(\mathscr{F}).$$

We obtain a family $P_{[-,\underline{t}]}$ of subcategories in $D^b_{(B)}(G/U, \mathbb{k})$, indexed by the dual $\mathbb{k}$-torus. We can think of them as "deformation" of $\mathscr{O}_{\mathrm{geom}}$ along $T^\vee_{\mathbb{k}}$.

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We say that objects of $P_{[-,\underline{t}]}$ have *exact monodromy* t.

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2. do we have a "Ringel duality" ?
3. can we obtain a Soergel-type description of $P_{[-,\underline{t}]}$?

Comparison of $P_{[-,\underline{t}]}$ and $\mathscr{O}_{\mathrm{geom}}$

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Actually, $P_{[-,\underline{t}]}$ does not come from pullback from any (partial) flag variety. We lack usual tools of homological algebra, and cannot prove directly that $P_{[-,\underline{t}]}$ has a highest weight structure.

Another point of view

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$$\begin{array}{ccc} D^b_{(B)}(\mathscr{B},\Bbbk) & \cong & D^b_{(B),T}(G/U,\Bbbk) \\ \cup\!\mid & & \cup\!\mid \\ \mathscr{O}_{\mathrm{geom}} & \cong & P_{(B),T}(G/U,\Bbbk). \end{array}$$

The pullback functor $D^b_{(B)}(\mathscr{B},\Bbbk) \to D^b_{(B)}(G/U,\Bbbk)$ identifies with the forgetful functor

$$\mathrm{For} : D^b_{(B),T}(G/U,\Bbbk) \to D^b_{(B)}(G/U,\Bbbk).$$

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Thus the objects of $\mathscr{O}_{\mathrm{geom}}$ in $D^b_{(B)}(G/U, \Bbbk)$ are those perverse $\mathscr{F}$ lying in the image of For. This suggests that maybe, $P_{[-,\underline{t}]}$ could identify with the heart of a t-structure on some "equivariant category", for some non-standard notion of equivariance.

Lusztig–Yun categories

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$$\mathfrak{D}(G/U /\!\!/ T)_{[-,\mathscr{L}]} \subseteq D^b_{(B),\widetilde{T}}(G/U, \Bbbk)$$

objects: $\mathscr{F}$ in $D^b_{(B),\widetilde{T}}(G/U, \Bbbk)$ such that the action of K is via $\chi_{\mathscr{L}}$.

Lusztig–Yun perverse sheaves

The perverse t-structure on $D^b_{(B),\widetilde{T}}(G/U,\Bbbk)$ restricts to a perverse t-structure on $\mathfrak{D}(G/U/\!\!/T)_{[-,\mathscr{L}]}$

$$\begin{array}{ccc} \mathfrak{D}(G/U/\!\!/T)_{[-,\mathscr{L}]} & \hookrightarrow & D^b_{(B),\widetilde{T}}(G/U,\Bbbk) \\ \cup\!| & & \cup\!| \\ \mathfrak{P}(G/U/\!\!/T)_{[-,\mathscr{L}]} & \hookrightarrow & P_{(B),\widetilde{T}}(G/U,\Bbbk). \end{array}$$

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The category

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is the category of Lusztig–Yun equivariant monodromic perverse sheaves.

One-dimensional $\Bbbk$-local systems on $T \overset{\sim}{\longleftrightarrow}$ elements of $T^\vee_\Bbbk$.

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We can define an equivariant Lusztig–Yun monodromic *triangulated* category, and a subcategory of perverse sheaves

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Now by definition, $\mathfrak{P}_{[-,\underline{t}]}$ is the heart of a t-structure on the Lusztig–Yun equivariant category $\mathfrak{D}_{[-,\underline{t}]}$.

Proposition

The restriction of the forgetful functor $D^b_{(B),\widetilde{T}}(G/U,\Bbbk) \to D^b_{(B)}(G/U,\Bbbk)$ *yields an equivalence*

$$\mathfrak{P}_{[-,\underline{t}]} \xrightarrow{\sim} P_{[-,\underline{t}]}.$$

Standard, costandard and simple objects

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For $t = 1$, we get back Δ_w, ∇_w and IC_w.

Highest weight structure

Theorem

The category $\mathfrak{P}_{[-,\underline{t}]}$ *admits a highest weight structure with weight poset* $(W, \leq)$. *The standard, costandard and simple objects are given by* $\Delta(w)_t$, $\nabla(w)_t$ *and* $\mathrm{IC}(w)_t$.

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Set $T(w)_t$ for the indecomposable tilting associated to w.

Socle of standard objects

Lemma (Beilinson–Bezrukavnikov–Mirković)

In $\mathscr{O}_{\mathrm{geom}}$, all the standard objects Δ_w share a common socle, namely IC_e.

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Fact 1: for $w \in W$, the right monodromy of $\Delta(w)_t$ is given by t, and the left monodromy by $w(t)$.

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Fact 1: for $w \in W$, the right monodromy of $\Delta(w)_t$ is given by t, and the left monodromy by $w(t)$.

Fact 2: if two objects $\mathscr{F}$ and $\mathscr{G}$ do not share a common left and right monodromy, then

$$\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = 0.$$

Consequence: if $w(t) \neq v(t)$ for $w, v \in W$, then $\Delta(w)_t$ and $\Delta(v)_t$ cannot share a common socle !

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Example: let $s_\alpha \in W$ be a simple reflection associated to a root α such that $\alpha(t) \neq 1$. Then

$$\Delta(s)_t \cong \mathrm{IC}(s)_t \cong \nabla(s)_t.$$

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Still too rough. The correct distinction came from Lusztig and Yun:

$$ W_t^{\circ} := \langle s_\alpha \mid \alpha^{\vee}(t) = 1 \rangle. $$

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We consider the quotient ${}_{t'}W_t / W_t^\circ$. Cosets there are called **blocks**.

For β a block in ${}_{t'}W_t/W_t^\circ$, let

$$\mathfrak{P}^{\beta}_{[t',\underline{t}]} := \langle \mathrm{IC}(w)_t \mid w \in \beta \rangle_{\mathrm{Serre}}$$

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Proposition

$$\mathfrak{P}_{[-,\underline{t}]} = \bigoplus_{t' \in W \cdot t} \left(\bigoplus_{\beta \in {}_{t'}W_t/W_t^\circ} \mathfrak{P}^{\beta}_{[t',\underline{t}]} \right).$$

In particular, there are no nonzero morphisms between two objects lying in different blocks.

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Fact 5: it suffices to understand the "neutral block subcategory": for any $\beta \in {}_{t'}W_t/W_t^\circ$, we have an equivalence of categories

$$\mathfrak{P}^\circ_{[t,\underline{t}]} \xrightarrow{\sim} \mathfrak{P}^\beta_{[t',\underline{t}]}$$

mapping standards to standards, costandards to costandards, tiltings to tiltings.

We let $\Phi_t^+ = \{\alpha \in \Phi^+ \mid \alpha^\vee(t) = 1\}$.

$$S_t := \{s = s_\alpha \mid \alpha \in \Phi_t^+,\ \alpha \text{ indecomposable in } \Phi_t^+\}.$$

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Fact 6: the pair (W_t°, S_t) is a Coxeter system.

Remark

(W_t°, S_t) is **not** a subsystem of (W, S): there may be simple roots in S_t that are not simple in S, and the two orders above do not coincide.

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Let $w_{\circ,t}$ be the longest element in (W_t°, S_t).

Intuition: the neutral block perverse subcategory

$$\mathfrak{P}^{\circ}_{[t,\underline{t}]}$$

is "governed by (W_t°, S_t)" the way $\mathscr{O}_{\mathrm{geom}}$ is by (W, S).

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The standard objects in $\mathfrak{P}^{\circ}_{[t,\underline{t}]}$ share a common socle, given by $\mathrm{IC}(e)_t$.

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Proposition (Socle)

The standard objects in $\mathfrak{P}^{\circ}_{[t,\underline{t}]}$ share a common socle, given by $\mathrm{IC}(e)_t$.

Proposition (Ringel duality)

There is an equivalence of categories

$$\mathrm{Tilt}\mathfrak{P}^{\circ}_{[t,\underline{t}]} \xrightarrow{\sim} \mathrm{Proj}\mathfrak{P}^{\circ}_{[t,\underline{t}]}$$

mapping $T(w)_t$ to $P(ww_{\circ,t})_t$ for any $w \in W_t^\circ$.

Proposition (Comparison tilting-projective)

We have an isomorphism

$$T(w_{\circ,t})_t \cong P(e)_t.$$

We thus want to study the functor

$$\mathbb{V}_t^\circ := \mathrm{Hom}_{\mathfrak{P}^\circ_{[t,\underline{t}]}}(T(w_{\circ,t})_t, -)$$

$$\mathrm{Tilt}\mathfrak{P}^\circ_{[t,\underline{t}]} \to \mathrm{Mod}^{\mathrm{fg}}(\mathrm{End}_{\mathfrak{P}^\circ_{[t,\underline{t}]}}(T(w_{\circ,t})_t)).$$

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1. determine $\mathrm{End}_{\mathfrak{P}^\circ_{[t,\underline{t}]}}(T(w_{\circ,t})_t)$,
2. is $\mathbb{V}_t^\circ$ fully faithful on tilting objects ?
3. what is its essential image ?

Strategy

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1. we can define a completed tilting category $\widehat{T}^{\circ}_{[t,t]}$,
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Main features for us:

1. we can define a completed tilting category $\widehat{T}^{\circ}_{[t,t]}$,
2. we have a projection functor $\widehat{T}^{\circ}_{[t,t]} \xrightarrow{\pi^t_\dagger} \mathrm{Tilt}\mathfrak{P}^{\circ}_{[t,\underline{t}]}$,
3. the indecomposable tilting objects in the Lusztig–Yun category lift to the completed category: we have objects $\widehat{T}_{w,t}$ for any $w \in W^{\circ}_t$ such that

$$\pi^t_\dagger(\widehat{T}_{w,t}) \cong T(w)_t.$$

We have two nontrivial monodromy morphisms for completed objects (left and right): one can think of the completed objects as projective limits of complexes for which the right and left monodromy morphisms factors through some quotient

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Pro-tilting objects are nice because of the following:

$$\mathrm{Hom}(\widehat{T}, \widehat{T}') \otimes_{\widehat{\mathsf{R}}_{\mathsf{t}}} \Bbbk \cong \mathrm{Hom}(\pi_\dagger^t(\widehat{T}), \pi_\dagger^t(\widehat{T}')).$$

Soergel theory II

In particular

$$\mathrm{End}(\widehat{T}_{w_\circ,t}) \otimes_{\widehat{\mathsf{R}}_{\mathsf{t}}} \Bbbk \cong \mathrm{End}(T(w_{\circ,t})_t).$$

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$$\mathrm{End}(\widehat{T}_{w_\circ,t}) \otimes_{\widehat{\mathsf{R}}_{\mathsf{t}}} \Bbbk \cong \mathrm{End}(T(w_{\circ,t})_t).$$

The strategy is then to determine first $\mathrm{End}(\widehat{T}_{w_\circ,t})$ and the essential image of

$$\widehat{\mathbb{V}}_t^\circ := \mathrm{Hom}(\widehat{T}_{w_\circ,t}, -)$$

on the tilting completed category, and to deduce results for the Luzstig–Yun case.

Theorem I

Assumption: the characteristic of $\Bbbk$ is not a torsion prime for the Langlands dual group $G_{\Bbbk}^{\vee}$.

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Theorem

The functor $\mathbb{V}^\circ_t$ *induces an equivalence of category*

$$\mathrm{Tilt}\mathfrak{P}^\circ_{[t,\underline{t}]} \xrightarrow{\sim} \mathbb{S}\mathrm{Mod}^{\mathrm{fg}}(\widehat{\mathsf{R}}_{\mathsf{t}} \otimes_{(\widehat{\mathsf{R}}_{\mathsf{t}})^{W^\circ_t}} \Bbbk)$$

where $\mathbb{S}\mathrm{Mod}^{\mathrm{fg}}(\widehat{\mathsf{R}}_{\mathsf{t}} \otimes_{(\widehat{\mathsf{R}}_{\mathsf{t}})^{W^\circ_t}} \Bbbk)$ *is the full subcategory of* $\mathrm{Mod}^{\mathrm{fg}}(\widehat{\mathsf{R}}_{\mathsf{t}} \otimes_{(\widehat{\mathsf{R}}_{\mathsf{t}})^{W^\circ_t}} \Bbbk)$ *generated under direct sums, direct summands and application of* $\widehat{\mathsf{R}}_{\mathsf{t}} \otimes_{(\widehat{\mathsf{R}}_{\mathsf{t}})^s} (-)$ *for* $s \in S_t$ *to the object* $\Bbbk$.

Theorem II

Theorem

The category $\mathfrak{P}^{\circ}_{[t,\underline{t}]}$ *is equivalent to* $\mathrm{Mod}^{\mathrm{fg}}(A)$ *for* A *an explicitly determined ring.*

Theorem III

Let H_t be the connected reductive algebraic group over $\mathbb{C}$ with maximal torus T and root system Φ_t (an *endoscopic group*). The positive subset Φ_t^+ defines a Borel subgroup B_t; the Weyl group identifies naturally with W_t° and the subset of simple roots is given by S_t.

Theorem

We have an equivalence of category

$$\mathfrak{P}^{\circ}_{[t,\underline{t}]} \xrightarrow{\sim} P_{(B_t)}(H_t/B_t, \Bbbk) =: \mathscr{O}_{\mathrm{geom}}(H_t).$$

swapping standard, costandard, simple and tilting objects.