Derived products and the left adjoint of derived parabolic induction

Claudius Heyer

Westfälische Wilhelms-Universität Münster

March 24, 2021
Journées du GDR TLAG

Smooth representations

- k = field,
- G = locally profinite group [Hausdorff topological group admitting a neighborhood basis of 1 consisting of compact open subgroups], [Example: $\mathrm{GL}_n(\mathbb{Q}_p)$ with $K_j = I_n + p^j \,\mathrm{Mat}_{n,n}(\mathbb{Z}_p)$]
- A G-representation [on a k-vector space] V is called smooth if every vector has open stabilizer in G, equivalently

$$V = V^\infty := \bigcup_{\substack{I \leq G \\ \text{compact open}}} V^I.$$

Example: $C(G,k) = \left\{ f\colon G \to k \;\middle|\; \begin{array}{l} \exists I \leq G \text{ compact open s.t.} \\ f(gx) = f(g)\ \forall g \in G, x \in I \end{array} \right\}$
with G acting by right translation [i.e., $(g \cdot f)(g') = f(g'g)$].

- $\mathrm{Rep}_k(G)$ = category of smooth representations of G
[it is an abelian Grothendieck category, i.e., AB5 and has a generator; it has enough injectives].

Smooth mod-p representations of a p-adic Lie group

- k = field of characteristic $p > 0$,
- G = p-adic Lie group (it is locally pro-p),
  [e.g., $\mathrm{GL}_n(\mathbb{Q}_p)$, $B_n(\mathbb{Q}_p)$, $\mathrm{GL}_n(\mathbb{Z}_p)$, etc.; but not $\mathrm{GL}_n(\mathbb{F}_p((t)))$!]

"Special features" (as opposed to $k = \mathbb{C}$)

Let $I \leq G$ be (open) pro-p.

1. $V \neq \{0\}$ in $\mathrm{Rep}_k(G) \implies V^I \neq \{0\}$.
2. The functor $V \mapsto V^I$ is left exact, but not exact.
   Example: take $V \neq V^I$ in $\mathrm{Rep}_k(I)$ [e.g., $V = C(I, k)$].
   Then $V \twoheadrightarrow V/V^I \neq \{0\}$ is surjective, but

$$V^I \xrightarrow{0} (V/V^I)^I \neq \{0\} \quad \text{is not!}$$

Upshot: To understand $\mathrm{Rep}_k(G)$, need to study $\mathrm{H}^i(I, V)$, $i > 0$.

Derived categories

Conceptually better: replace $\mathsf{Rep}_k(G)$ by

$$D(G) = \text{derived category of } \mathsf{Rep}_k(G) \text{ of (unbounded) cochain complexes,}$$

obtained from $C(\mathsf{Rep}_k(G))$ by inverting quasiisomorphisms
[i.e., those $f\colon X^\bullet \to Y^\bullet$ with $\mathrm{H}^i(f)\colon \mathrm{H}^i(X^\bullet) \xrightarrow{\cong} \mathrm{H}^i(Y^\bullet)$, for all $i \in \mathbb{Z}$].
$D(G)$ is a triangulated category [it is not abelian].

$V \mapsto V^I$ gives rise to an exact functor $\mathrm{RH}^0(I,-)\colon D(G) \to D(\mathsf{Vect}_k)$ such that

$$\mathrm{H}^i(I,V) = \mathrm{H}^i\big(\mathrm{RH}^0(I,V[0])\big), \qquad \text{for all } i \in \mathbb{Z}.$$

[$V[0] = V$ regarded as a complex concentrated in degree 0.]

Parabolic induction

- $F/\mathbb{Q}_p$ finite extension,
- $G = F$-points of a connected reductive F-group
  [e.g. $\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$, $\mathrm{Sp}_{2n}(F)$,...]
- $P = UM \leq G$ parabolic with Levi M and unipotent radical U
  [e.g. $\begin{pmatrix} \mathrm{GL}_{n_1}(F) & * & * \\ & \ddots & \\ 0 & & * \\ 0 & 0 & \mathrm{GL}_{n_r}(F) \end{pmatrix}$ in $\mathrm{GL}_n(F)$, where $n = n_1 + \cdots + n_r$],
- parabolic induction is the functor

$$i_M^G \colon \mathrm{Rep}_k(M) \xrightarrow{\mathrm{Inf}_P^M} \mathrm{Rep}_k(P) \xrightarrow{\mathrm{Ind}_P^G} \mathrm{Rep}_k(G),$$

where $\mathrm{Inf}_P^M V = V$ with U acting trivially and

$$\mathrm{Ind}_P^G V = \left\{ f \colon G \to V \;\middle|\; \begin{array}{l} f \text{ locally constant, and} \\ f(xg) = x \cdot f(g),\ \forall x \in P, g \in G \end{array} \right\}$$

[G acts by right translation].

Derived parabolic induction

$i_M^G\colon \mathrm{Rep}_k(M) \to \mathrm{Rep}_k(G)$ is exact, hence extends to a functor

$$\mathrm{R}i_M^G\colon D(M) \longrightarrow D(G).$$

Question

Do there exist left or right adjoints of $\mathrm{R}i_M^G$?

- i_M^G admits a right adjoint and this extends to a derived adjunction.
  [Because $\mathrm{Rep}_k(G)$ has enough injectives.]
- i_M^G has a left adjoint $L^0(U, -)$, given by

$$L^0(U, V) = V_U := V/\langle u \cdot v - v \mid u \in U, v \in V\rangle.$$

  [U-coinvariants/Jacquet functor/parabolic restriction]
- Problem: $\mathrm{Rep}_k(G)$ lacks projective objects, so the left derived functor of $L^0(U, -)$ may not even exist.

Brown representability (Brown, Neeman, Krause, Franke)

Let $\mathcal{C}$ = compactly generated, triangulated, with infinite $\bigoplus$.

1. $H\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Ab})$ homological and $H(\bigoplus_{j\in J} X_j) = \prod_{j\in J} H(X_j)$
   $\implies H = \mathcal{C}(-, X)$, some $X \in \mathcal{C}$.
2. $\mathcal{C}$ admits infinite $\prod$ [apply 1. with $H = \prod_{j\in J} \mathcal{C}(-, X_j)$].
3. $H\colon \mathcal{C} \to (\mathrm{Ab})$ homological and $H(\prod_{j\in J} X_j) = \prod_{j\in J} H(X_j)$
   $\implies H = \mathcal{C}(X, -)$, some $X \in \mathcal{C}$.

$D(M)$ and $D(G)$ satisfy Brown representability, hence:

- Infinite products exist in $D(M)$.
- If $\mathrm{R}i_M^G\colon D(M) \to D(G)$ commutes with products, then $\mathrm{R}i_M^G$ admits a left adjoint [apply 3. with $H = D(G)(X, \mathrm{R}i_M^G(-))$].

Note: Infinite products in $\mathrm{Rep}_k(G)$ are not exact

[they are of the form $(\prod_{j\in J} V_j)^\infty$, and $(\,\cdot\,)^\infty = \varinjlim_{I\le G} (\,\cdot\,)^I$ is not exact]

$\implies$ products in $D(G)$ cannot be computed "degreewise".

An auxiliary category

- Put $\Omega_G = \{$compact open subgroups of $G\}$ [it is a G-poset]
- Define a category $\mathrm{Fun}_k(G)$: objects are
  - functors $\omega\colon \Omega_G^{\mathrm{op}} \to \mathrm{Vect}_k$ and
  - natural maps $\rho_I(g)\colon \omega(I) \xrightarrow{\cong} \omega(gIg^{-1})$ for $g \in G$, $I \in \Omega_G$, such that $\rho_I(gh) = \rho_{hIh^{-1}}(g)\rho_I(h)$ and $\rho_I(x) = \mathrm{id}$, for $x \in I$.
- $\mathrm{Fun}_k(G)$ is Grothendieck abelian and arbitrary $\prod$ are exact.

Example

- Put $\Gamma_G(V)(I) \coloneqq V^I$ and $\rho_I(g)\colon V^I \xrightarrow{v \mapsto gv} V^{gIg^{-1}}$; this defines a [fully faithful left exact] embedding $\Gamma_G\colon \mathrm{Rep}_k(G) \to \mathrm{Fun}_k(G)$.
- $\Phi_G(\omega) \coloneqq \varinjlim_{I \in \Omega_G} \omega(I)$ is exact and left adjoint to Γ_G and satisfies $\Phi_G \circ \Gamma_G \cong \mathrm{id}_{\mathrm{Rep}_k(G)}$.
- $(\prod_{j \in J} V_j)^\infty = \varinjlim_{I \in \Omega_G} \prod_{j \in J} V_j^I = \Phi_G \prod_{j \in J} \Gamma_G(V_j)$.

Computing derived products

Theorem (H.)

1. The derived functors

$$D(G) \underset{\mathrm{R}\Phi_G}{\overset{\mathrm{R}\Gamma_G}{\rightleftarrows}} D(\mathsf{Fun}_k(G)) \quad (\top)$$

   exist, are adjoint, and satisfy $\mathrm{R}\Phi_G \circ \mathrm{R}\Gamma_G \cong \mathrm{id}_{D(G)}$.
2. $R^i\Gamma_G(V)(I) = \mathrm{H}^i(I, V)$, for $i \geq 0$.
3. $\prod_{j\in J} V_j^\bullet \cong \mathrm{R}\Phi_G \prod_{j\in J} \mathrm{R}\Gamma_G(V_j^\bullet)$ in $D(G)$.

Example

$$\mathrm{H}^i(\prod_{j\in J} V_j[0]) = \varinjlim_{I\in\Omega_G} \prod_{j\in J} \mathrm{H}^i(I, V_j).$$

Theorem (H.)

$\mathrm{R}i_M^G \colon D(M) \to D(G)$ commutes with products.

Proof sketch:

- $\mathrm{R}i_M^G = \mathrm{R}\,\mathrm{Ind}_P^G \circ \mathrm{R}\,\mathrm{Inf}_P^M$,
- $\mathrm{R}\,\mathrm{Ind}_P^G$ commutes with products [since $\mathrm{R}\,\mathrm{Res}_P^G \dashv \mathrm{R}\,\mathrm{Ind}_P^G$],
- remains to show

$$\alpha \colon \mathrm{R}\,\mathrm{Inf}_P^M \prod_{j \in J} V_j^\bullet \xrightarrow{\cong} \prod_{j \in J} \mathrm{R}\,\mathrm{Inf}_P^M V_j^\bullet \qquad \text{in } D(P),$$

- reduce to "baby case": for $V_j^\bullet = V_j[r_j]$ [sitting in degree $-r_j$]

$$\mathrm{H}^i(\alpha) \colon \varinjlim_{I_M \in \Omega_M} \prod_{j \in J} \mathrm{H}^{i+r_j}(I_M, V_j) \xrightarrow[\cong]{\text{inflation}} \varinjlim_{I_P \in \Omega_P} \prod_{j \in J} \mathrm{H}^{i+r_j}(I_P, V_j).$$

[This uses that P is a p-adic Lie group.] □

Left adjoint of derived parabolic induction

Corollary (Brown representability)

$\mathrm{R}i_M^G$ admits a left adjoint $\mathrm{L}(U,-)\colon D(G) \to D(M)$.

Question: What does $\mathrm{L}(U,-)$ look like? Can it be computed?

- $L^i(U,V) \coloneqq \mathrm{H}^i(\mathrm{L}(U,V[0])) = \begin{cases} V_U, & \text{if } i=0, \\ 0, & \text{if } i>0. \end{cases}$
- $\mathrm{L}(U,-) = \mathrm{L}_U \circ \mathrm{R}\,\mathrm{Res}_P^G$, where $\mathrm{L}_U\colon D(P) \to D(M)$ is the left adjoint of $\mathrm{R}\,\mathrm{Inf}_P^M$ [it exists for any p-adic Lie group $P = U \rtimes M$].
- $\mathrm{L}_U \circ \mathrm{R}\,\text{c-Ind}_{I_P}^P \cong \mathrm{R}\,\text{c-Ind}_{I_M}^M \circ \mathrm{L}_{I_P \cap U}$, for $I_P \leq P$ compact open. [$\text{c-Ind}_{I_P}^P \dashv \mathrm{Res}_{I_P}^P\colon \mathrm{Rep}_k(P) \to \mathrm{Rep}_k(I_P)$.]

Theorem (H.)

$\mathrm{Hom}_k(L^{-i}(U,V),W) \cong \mathrm{Ext}^i_{\mathrm{Rep}_k(U)}(V,W)$ naturally [for $V \in \mathrm{Rep}_k(G)$, $W \in \mathrm{Vect}_k$; U acts trivially on W].

If $P = U \rtimes M$ is compact and torsion-free, one can say more:
Let $d = \dim U$ and write $\omega := \mathrm{Hom}_k(\mathrm{H}^d(U, k), k)[d] \in D(P)$
[it is an invertible complex concentrated in degree $-d$].
Applying a result of Balmer–Dell'Ambrogio–Sanders we deduce:

Theorem (H., Schneider)

The left adjoint of $\mathrm{R}\,\mathrm{Inf}_P^M \colon D(M) \to D(P)$ is

$$\mathrm{L}_U = \mathrm{RH}^0(U, \omega \otimes_k -) \colon D(P) \longrightarrow D(M).$$

It follows that there is an infinite chain of adjunctions

$$\cdots \dashv \mathrm{R}\,\mathrm{Inf}_P^M(-) \otimes \omega^{-1} \dashv \mathrm{RH}^0(U, \omega \otimes_k -) \dashv \mathrm{R}\,\mathrm{Inf}_P^M \dashv \mathrm{RH}^0(U, -) \dashv \cdots$$