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Weak Holonomicity for Equivariant $\mathcal{D}$-Modules on Rigid Analytic Spaces

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Weak holonomicity for equivariant $\mathcal{D}$-modules on rigid analytic spaces

by

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Contents

Chapter 1

Introduction

The category of coherent $\mathcal{D}_X$-modules (differential equations) on a smooth $\mathbb{C}$-analytic variety X is a classical object. Among its many applications to representation theory, we mention the Beilinson-Bernstein theorem, which relates the representations of a given semi-simple complex Lie-algebra to $\mathcal{D}$-modules on its flag variety. Many interesting representations correspond thereby to so-called holonomic modules and satisfy many finiteness properties. A non-zero coherent $\mathcal{D}_X$-module M is called holonomic, if the dimension of its associated characteristic variety $Char(M)$ is exactly $\dim X$ (note that one always has $\dim(Char(M)) \geq \dim X$, by Bernstein's inequality). An equivalent definition makes use of the *duality functor*

$$\mathbb{D}: \ D^-(\mathcal{D}_X) \longrightarrow D^+(\mathcal{D}_X)^{op}, \ \ M^\cdot \mapsto R\mathcal{H}om_{\mathcal{D}_X}(M^\cdot, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[\dim X]$$

on the derived category $D^-(\mathcal{D}_X)$. A module M is then holonomic if and only if $H^i(\mathbb{D}M) = 0$, for all $i \neq 0$.

In the arithmetic setting, let K be a discretely valued complete non-archimedean field of mixed characteristic $(0, p)$ with valuation ring $\mathcal{R}$ and uniformiser π. Let $\mathbf{X}$ be a smooth rigid-analytic variety over K. In [6] Ardakov-Wadsley introduced a certain sheaf of infinite order differential operators $\widehat{\mathcal{D}}_{\mathbf{X}}$ on $\mathbf{X}$ and used it to define the abelian category $\mathcal{C}_{\mathbf{X}}$ of coadmissible $\widehat{\mathcal{D}}_{\mathbf{X}}$-modules. It is an arithmetic analogue of the category of coherent complex-analytic $\mathcal{D}$-modules. The sheaf $\widehat{\mathcal{D}}_{\mathbf{X}}$ is in fact a certain Fréchet completion of the sheaf of usual finite order (algebraic) differential operators $\mathcal{D}_{\mathbf{X}}$ [1].

In the context of $\mathcal{D}$-modules on smooth rigid analytic varieties, the notion of characteristic variety is much more complicated and not yet developed. In order to define a notion of holonomicity for $\widehat{\mathcal{D}}$-modules, the authors in [2], introduced a dimension theory for coadmissible $\widehat{\mathcal{D}}$-modules by using the homological grade of a module as its codimension. This is based on the key fact that whenever $\mathbf{X}$ is affinoid with free tangent module $\mathcal{T}(\mathbf{X})$, then $\widehat{\mathcal{D}}(\mathbf{X})$ is almost Auslander-Gorenstein (it is a well-behaved inverse limit of Auslander-Gorenstein K-algebras). They then proved Bernstein's inequality in this setting and characterize weak holonomicity as being of minimal dimension. It should be pointed out that the abelian subcategory $\mathcal{C}_{\mathbf{X}}^{wh} \subset \mathcal{C}_{\mathbf{X}}$ of weakly holonomic modules does not yet satisfy all desired finiteness properties and serves only as a first well-behaved approximation (hence the adjective 'weak').

Recently, K. Ardakov introduced in [4] the category of coadmissible equivariant $\mathcal{D}_{\mathbf{X}}$-modules on smooth rigid analytic spaces endowed with suitable group actions. Let us explain what the

[1] In the case of the flag variety, the sheaf on the Zariski-Riemann space of X associated with the sheaf $\widehat{\mathcal{D}}_X$ was introduced and studied by Huyghe-Patel-Schmidt-Strauch in [13], where it is called $\mathscr{D}_\infty$.

equivariant setting is. Let $\mathbf{X}$ be a smooth rigid K-analytic variety and G a p-adic Lie group (such as $\mathrm{GL}_n(\mathbb{Q}_p)$) which acts continuously on $\mathbf{X}$ [2]. A coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-module on $\mathbf{X}$ is, vaguely speaking, a G-equivariant $\mathcal{D}_{\mathbf{X}}$-module (in the usual sense) which satisfies additional finiteness conditions. These modules form an abelian category $\mathcal{C}_{\mathbf{X}/G}$. If the group $G = 1$ is trivial, we recover the category $\mathcal{C}_{\mathbf{X}}$.

Motivated by these results, the aim of this thesis is to develop a notion of weak holonomicity in this equivariant setting, i.e. to define an equivariant analogue of the category $\mathcal{C}^{wh}_{\mathbf{X}}$, at least in the case of rigid analytic flag varieties. The main result is the following theorem, which is proved in chapter 4:

Theorem 1 (Bernstein's inequality for rigid analytic flag varieties):
Let $\mathbb{G}$ be a connected, simply connected, split semi-simple algebraic group over K and let $G := \mathbb{G}(K)$. Let $\mathbf{X}$ be the rigid analytification of the flag variety of $\mathbb{G}$, endowed with its natural G-action (by conjugating Borel subgroups of $\mathbb{G}$). Then Bernstein's inequality holds for any non-zero module $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$, i.e. $\dim(\mathcal{M}) \geq \dim \mathbf{X}$.

We emphasize that the arguments used in the proof of this theorem can in fact be applied to larger classes of spaces, for example, poly-discs, affine spaces (with suitable actions of compact Lie groups) or G-projective varieties (Zariski-closed stable subspaces of analytic projective space $\mathbb{P}^{n,an}_K$). This establishes Bernstein's inequality in all these cases. We hope to extend this results in the near future in order to include even more spaces.

In chapter 2 we recall some basic notions and properties of rigid analytic geometry and of p-adic Lie groups, then we summarize the theory of coadmissible equivariant $\mathcal{D}$-modules developed by K.Ardakov in [4]. Chapter 3 and chapter 4 are dedicated to the development of a dimension theory for coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules. The main point is the following key proposition. In order to formulate it, we assume that $\mathbf{X}$ is affinoid and G is compact such that $(\mathbf{X}, G)$ is small (see for the main body of the text for a precise definition of this technical condition). There is a K-Fréchet-Stein algebra $\widehat{\mathcal{D}}(\mathbf{X}, G)$, which can be viewed as a certain completion of the skew-group K-algebra $\mathcal{D}(\mathbf{X}) \rtimes G$. Here, $\mathcal{D}(\mathbf{X}) \rtimes G$ is a certain crossed product which contains $\mathcal{D}(\mathbf{X})$ as a subring and G as a subgroup in the group of invertible elements $(\mathcal{D}(\mathbf{X}) \rtimes G)^{\times}$.

Key proposition: Let $\mathbf{X}$ be a smooth affinoid variety of dimension d and G be a compact p-adic Lie group acting continuously on $\mathbf{X}$ such that $(\mathbf{X}, G)$ is small. Then the Fréchet-Stein K-algebra $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is isomorphic to the inverse limit $\varprojlim_n D_{n,G}$, where each $D_{n,G}$ is an Auslander-Gorenstein ring of self-injective dimension at most $2d$.

The proposition allows us to follow the non-equivariant setting and obtain the grade as a codimension function. This leads to a well-behaved definition of dimension for coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules. Then we can define correctly the dimension for coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules on a general rigid analytic variety $\mathbf{X}$ using globalization via admissible affinoid coverings.

After having introduced the dimension theory on the category $\mathcal{C}_{\mathbf{X}/G}$, we then study the question whether Bernstein's inequality holds for all G-equivariant $\mathcal{D}$-modules of $\mathcal{C}_{\mathbf{X}/G}$. If it is satisfied, we can define the notion of an equivariant weakly holonomic module on $\mathbf{X}$, and hence form the subcategory $\mathcal{C}^{wh}_{\mathbf{X}/G}$ of $\mathcal{C}_{\mathbf{X}/G}$ of equivariant weakly holonomic modules.

[2] see [4, Section 3.1] for a precise definition

As noticed above, on a complex smooth algebraic variety X, the restriction of the dual functor $\mathbb{D}$ to the category of holonomic modules is isomorphic to

$$\mathcal{E}xt^{\dim X}_{\mathcal{D}_X}(-,\mathcal{D}_X)\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1}.$$

Even if a full derived dual functor in the rigid-analytic setting has not yet been defined, we go on and construct, for all non negative integers $i\in\mathbb{N}$, analogous 'Ext'-functors $E^i:\mathcal{C}_{\mathbf{X}/G}\to\mathcal{C}^r_{\mathbf{X}/G}$, where $\mathcal{C}^r_{\mathbf{X}/G}$ denotes the category of coadmissible G-equivariant right $\mathcal{D}_{\mathbf{X}}$-modules. Let us explain briefly their definition. Let $\mathcal{M}\in\mathcal{C}_{\mathbf{X}/G}$. Using the localisation functor $Loc_{\mathbf{X}}(-)$ from [4], the sheaf $E^i(\mathcal{M})$ is defined, locally, as follows (cf. 4.2.5). For each $\mathbf{U}$ in the set $\mathcal{B}$ of affinoid subdomains of $\mathbf{X}$ such that the tangent $\mathcal{O}(\mathbf{U})$-module $\mathcal{T}(\mathbf{U})$ admits a free Lie lattice, then

$$E^i(\mathcal{M})(\mathbf{U}):=\lim_{H} Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}),\widehat{\mathcal{D}}(\mathbf{U},H)),$$

where H runs over the set of all open compact subgroups of G such that $(\mathbf{U},H)$ is small. We will prove in the first part of chapter 4 that this is well-defined, which means that the limit exists and all transition maps are bijections. Furthermore, we will prove the following (cf. Theorem 4.2.22):

Theorem 2: For every $i\in\mathbb{N}$, $E^i(\mathcal{M})$ is a sheaf of coadmissible G-equivariant right $\mathcal{D}_{\mathbf{X}}$-module for every coadmissible G-equivariant left $\mathcal{D}$-module $\mathcal{M}\in\mathcal{C}_{\mathbf{X}/G}$.

We then define the functors

$$\mathcal{E}^i:\mathcal{C}_{\mathbf{X}/G}\longrightarrow\mathcal{C}_{\mathbf{X}/G}$$

for $i\geq 0$ by composing E^i with the side-changing functor $\mathcal{H}om_{\mathcal{O}_{\mathbf{X}}}(\Omega_{\mathbf{X}},-)$. Note that $\mathcal{E}^i$ is an analogue of the classical Ext-functor $\mathcal{E}xt^{\dim X}_{\mathcal{D}_X}(-,\mathcal{D}_X)\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1}$. We then easily verify that once Bernstein's inequality holds for $\mathcal{C}_{\mathbf{X}/G}$, $\mathcal{E}^i(\mathcal{M})=0$ for every equivariant weakly holonomic $\mathcal{D}_{\mathbf{X}}$-module $\mathcal{M}\in\mathcal{C}^{wh}_{\mathbf{X}/G}$ and every $i\neq d=\dim\mathbf{X}$. Furthermore

Theorem 3: The functor

$$\mathbb{D}:=\mathcal{E}^d|_{\mathcal{C}^{wh}_{\mathbf{X}/G}}$$

induces an endofunctor on the category $\mathcal{C}^{wh}_{\mathbf{X}/G}$ and satisfies $\mathbb{D}^2=id$.

The functor $\mathbb{D}$ can therefore be regarded as the correct analogue of the classical duality functor. We call it the *duality functor* on the category $\mathcal{C}^{wh}_{\mathbf{X}/G}$.

In the last chapter, we will give some concrete examples of equivariant weakly holonomic modules. Throughout we always assume that Bernstein's inequality is valid for the category $\mathcal{C}_{\mathbf{X}/G}$. We first present a natural way to construct objects $\mathcal{C}^{wh}_{\mathbf{X}/G}$ via an extension functor $E_{\mathbf{X}/G}$ from G-equivariant (coherent) $\mathcal{D}_{\mathbf{X}}$-modules of minimal dimension to $\mathcal{C}_{\mathbf{X}/G}$, and then prove that this functor preserves equivariant weak holonomicity. We recall here that in the classical theory over complex algebraic varieties, all integrable connections are actually holonomic. In our setting, and if $G=1$ is trivial, all integrable connections on a smooth rigid-analytic space $\mathbf{X}$ are known to be weakly holonomic [2]. The point is that, for any affinoid subdomain $\mathbf{U}$ such that $\mathcal{T}(\mathbf{U})$ admits a free Lie lattice:

1. The $\mathcal{D}(\mathbf{U})$-action on $\mathcal{M}(\mathbf{U})$, where $\mathcal{M}$ is an integrable connection, extends naturally to a $\widehat{\mathcal{D}}(\mathbf{U})$-action under which $\mathcal{M}(\mathbf{U})$ becomes a coadmissible $\widehat{\mathcal{D}}(\mathbf{U})$-module.

2. The ring homomorphism $\mathcal{D}(\mathbf{U})\longrightarrow\widehat{\mathcal{D}}(\mathbf{U})$ is faithfully flat.

When working with a non-trivial p-adic group $G \neq 1$, to have a $\widehat{\mathcal{D}}(\mathbf{X}, G)$-action on a G-equivariant integrable connection extending its given $(\mathcal{D} - G)$-structure, needs a real condition, which we call 'strongly equivariant' (cf. Proposition 5.1.4).

We conclude the chapter 5 by constructing a large class of equivariant weakly holonomic modules on rigid flag varieties. Let $\mathbf{X}$ be the rigid flag variety associated to a connected, simply connected, split semi-simple algebraic group $\mathbb{G}$ over K. Let $\mathbb{P}$ be a parabolic subgroup of $\mathbb{G}$. Let $\mathfrak{g}$, $\mathfrak{p}$ be the Lie algebras of $\mathbb{G}$ and $\mathbb{P}$, respectively. Let $G := \mathbb{G}(K)$ and $P := \mathbb{P}(K)$. In [4], K.Ardakov has proved an analogue of the Beilinson-Bernstein theorem for trivial character in this p-adic setting. More precisely, he defined the Fréchet-Stein K-algebra[3] $\widehat{U}(\mathfrak{g}, G)$,which is, roughly speaking, a certain completion of the skew-group algebra $U(\mathfrak{g}) \rtimes G$, then proved that the localization functor on the category of coadmissible $\widehat{U}(\mathfrak{g}, G)_0$-modules is an equivalence of categories with the category $\mathcal{C}_{\mathbf{X}/G}$. In [19], the authors constructed a functor $M \mapsto D(G, K) \otimes_{D(\mathfrak{g},P)} M$ from the parabolic BGG category $\mathcal{O}_0^{\mathfrak{p}}$ to coadmissibe modules over the locally analytic distribution algebra $D(G, K)$. These modules are locally analytic globalizations of the classical Verma modules and their simple constituents. We show that Orlik-Strauch modules localize to G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules which are weakly holonomic.

Theorem 4: The localization $Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},G)}(D(G, K) \otimes_{D(\mathfrak{g},P)} M))$ is a G-equivariant weakly holonomic module for any $U(\mathfrak{g})$-module $M \in \mathcal{O}_0^{\mathfrak{p}}$.

Notation: Throughout this paper, we fix a complete discrete valuation field K of mixed characteristic $(0, p)$ with valuation ring $\mathcal{R}$ and a uniformiser $\pi \in \mathcal{R}$. For any ring R, all R-modules, if not further specified, are left modules.

[3] here, we should understand in more general sense, as defined in [25]

Chapter 2

Background material

2.1 Rigid analytic varieties

We begin by collecting some notions and standard results about rigid analytic varieties. We refer to [8, 7] for a quite complete and systematic treatments on the theory. Conrad's note [10] will be an interesting reference for those who want a brief overview.

2.1.1 Affinoid K-spaces and affinoid subdomains

Let $(K, |\ .\ |)$ be a complete non-Archimedean field. Its algebraic closure will be denoted by $\overline{K}$. Note that the absolute value $|\ .\ |$ of K extends uniquely to $\overline{K}$ and we still denote it by $|\ .\ |$.

Definition 2.1.1. *The Tate algebra in n-variables $T_n := K\langle x_1, ..., x_n\rangle$ is the K-algebra of all formal power series*

$$\sum_{\nu\in\mathbb{N}^n} a_\nu x^\nu \in K[\![x_1, ..., x_n]\!],\ a_\nu \in K \text{ such that } \lim_{|\nu|\longrightarrow 0} |\ a_\nu\ |= 0,$$

where $x^\nu = x_1^{\nu_1}...x_n^{\nu_n}$ and $|\ i\ |= i_1 + ... + i_n$ for all n-tuple $i = (i_1, ..., i_n) \in \mathbb{N}$.

We may consider T_n as the K-algebra of convergent power series on the n-dimensional unit ball $\mathbb{B}^n(\overline{K})$. We equip T_n with a norm as follows. Let $f(x) = \sum_\nu a_\nu x^\nu$, then

$$|\ f\ |:= \max_\nu |\ a_\nu\ |< \infty.$$

This norm is called the *Gauss norm* and it is well-known that T_n is a Banach K-algebra with respect to the Gauss norm. Here, by Banach K-algebra, we mean a normed K-algebra which is complete under the given norm on it.
Concerning the algebraic properties, the Tate algebras T_n are noetherian. Similarly to the ring of polynomials in n-variables over field, each T_n has Krull dimension n.

Definition 2.1.2. *Let A be a K-algebra. Then A is called an affinoid K-algebra if there is an epimorphism of K-algebras*

$$\alpha : T_n \longrightarrow A \text{ for some } n \in \mathbb{N}.$$

An *affinoid K-space* is a set $Sp(A)$ consisting of the maximal ideals in an affinoid K-algebra A. For a point $x \in Sp(A)$, we let $\mathfrak{m}_x$ denote the corresponding maximal ideal in A.
Note that each element $f \in A$ can be considered as a function on $Sp(A)$ in the following way. For any $x \in Sp(A)$, then $f(x)$ is the residue class of f in $A/\mathfrak{m}_x$, which is a finite extension field of K.

After embedding $A/\mathfrak{m}_x$ in to an algebraic closure $\bar{K}$ of K, we may consider $f(x)$ as an element of $\overline{K}$. Therefore, to every $f \in A$ one associates the following function:

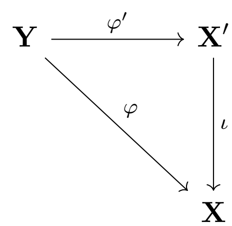
$$\begin{aligned} Sp(A) &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto |f(x)|. \end{aligned}$$

There is a natural (Zariski) topology on $Sp(A)$ generated by the subsets of the form

$$D_f = \{x \in Sp(A) : f(x) \neq 0\}, \quad \text{with } f \in A.$$

When studying sheaves on an affinoid K-space or more generally on rigid analytic spaces, it is much more convenient to introduce a 'new topology' namely the Grothendieck topology rather than to work with the Zariski topology. We will explain this more precisely in the next subsection.

Definition 2.1.3. *Let $\mathbf{X} = Sp(A)$ be an affinoid K-space. By affinoid subdomain of $\mathbf{X}$, we mean a subset $\mathbf{U} \subset \mathbf{X}$ such that there is a morphism of affinoid K-spaces $\iota : \mathbf{X}' \longrightarrow \mathbf{X}$ such that $\iota(X') \subset \mathbf{U}$ and which satisfies the following universal property:*
For any morphism of affinoid K-spaces $\varphi : \mathbf{Y} \longrightarrow \mathbf{X}$ satisfying $\varphi(\mathbf{Y}) \subset \mathbf{U}$, there exists a unique morphism $\varphi' : \mathbf{Y} \longrightarrow \mathbf{X}'$ such that the diagram



is commutative. We then say that the morphism $\iota : \mathbf{X}' \longrightarrow \mathbf{X}$ represents $\mathbf{U}$.

The set of all affinoid subdomains of $\mathbf{X}$ is denoted by $\mathbf{X}_w$. It is proved that if $U \in \mathbf{X}_w$, then the morphism $\iota : \mathbf{X}' \longrightarrow \mathbf{X}$ representing U is a bijection between $\mathbf{X}'$ and $\mathbf{U}$, so that $\mathbf{U}$ is equipped with a structure of affinoid K-space inherited from $\mathbf{X}'$.
Below we have some examples of (special) affinoid subdomains of $\mathbf{X}$:

Example 2.1.4. *([8, Definition 3.7, Proposition 3.11])*
Let $\mathbf{X} = Sp(A)$ and $f_0, f_1, ..., f_r, g_1, ..., g_s \in A$. For every $d \in \mathbb{N}$, we denote

$$A\langle\xi\rangle = A\langle\xi_1, ..., \xi_d\rangle = \{\sum_{\nu} a_\nu \xi^\nu : a_\nu \in A, \lim_{|\nu|\to\infty} |a_\nu| = 0\}$$

the algebra of restricted power series in ξ with coefficients in A.

1. ***Weierstrass subdomain***

$$\mathbf{X}(f_1, ..., f_r) = \{x \in X : |f_i(x)| \leq 1\}.$$

Then $\mathbf{X}(f_1, ..., f_r) \cong Sp(A\langle f\rangle)$ with $A\langle f\rangle$ is the affinoid K-algebra

$$A\langle f\rangle := A\langle\xi_1, ..., \xi_r\rangle/(\xi_1 - f_1, ..., \xi_r - f_r).$$

2. ***Laurent subdomain***

$$\mathbf{X}(f_1,...,f_r,g_1^{-1},...,g_s^{-1}) = \{x \in \mathbf{X} : |f_i(x)| \leq 1, |g_j(x)| \geq 1\}.$$

Then $\mathbf{X}(f_1,...,f_r,g_1^{-1},...,g_s^{-1}) = Sp(A\langle f, g^{-1}\rangle)$ *with* $A\langle f, g^{-1}\rangle$ *is the affinoid* K*-algebra*

$$A\langle f\rangle := A\langle \xi_1,...,\xi_r,\zeta_1,...,\zeta_s\rangle/(\xi_1 - f_1,...,\xi_r - f_r, 1 - g_1\zeta_1,...,1 - g_s\zeta_s).$$

Then $\boldsymbol{X}\left(\frac{f_1}{f_0},...,\frac{f_r}{f_0}\right) \cong Sp(A\langle\frac{f}{f_0}\rangle)$ *with* $A\langle\frac{f}{f_0}\rangle$ *is the affinoid* K*-algebra*

$$A\langle\frac{f}{f_0}\rangle := A\langle\xi_1,...,\xi_r\rangle/(f_1 - f_0\xi_1,...,f_r - f_0\xi_r).$$

2.1.2 Rigid analytic varieties

Definition 2.1.5. *A Grothendieck toplology* $\mathfrak{T}$ *consists of a category* $Cat\mathfrak{T}$ *and a set* $Cov\mathfrak{T}$ *of families* $(U_i \longrightarrow U)_{i\in I}$ *of morphisms in* $Cat\mathfrak{T}$*, called coverings, such that the following conditions hold:*

(i) If $\Phi : U \longrightarrow V$ *is an isomorphism in* $Cat\mathfrak{T}$*, then* $(\Phi) \in Cov\mathfrak{T}$*.*

(ii) If $(U_i \longrightarrow U)_{i\in I}$ *and* $(V_{ij} \longrightarrow U_i)_{j\in J_i}$ *belong to* $Cov\mathfrak{T}$ *for all* i*, then so is the composition* $(V_{ij} \longrightarrow U_i \longrightarrow U)_{i\in I, j\in J}$*.*

(iii) If $(U_i \longrightarrow U)_{i\in I}$ *is in* $Cov\mathfrak{T}$ *and* $V \longrightarrow U$ *is a morphism in* $Cat\mathfrak{T}$*, then the fiber product* $U_i \times_U V$ *exists in* $Cat\mathfrak{T}$ *and* $(U_i \times_U V \longrightarrow V)_{i\in I}$ *is in* $Cov\mathfrak{T}$*.*

An ordinary topology on a set X is a first (and natural) example of Grothendieck topology. Indeed, if X is a set and $Cat\mathfrak{T}$ is a category of certain subsets in X with inclusion morphisms, then the first condition in the definition is trivial while the last two conditions can be interpreted as

(ii) If $U = \cup_{i\in I}U_i$ and $U_i = \cup_{j\in J_i}V_{ij}$ are coverings in $Cat\mathfrak{T}$, then so is $U = \cup_{i,j}V_{ij}$.

(iii) If $U = \cup_{i\in I}U_i$ is a covering and $V \hookrightarrow U$ is an inclusion, then $V \cap U_i \in Cat\mathfrak{T}$ for all i and $V = \cup_{i\in I}V \cap U_i$ is a covering.

A set X which is equipped with a Grothendieck topology $\mathfrak{T}$ is called **G**-topological space. If $\mathbf{U} \in Cat\mathfrak{T}$, then U is called an *admissible open.* If $(U \longrightarrow U_i)_{i\in I}$ is an element of $Cat\mathfrak{T}$, then it is called an *admissible covering.*

Let $\mathbf{X}$ be an affinoid K-space. Then the *weak Grothendieck topology* on $\mathbf{X}$ is the Grothendieck topology given by the category $Cat\mathfrak{T}$ of affinoid subdomains of $\mathbf{X}$ with inclusions as morphisms and the set $Cov\mathfrak{T}$ consisting of finite families $(U_i \longrightarrow U)$ of inclusion of affinoid subdomains in $\mathbf{X}$ such that $U = \cup_i U_i$. The *strong Grothendieck topology* on the affinoid K-space $\mathbf{X}$ is the Grothendieck topology induced from the weak Grothendieck topology by adding more admissible open sets (not only affinoid subdomains) and more admissible coverings (not only finite coverings) in a certain way. More generally, we allow ourselves to give the definition of strong Grothendieck topology as follows. A Grothendieck topology on a set X is called strong if it satisfies the following conditions:

(G_0) $\emptyset$ and X are admissible open.

(G_1) If $(U_i)_{i\in I}$ is an admissible covering of an admissible open subset $U \subset X$ and $V \subset U$ is a subset such that $V \cap U_i$ is admissible open for all i, then V is admissible open in X.

(G_3) If $(U_i)_{i\in I}$ is a covering of an admissible open subset $U \subset X$ by admissible open subsets $U_i \subset U$ such that $(U_i)_{i\in I}$ admits an admissible covering of U as refinement. Then $(U_i)_{i\in I}$ itself is an admissible covering.

(see [8, Definition 5.1.4] for more details).

Let X be a $\mathbf{G}$-topological space. As usual, there are notions of presheaves and sheaves on X. Ignoring the technical tricks, the main diffirence here is that in stead of working with open subsets and coverings in an ordinary topology, we work with admissible open subsets and admissible coverings in a Grothendieck topology. Then the basic definitions and properties stay the same.

Let us now describe the structure sheaf $\mathcal{O}_{\mathbf{X}}$ on an affinoid K-space $\mathbf{X}$. For any affinoid subdomain $U \subset \mathbf{X}$, let $\mathcal{O}(U)$ denote the affinoid K-algebra corresponding to U. If $V \subset U$ is another affinoid subdomain of $\mathbf{X}$, then there is a canonical morphism of affinoid K-algebras (which is called a restriction map)

$$r_V^U : \mathcal{O}(U) \longrightarrow \mathcal{O}(V).$$

Then $\mathcal{O}_{\mathbf{X}}$ is a presheaf of affinoid functions on $\mathbf{X}$ such that for any $x \in X$ the stalk

$$\mathcal{O}_{\mathbf{X},x} := \varinjlim_{x\in U} \mathcal{O}(U),$$

where U runs over the set of affinoid subdomains of $\mathbf{X}$ containing x, is a local ring with maximal ideal $\mathfrak{m}_x\mathcal{O}_{X,x}$ ([8, Proposition 4.1.1]). Moreover:

Theorem 2.1.6. *(Tate) The presheaf $\mathcal{O}_{\mathbf{X}}$ of affinoid functions on the affinoid K-space $\mathbf{X}$ is a sheaf with respect to the weak Grothendieck topology. Furthermore, any finite covering $\mathcal{U}$ of $\mathbf{X}$ by affinoid subdomains is acyclic with respect to $\mathcal{O}_{\mathbf{X}}$.*

The structure sheaf $\mathcal{O}_{\mathbf{X}}$ on $\mathbf{X}$ together with the weak Grothendieck topology extends in a natural way to a sheaf on $\mathbf{X}$ together with the strong Grothendieck topology by [8, Corollary 5.2.5].
The notion of (locally) ringed K-spaces and morphisms between them can be naturally adapted to $\mathbf{G}$-topological spaces.
A trivial example to us will be the affinoid K-space $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ with the strong Grothendieck topology.
We can now state the definition of rigid analytic K-spaces as follows.

Definition 2.1.7. *A rigid analytic K-space is a locally $\mathbf{G}$-ringed K-space $(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ such that*

(i) The Grothendieck topology on $\mathbf{X}$ is a strong Grothendieck topology.

(ii) $\mathbf{X}$ admits an admissible covering $(U_i)_{i\in I}$, where each $(X_i, \mathcal{O}_{\mathbf{X}}|_{U_i})$ is an affinoid K-space for all i.

A morphism of rigid K-spaces $(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \longrightarrow (\mathbf{Y}, \mathcal{O}_{\mathbf{Y}})$ is a morphism between locally $\mathbf{G}$-ringed K-spaces.

2.1.3 Coherent sheaves on rigid analytic spaces

Let $\mathbf{X} = Sp(A)$ be an affinoid K-space and M be a A-module. Then we define

$$\tilde{M} := \mathcal{O}_{\mathbf{X}} \otimes_A M.$$

This is a sheaf of $\mathcal{O}_{\mathbf{X}}$-modules on $\mathbf{X}$ and we call it the $\mathcal{O}_{\mathbf{X}}$-module associated to M. The functor $\tilde{(.)}$ is exact, fully faithful from the category of A-modules to $\mathcal{O}_{\mathbf{X}}$-modules ([8, Proposition 6.1]).

Now let $\mathbf{X}$ be a rigid analytic space. As usual, we say that an $\mathcal{O}_{\mathbf{X}}$-module $\mathcal{M}$ on $\mathbf{X}$ is quasi-coherent if for any $x \in X$ there exists an admissible open subset U such that there is an exact sequence

$$\mathcal{O}_U^I \longrightarrow \mathcal{O}_U^J \longrightarrow \mathcal{M}|_U \longrightarrow 0.$$

$\mathcal{M}$ is called coherent if and only if there exists an admissible affinoid covering $\mathcal{U} = (U_i)_{i\in I}$ of $\mathbf{X}$ such that $\mathcal{M}|_{U_i}$ is an $\mathcal{O}_{U_i}$-module associated to a finitely generated $\mathcal{O}(U_i)$-module for all $i \in I$. More precisely, we then say that $\mathcal{M}$ is $\mathcal{U}$-coherent.

Theorem 2.1.8. *(Kiehl) Let $\boldsymbol{X} = Sp(A)$ be an affinoid K-space and $\mathcal{M}$ is an $\mathcal{O}_{\mathbf{X}}$-module. Then $\mathcal{M}$ is coherent if and only if $\mathcal{M}$ is associated to a finitely generated A-module.*

Let X_w denote the set of all affinoid subdomains of a rigid analytic space $\mathbf{X}$. Unlike in the case of affinoid K-spaces, X_w is not a Grothendieck topology on $\mathbf{X}$. However, $\mathbf{X}_w$ forms a basis for the Grothendieck topology on $\mathbf{X}$. Being a basis for the Grothendieck topology on $\mathbf{X}$ means that every admissible open subset has an admissible covering by elements in $\mathbf{X}_w$. In general, There is a natural way to construct a sheaf on $\mathbf{X}$ from a sheaf defined on a certain basis of (the Grothendieck topology) on $\mathbf{X}$. We state the following theorem:

Theorem 2.1.9. *([6, Theorem 9.1]) Let $\mathcal{B}$ be a basis for the Grothendieck topology on $\mathbf{X}$. Then the restriction functor is an equivalence of categories between the category of sheaves on $\mathbf{X}$ and the category of sheaves on $\mathcal{B}$.*

2.1.4 Construction of rigid analytic spaces

We explain in this section the ways of defining a rigid analytic variety from a scheme of locally finite type over K (which is known as the analytification functor or GAGA Serre's functor in the complex setting) and from a formal $\mathcal{R}$-scheme.
First, let us recall the construction of the analytificaltion functor in the rigid analytic setting. This is a functor which associates to each K-scheme X of locally finite type a rigid analytic K-space X^{an} (in [8, 5.4] it is denoted by X^{rig}).
Let $X = SpecK[\xi_1, ..., \xi_n]/\mathfrak{a}$ be an affine scheme with an ideal $\mathfrak{a} \subset K[\xi_1, ..., \xi_n]$. For all $i \in \mathbb{N}$, there is an inclusion of affinoid K-spaces

$$Sp\left(T_n^{(i)}/(\mathfrak{a})\right) \hookrightarrow Sp\left(T_n^{(i+1)}/(\mathfrak{a})\right),$$

where for all $i \leq 0$ and for some scalar $c \in K$ such that $|c| > 1$, $T_n^{(i)}$ denotes the Tate algebra $T\langle c^{-i}\xi_1, ..., c^{-i}\xi_n\rangle$. It is worth pointing out that this K-algebra contains all power series converging on the closed n-dimensional ball of radius $|c^i|$. Each affinoid K-space $Sp\left(T_n^{(i)}/(\mathfrak{a})\right)$ is contained in X. Now define:

$$X^{an} = \bigcup_{i=0}^{\infty} Sp\left(T_n^{(i)}/(\mathfrak{a})\right).$$

Then the set X^{an} can be equipped with a structure of locally $\mathbf{G}$-ringed K-space such that the natural morphism

$$\rho : X^{an} \longrightarrow X$$

is an bijection of X^{an} onto the closed points of X. In particular when $X = \mathbb{A}_K^n$, then X^{an} is the union of all the n-dimensional balls of radius $|c|^i$.

More generally, we have the following theorem:

Theorem 2.1.10. *([8, Definition and Proposition 5.4.3])*
Let $(X, \mathcal{O}_X)$ be a K-scheme of locally finite type. Then there is a rigid analytic K-space $(X^{an}, \mathcal{O}_{X^{an}})$ together with a morphism of locally $\mathbf{G}$-ringed K-spaces

$$(\rho, \rho^*) : (X^{an}, \mathcal{O}_{X^{an}}) \longrightarrow (X, \mathcal{O}_X)$$

satisfying the following universal property: For any rigid analytic K-space $(Y, \mathcal{O}_Y)$ and any morphism of locally $\mathbf{G}$-ringed K-spaces $(Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$, there exists a unique morphism of rigid analytic K-spaces $(Y, \mathcal{O}_Y) \longrightarrow (X^{an}, \mathcal{O}_{X^{an}})$ such that the following diagram is commutative:

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \longrightarrow & (X^{an}, \mathcal{O}_{X^{an}}) \\ & \searrow & \downarrow {\scriptstyle (\rho,\rho^*)} \\ & & (X, \mathcal{O}_X) \end{array}$$

Recall that the morphism $\rho : X^{an} \longrightarrow X$ induces a functor

$$\rho^* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_{X^{an}})$$
$$\mathcal{M} \longmapsto \mathcal{O}_{X^{an}} \otimes_{\rho^{-1}\mathcal{O}_X} \rho^{-1}\mathcal{M}.$$

The following result is due to [3, Proposition 2.2.1]

Proposition 2.1.11. *(i) The functor ρ^* is exact and faithful.*

(ii) If X is proper, then one has

$$H^i(X^{an}, \rho^*\mathcal{M}) = H^i(X, \mathcal{M})$$

for all $i \geq 0$ and all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{M}$.

Now, we look at the construction of rigid analytic spaces from *formal schemes*. Recall that the valuation ring $\mathcal{R}$ of K is π-adically complete. We may define the $\mathcal{R}$-algebra $\mathcal{R}\langle\xi_1, ..., \xi_n\rangle$ of restricted power series in the variable $\xi_1, ..., \xi_n$ as the subalgebra if the $\mathcal{R}$-algebra $\mathcal{R}[[\xi_1, ..., \xi_n]]$ of formal power series consisting of all power series $\sum_\nu c_\nu \xi^\nu$ with coefficients $c_\nu \in \mathcal{R}$ constituting a zero sequence in $\mathcal{R}$. Note that $\mathcal{R}\langle\xi_1, ..., \xi_n\rangle$ is noetherian([8, Remark 7.3.1]). Furthermore

$$\mathcal{R}\langle\xi_1, ..., \xi_n\rangle \cong \varprojlim_n \mathcal{R}[\xi_1, ..., \xi_n]/(\pi^n).$$

Definition 2.1.12. *(i) A topological R-algebra is called of topologically finite type if it is isomorphic to an $\mathcal{R}$-algebra of the form $\mathcal{R}\langle\xi_1, ..., \xi_n\rangle/I$ with an ideal I of $\mathcal{R}\langle\xi_1, ..., \xi_n\rangle$.*

(ii) A formal $\mathcal{R}$-scheme $\mathcal{X}$ is called ***locally of topologically finite type*** *if there is an open affine covering $(\mathcal{U}_i)_{i\in I}$ of $\mathcal{X}$ with $\mathcal{U}_i = Spf A_i$, where each A_i is an $\mathcal{R}$-algebra of topologically finite type.*

Let $\mathcal{X}$ be a formal $\mathcal{R}$ scheme of locally topologically finite type. Then there is a rigid analytic K-variety $\mathcal{X}_{rig}$ associated to $\mathcal{X}$, which is defined locally as follows. Suppose that $\mathcal{X} = Spf(A)$, where $A = \mathcal{R}\langle \xi_1, ..., \xi_n \rangle / I$. Then $A \otimes_{\mathcal{R}} K$ is an affinoid K-algebra ([8, 7.4]. In fact

$$A \otimes_{\mathcal{R}} K \cong K\langle \xi_1, ..., \xi_n \rangle / IK\langle \xi_1, ..., \xi_n \rangle.$$

We define

$$\mathcal{X}_{rig} := Sp(A \otimes_{\mathcal{R}} K).$$

If $\varphi : Spf(A) \longrightarrow SpfB$ is a morphism of affine formal $\mathcal{R}$-scheme. Then it is induced from a unique $\mathcal{R}$-homomorphism $\varphi^* : B \longrightarrow A$. Then the corresponding generic fiber

$$\varphi^*_{rig} : B \otimes_{\mathcal{R}} K \longrightarrow A \otimes_{\mathcal{R}} K$$

determines a morphism of affinoid K-varieties

$$\varphi_{rig} : Sp(A \otimes_{\mathcal{R}} K) \longrightarrow Sp(B \otimes_{\mathcal{R}} K).$$

More generally

Proposition 2.1.13. *([8, Proposition 7.4.3]) The functor $A \longmapsto A \otimes_{\mathcal{R}} K$ on R-algebra of topological finite type gives rise to a functor $\mathcal{X} \longmapsto \mathcal{X}_{rig}$ from the category of formal $\mathcal{R}$-schemes that are locally of topologically finite type to the category of rigid analytic K-varieties.*

Given a rigid analytic K-variety $\mathbf{X}$. A formal $\mathcal{R}$-scheme of locally topologically finite type $\mathcal{X}$ is called a **formal $\mathcal{R}$-model** of $\mathbf{X}$ if $\mathcal{X}_{rig} = \mathbf{X}$. When $\mathbf{X} = Sp(A)$ is an affinoid K-variety. An $\mathcal{R}$-algebra of topologically finite type is called an **affine formal model** in A if $\mathcal{A} \otimes_{\mathcal{R}} K = A$.

2.2 Crossed products

Since we will usually be working with the notion of a crossed product, this subsection is devoted to recalling some basic facts concerning its definition and properties. For more details, the reader is recommended to take a look at [20], [17] and also [4].

All rings appearing in this subsection are supposed to be unital. For a ring R, we let $R^\times$ denote the set of all units in R.

Definition 2.2.1. *Let R be a ring and G be a group. Then a crossed product $R * G$ of R and G is a ring containing R and a set of units $\bar{G} = \{\bar{g}, g \in G\} \subset (R * G)^\times$ which is isomorphic to G such that:*

(i) $R * G$ is free as a right R-module with basis $\bar{G}$ with $\bar{1_G} = 1_R$,

(ii) $\bar{g_1} R = R \bar{g_1}$ and $\bar{g_1}\bar{g_2} R = \overline{g_1 g_2} R$ for all $g_1, g_2 \in G$.

Let $R * G$ be a crossed product. Thanks to (ii), the ring $R * G$ is also free as a left R-module. Given a crossed product $R * G$ of R and G, there are associated maps $\sigma : G \longrightarrow Aut(R)$ and $\tau : G \times G \longrightarrow R^\times$, defined as follow:

$$\sigma(g)(r) := \bar{g}^{-1} r \bar{g} \quad \text{and} \quad \tau(g_1, g_2) := (\overline{g_1 g_2})^{-1} \bar{g_1} \bar{g_2}, \quad \text{for all } r \in R \text{ and } g_1, g_2 \in G.$$

These maps yield the following properties:

$$\tau(g_1g_2, g_3)\tau(g_1, g_2)^{\sigma(g_3)} = \tau(g_1, g_2g_3)\tau(g_2, g_3) \tag{2.1}$$

and

$$\sigma(g_1)\sigma(g_2) = \sigma(g_1g_2)\eta(g_1, g_2), \tag{2.2}$$

where $\eta(g_1, g_2) \in Aut(R)$ denotes the multiplication by $\tau(g_1, g_2)$ on R and $\tau(g_1, g_2)^{\sigma(g_3)}$ denotes the right action of $\sigma(g_3) \in Aut(R)$ on $\tau(g_1, g_2) \in R$.

Conversely, given two maps $\sigma : G \longrightarrow Aut(R)$ (which is not necessary a group homomorphism) and $\tau : G \times G \longrightarrow R^\times$ satisfying (2.1) and (2.2), then we may define a crossed product

$$R * G = \{\sum_{\text{finite}} \bar{g}r_g \mid g \in G, r_g \in R\}$$

of G over R, where its addition is defined as usual and the multiplication law is determined by the following rules:

$$\bar{g_1}\bar{g_2} = \overline{g_1g_2}\tau(g_1, g_2) \tag{2.3}$$

and

$$r\bar{g_1} = \bar{g_1}\sigma(g_1^{-1})(r) \tag{2.4}$$

for all $r \in R$ and $g_1, g_2 \in G$.

The first example of a crossed product will be the group ring $R[G]$ of G over R. In that case the maps σ and τ are both trivial, which means that $\sigma(g_1) = 1$ and $\tau(g_1, g_2) = 1$ for all $g_1, g_2 \in G$.

Another important example to us is when σ is a homomorphism of groups (thus the group G acts on R via σ) and τ is trivial, we obtain the skew product $R \rtimes G$. By definition, it is the free right R-module with basis G:

$$R \rtimes G = \{\bar{g}_0r_0 + ... + \bar{g}_nr_n, \, r_i \in R, g_i \in G, n \in \mathbb{N}\}$$

Now the equalities (2.1) and (2.2) become

$$\bar{g_1}\bar{g_2} = \bar{g_1g_2} \quad \text{and} \quad r\bar{g_1} = \bar{g_1}\sigma(g_1^{-1})(r).$$

By consequence, we can drop the overbars of $\bar{g} \in R$ and write it simply by $g \in G$. It follows that $R \rtimes G$ contains G as a subgroup of the group of units $(R \rtimes G)^\times$. For $g \in G, r \in R$, in the sequel, we let $g.r$ (resp. $r.g$) denote the image of r under $\sigma(g)$ (resp. $\sigma(g^{-1})$). This corresponds to the left (resp. right) action of G on R. The multiplication in $R \rtimes G$ is then described by:

$$(g_1r_1)(g_2r_2) = (g_1g_2)((g_2^{-1}.r_1)r_2)$$

for any $r_1, r_2 \in R$ and $g_1, g_2 \in G$. The ring $R \rtimes G$ naturally contains R as a subring. Furthermore one has the following relation in $R \rtimes G$:

$$grg^{-1} = g.r, \text{ for any } g \in G, r \in R.$$

Remark 2.2.2. *If we consider the right action of the group G on R, then we can see that the skew group ring $R \rtimes G$ can be also considered as a free left R-module with basis G. More precisely, each* *element in $R \rtimes G$ has a unique representation $\sum_{g\in G} r_g g$, where $r_g \in R$ is zero for all but finitely many $g \in G$. Indeed, the relation $grg^{-1} = g.r$ implies that*

$$sg = gg^{-1}sg = g(g^{-1}sg) = g(g^{-1}.s).$$

Under this representation, one can rewrite the multiplication on $R \rtimes G$ as follows:

$$(rg)(r'g') = (r(g.r'))(gg'). \tag{2.5}$$

Note that in [4, 2.2], the author has considered $R \rtimes G$ as a free left R-module with basis G. Hence he defined its multiplication by using (2.3).

Recall [4, Definition 2.2.1] that a trivialisation (of the skew-group ring $R \rtimes G$) is a group homomorphism $\beta : G \longrightarrow R^\times$ such that

$$\beta(g)r\beta(g)^{-1} = g.r \text{ for all } g \in G \text{ and } r \in R.$$

Note that whenever there is a trivialisation $\beta : G \longrightarrow R^\times$, then the skew-group ring $R \rtimes G$ is naturally isomorphic to the group ring $R[G]$ [4, Lemma 2.2.2]. The isomorphism is explicited by

$$\begin{aligned}\tilde{\beta} : R[G] &\longrightarrow R \rtimes G\\ r &\longmapsto r\\ g &\longmapsto \beta(g)^{-1}g\end{aligned}$$

for any $r \in R$ and $g \in G$.

Definition 2.2.3. *Let N be a normal sugroup of G and $\beta : N \longrightarrow R^\times$ be a trivialisation of $R \rtimes N$. We define*

$$R \rtimes_N G = R \rtimes_N^\beta G := \frac{R \rtimes G}{(R \rtimes G)(\tilde{\beta}(N) - 1)}$$

It is proved (loc.cit Lemma 2.2.4) that when β is G-equivariant, which means that $\beta(gng^{-1}) = g.\beta(n)$ for every $n \in N$ and $g \in G$, then $R \rtimes_N G$ is an associative ring containing R as a subring and there is a natural group homomorphism $G \longrightarrow (R \rtimes_N G)^\times$ by definition.

The following lemma will be useful for the next chapters. This is due to [22, Lemma 2.2]

Lemma 2.2.4. *Let $\varphi : R \longrightarrow A$ be a morphism of rings such that φ is left (resp. right) flat and that it factors through*

$$R \longrightarrow R * G \longrightarrow A.$$

Then the morphism $R * G \longrightarrow A$ is left (resp. right) flat.

2.3 Review on p-adic Lie groups

Similarly to real (or complex) Lie groups, a Lie group over a non-archimedean fields K (or p-adic Lie groups) is, rougly speaking, a manifold over K which admits a group structure compatible with its 'analytic structure'. In this subsection we recall some basic definitions and properties that may be used in the future. For more details, the reader are recommended to take a look at [23], [9] and [11].

Definition 2.3.1. *Let $U \subset K^n$ be an open subset, a map $f : U \longrightarrow K^n$ is called **locally analytic** if it is locally given by a convergent power series around each point in U. More precisely, if for any $x_0 \in U$, there exists a ball $B_r(x_0) =\subset U$ and a power series $F(X) \sum_\alpha v_\alpha X^\alpha$ satisfying $\lim_{|\alpha|\leftarrow 0} |v_\alpha| r^{|\alpha|}$ and such that $f(x) = F(x - x_0)$ for any $x \in B_r(x_0)$.*

We can define a n-dimensional (locally analytic) manifold over K in the usual way, namely a Hausdorff topological space M equipped with a (maximal) atlas $\mathcal{A}$ consisting of homomorphisms from open subsets of M onto open subset of K^n such that the transition map $\varphi \circ \phi^{-1}$ is locally analytic for all $\varphi, \phi \in \mathcal{A}$.
Analytic mappings between (locally analytic) manifolds are defined as usual (by checking analyticity on local charts). The set $C^{an}(M, K)$ of all locally analytic functions $f : M \longrightarrow K$ is a K-vector space with respect to pointwise addition and scalar multiplication and is functorial in M. Furthermore, $C^{an}(M, K)$ can be equipped with the structure of topological vector space.

Definition 2.3.2. *A p-adic Lie group is a manifold over K which carries the structure of a group such that the mulitiplication*

$$\begin{aligned} m_G : \ G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh \end{aligned}$$

and inverse map

$$\begin{aligned} i_G : G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are locally analytic.

Any p-adic Lie group is a totally disconnected locally compact topological group.

Definition 2.3.3. *Let G be a p-adic Lie group over K. Then the strong dual*

$$D(G, K) := C^{an}(G, K)'_b,$$

of the locally convex K-vector space $C^{an}(G, K)$ is called the (locally convex) vector space of K-valued distributions on G.

It is proved ([24, Proposition 2.3] that $D(G, K)$ can be equipped with a structure of an associative K-algebra. Furthermore, if G is compact, then $D(G, K)$ is a Fréchet K-algebra (i.e the underlining topology is Fréchet which is compatible with the K-algebra structure).

2.4 Equivariant sheaves on rigid analytic spaces

2.4.1 Group actions on rigid analytic spaces

Let $\mathbf{X}$ be a rigid analytic space over K. The spirit of the theory of equivariant sheaves on $\mathbf{X}$ is exactly the same as usual (when working on usual topological spaces). Let $\mathbf{X}, \mathbf{Y}$ be rigid analytic spaces. Let us recall below some essential definitions. Fix an abstract group G with unit element 1. Then G acts on $\mathbf{X}$ if there is a group homomorphism $\rho : G \longrightarrow Homeo(\mathbf{X})$ from G to the group $Homeo(\mathbf{X})$ of continuous bijections on $\mathbf{X}$. In this case, each g gives rise to a pair $(\rho(g)_*, \rho(g)^*)$ of equivalences of categories from the category of abelian sheaves on $\mathbf{X}$ to itself. More precisely, if $\mathcal{F}$ is a sheaf on $\mathbf{X}$ then $(\rho(g))^*(\mathcal{F})$ is the sheaf whose local sections are defined by $(\rho(g))^*(\mathcal{F})(U) =$

$\mathcal{F}(gU)$ for all admissible open subsets $U \subset \mathbf{X}$. Similarly, the sheaf $\rho(g)_*(\mathcal{F})$ is defined locally as $\rho(g)_*(\mathcal{F})(U) := \mathcal{F}(g^{-1}U)$ for every admissible open subset $U \subset \mathbf{X}$ (here we denote gU the image of U via the bijection $\rho(g)$ for all $g \in G$). In the sequel, we write g_* and g^* instead of $\rho(g)_*$ and $\rho(g)^*$ for short.
We recall the following definition from [4, Section 2.3]:

Definition 2.4.1. (*i*) *Let R-be a ring and $\mathcal{F}$ be a sheaf (of groups, of rings, of R-modules,etc) on $\mathbf{X}$. Then $\mathcal{F}$ is called G-equivariant if for each $g \in G$, there is an isomorphism of sheaves (of groups, of rings, of R-modules,etc) $g^{\mathcal{F}} : \mathcal{F} \xrightarrow{\sim} g^*\mathcal{F}$ such that $1^{\mathcal{F}} = Id$ and $(gh)^{\mathcal{F}} = h^*(g^{\mathcal{F}}) \circ h^{\mathcal{F}}$ for any $g, h \in G$.*

(*ii*) *Let $\mathcal{A}$ be a G-equivariant sheaf of R-algebras on $\mathbf{X}$. A G-equivariant sheaf of R-modules $\mathcal{M}$ is called G-equivariant sheaf of $\mathcal{A}$-modules if for any $g \in G$, $a \in \mathcal{A}$, one has*

$$g^{\mathcal{M}}(a.m) = g^{\mathcal{A}}(a).g^{\mathcal{M}}(m), \ (resp. \ g^{\mathcal{M}}(m.a) = g^{\mathcal{M}}(m).g^{\mathcal{A}}(a)).$$

Remark 2.4.2.

(i) Let U be a G-stable admissible open subset of $\mathbf{X}$. Then there is a left (resp. right) action of G on $\mathcal{A}(U)$ determined by

$$g.a := g^{\mathcal{A}}(a) \quad (resp. \quad a.g := (g^{-1})^{\mathcal{A}}(a))$$

for any $g \in G$ and $a \in \mathcal{A}(U)$.

(ii) Suppose that $V \subset U$ are G-stable admissible subsets of $\mathbf{X}$, then the restriction map $\mathcal{A}(U) \longrightarrow \mathcal{A}(V)$ is left (resp. right) G-equivariant.

The notion of equivariant sheaf of algebras on $\mathbf{X}$ is related to the notion of skew-group rings in the following way. Let $\mathcal{A}$ be a G-equivariant sheaf of R-algebras on $\mathbf{X}$. By using Remark 2.2.9(i) we can form the skew-group ring $\mathcal{A}(U) \rtimes G$ for any G-stable admissible open subset U of $\mathbf{X}$. The following proposition is just restated from [4, Proposition 2.3.5] but is also applied to G-equivariant right $\mathcal{A}$-modules.

Proposition 2.4.3. *[4, Proposition 2.3.5] If $\mathbf{X}$ is an admissible open subset with respect to the G-topology on $\mathbf{X}$. Then the functor of global sections $\Gamma(\mathbf{X}, -)$ sends G-equivariant left (resp. right) $\mathcal{A}$-modules to left (resp. right) $\mathcal{A}(\mathbf{X}) \rtimes G$-modules.*

Suppose for the moment that $\mathbf{X}$ is quasi-compact and quasi-separated. Then there is a Hausdorff topology on the group $Aut_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ of K-linear automorphisms of $\mathbf{X}$, which is described as follows. First, following [8, Theorem 4.1] there exists a formal model $\mathcal{X}$ for $\mathbf{X}$, which means that $\mathcal{X}$ is a quasi-compact admissible formal scheme over $\mathcal{R}$ such that $\mathbf{X} = \mathcal{X}_{rig}$, where rig is the functor which associates to each admissible formal scheme its generic fibre. Next, consider the group $\mathcal{G}(\mathcal{X}) := Aut_{\mathcal{R}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Then for each $n \geq 0$, the n-th congruence subgroup of $\mathcal{G}_{\mathcal{X}}$ is

$$\mathcal{G}_{\pi^n}(\mathcal{X}) := \ker[\mathcal{G}(\mathcal{X}) \longrightarrow Aut_{\mathcal{R}_n}(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})],$$

where $\mathcal{R}_n := \mathcal{R}/\pi^n\mathcal{R}$ and $\mathcal{X}_n := \mathcal{X} \otimes_R \mathcal{R}_n$. These subgroups are normal in $\mathcal{G}(\mathcal{X})$ and form a descending filtration of the group $\mathcal{G}(\mathcal{X})$, which will equip $\mathcal{G}(\mathcal{X})$ with a topological group structure. Since $\bigcap_n \mathcal{G}_{\pi^n}(\mathcal{X}) = 0$, $\mathcal{G}(X)$ is indeed Hausdorff. This topology induces a Hausdorff topology on $Aut_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ via the injective homomorphism of groups $\mathcal{G}(\mathcal{X}) \longrightarrow Aut_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ which is induced by the functor rig and we have the following theorem:

Theorem 2.4.4. *Let* $\mathbf{X}$ *be a quasi-compact quasi-separated rigid analytic variety over* K*. Then for any formal model* $\mathcal{X}$ *of* $\mathbf{X}$*, the congruence subgroups*

$$\mathcal{G}_{\pi^n}(\mathcal{X})_{rig} \quad \textit{for all } n \geq 0$$

generate a Hausdorff topology on $Aut_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ *such that* $Aut_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ *is a topological group. Furthermore, this topology is independent of the choice of a fomal model* $\mathcal{X}$ *of* $\mathbf{X}$*.*

Proof. [4, Theorem 3.1.5] □

Now let G be a topological group and $\mathbf{X}$ be a (general) rigid analytic space over K. The following definition is due to [4, Definition 3.1.8].

Definition 2.4.5. *We say that* G *acts continuously on* $\mathbf{X}$ *if there is a group homomorphism* $\rho : G \longrightarrow Aut_K(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ *such that for every quasi-compact quasi-separated admissible open subset* U *of* $\mathbf{X}$*, the following conditions hold:*

(a) The stabiliser G_U *of* U *is open in* G*,*

(b) The induced group homomorphism $\rho_U : G_U \longrightarrow Aut_K(U, \mathcal{O}_U)$ *is continuous with respect to the induced topology on* G_U *and the topology on* $Aut_K(U, \mathcal{O}_U)$ *defined in Theorem* 2.2.8

The following example is due to [3, Proposition 3.1.12]:

Example 2.4.6. *Let* $\mathbb{X}$ *be flat* $\mathcal{R}$*-scheme of finite presentation and* $\mathbb{G}$ *be a* $\mathcal{R}$*-group scheme which acts on* $\mathbb{X}$ *via* $\rho : \mathbb{G} \longrightarrow Aut(\mathbb{X})$*. Let* $\mathcal{X}$ *be the formal completion of* $\mathbb{X}$ *and* $\mathbf{X} := \mathcal{X}^{rig}$ *be its generic fibre (which is a rigid analytic variety [8, Proposition* 7.4.3*]). Write* $G := \mathbb{G}(K)$*. Then* G *acts continuously on* $\mathbf{X}$*.*

2.4.2 The completed skew-group algebra $\widehat{\mathcal{D}}(\mathbf{X}, G)$

We begin this section by recalling the notion of Lie-Rinehart algebras and its envelopping algebras, as introduced in [6]. Let R be a commutative ring and A be a commutative R-algebra. A R-Lie algebra L is called *Lie-Rinehart algebra* or a (R, A)*-Lie algebra* if it is also an A-module equipped with an A-linear Lie algebra homomorphism $\rho : L \longrightarrow Der_R(A)$ such that

$$[x, ay] = a[x, y] + \rho(x)(a)y$$

for all $x, y \in L$ and $a \in A$. Let (L, ρ) be an (R, A)-Lie algebra. The enveloping algebra of L is the unique associative R-algebra $U(L)$ which comes equipped with the canonical homomorphisms

$$i_A : A \longrightarrow U(L) \text{ and } \; i_L : L \longrightarrow U(L)$$

satisfying the following universal property: Let S be an associative R-algebra with an R-algebra homomorphism $j_A : A \longrightarrow S$ and an R-Lie algebra homomorphism $j_L : L \longrightarrow S$ such that $j_L(ax) = j_A(a)j_L(x)$ and $[j_L(x), j_A(a)] = j_A(\rho(x)(a))$ for any $a \in A, x \in L$. Then there is a unique R-algebra homomorphism $\varphi : U(L) \longrightarrow S$ such that $\varphi \circ i_A = j_A$, and $\varphi \circ i_L = j_L$.

Note that if L is *smooth* over A, which means that L is finitely generated projective as an A-module, then the morphisms i_A and i_L are injective. We can therefore identify A and L with its images in $U(L)$ via these morphisms.

A natural example of an (R, A)-Lie algebra is when $L = Der_R(A)$ and ρ is the identity.

It is proven in [21] that if A is a noetherian ring and L is a finitely generated A-module, then $U(L)$ is a (left and right) noetherian ring.

If $\varphi : A \longrightarrow B$ is a morphism of R-algebras. We say that the action of L on A lifts to B if there exists an A-linear Lie algebra homomorphism $\sigma : L \longrightarrow Der_R(B)$ such that for every $x \in L$, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho(x)} & A \\ \varphi \downarrow & & \downarrow \varphi \\ B & \xrightarrow{\sigma(x)} & B \end{array}$$

is commutative. If this is the case, then we obtain that $(B \otimes_A L, 1 \otimes \sigma)$ is an (R, B)-Lie algebra ([6, Lemma 2.2]).

Now, let $\mathbf{X}$ be an affinoid K-variety and G be a compact p-adic Lie group which acts continuously on $\mathbf{X}$. Let us fix a G-stable affine formal model $\mathcal{A}$ in $A := \mathcal{O}(\mathbf{X})$. Let $L := Der_K(A)$ denote the (K, A)-Lie algebra of K-derivations endowed with the natural action of G. An $\mathcal{A}$-submodule $\mathcal{L}$ of L is called *G-stable $\mathcal{A}$-Lie lattice* in L if it is a finitely presented $\mathcal{A}$-module which spans L as a K-vector space and is stable under the G-action and the Lie bracket on L.
For such a G-stable $\mathcal{A}$-Lie lattice $\mathcal{L}$, we denote by $\widehat{U(\mathcal{L})}$ the π-adic completion of the envelopping algebra $U(\mathcal{L})$ and write $\widehat{U(\mathcal{L})}_K := \widehat{U(\mathcal{L})} \otimes_{\mathcal{R}} K$. It is proved in [6] that $\widehat{U(\mathcal{L})}_K$ is an associative K-Banach algebra.

In the sequel, we suppose in addition that $\mathcal{L}$ is *smooth* as an $\mathcal{A}$-module. This extra condition ensures that the unit ball of the K-Banach algebra $\widehat{U(\mathcal{L})}_K$ is isomorphic to $\widehat{U(\mathcal{L})}$.
The G-action on $\mathcal{L}$ extends naturally on $U(\mathcal{L})$, hence on it π-adic completion $\widehat{U(\mathcal{L})}$ and on $\widehat{U(\mathcal{L})_K}$. Thus we may form the skew product $\widehat{U(\mathcal{L})_K} \rtimes G$. Now, since $\mathcal{A}$ is G-stable, the morphism $\rho : G \longrightarrow Aut(A)$ factors through $Aut(\mathcal{A})$. Write

$$G_{\mathcal{L}} := \rho^{-1}(\exp(p^{\epsilon}\mathcal{L})) \subset G. \tag{2.6}$$

Here $\epsilon = 1$ of $p = 1$; $\epsilon = 2$ if $p > 2$ and $\rho : G \longrightarrow Aut(\mathcal{A})$. Then it is proved ([4, Theorem 3.2.12] that there is a G-equivariant trivialisation

$$\beta_{\mathcal{L}} : G_{\mathcal{L}} \longrightarrow \widehat{U(\mathcal{L})_K}^{\times}$$

of the $G_{\mathcal{L}}$-action on $\widehat{U(\mathcal{L})_K}$. This implies that for any open normal subgroup H of G which is contained in $G_{\mathcal{L}}$, we may form the quotient $\widehat{U(\mathcal{L})_K} \rtimes_H G$ as defined in Definition 2.2.3. That is why we need the following definition (see [4, Definition 3.2.13] for more details):

Definition 2.4.7. *Let $\mathcal{A}$ be a G-stable affine formal model in A. Then a pair $(\mathcal{L}, J)$ is called an $\mathcal{A}$-trivialising pair if $\mathcal{L}$ is a G-stable $\mathcal{A}$-Lie lattice in L and J is an open normal subgroup of G contained in the subgroup $G_{\mathcal{L}}$ of G (which generally depends on $\mathcal{L}$).*

The set $\mathcal{I}(\mathcal{A}, \rho, G)$ of all $\mathcal{A}$-trivialising pairs is a directed poset with respect to the following order:

$$(\mathcal{L}_1, N_1) \leq (\mathcal{L}_2, N_2) \;\text{ iff }\; \mathcal{L}_2 \subset \mathcal{L}_1 \text{ and } N_2 \subset N_1.$$

At this point we can form the *completed skew-group algebra*

$$\widehat{\mathcal{D}}(\mathbf{X}, G) = \varprojlim_{(\mathcal{L},J)} \widehat{U(\mathcal{L})_K} \rtimes_J G,$$

where $(\mathcal{L}, J)$ runs over the set $\mathcal{I}(\mathcal{A}, \rho, G)$ of $\mathcal{A}$-trivialising pairs.

It is proved in [loc.cite] that this definition is independent of the choice of the fomal model $\mathcal{A}$ in A and $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is equipped with a structure of K-Fréchet algebra.
Since we want to equip $\widehat{\mathcal{D}}(\mathbf{X}, G)$ with a structure of two-sided Fréchet-Stein algebra, it is necessary to make use of the following definitions:

Definition 2.4.8. *([25, 3]) Let U be a K-algebra. Then U is called a (two-sided)* ***Fréchet-Stein algebra*** *if for any non-negative integer $n \geq 0$, there exists a Banach K-algebra U_n which is (two-sided) noetherian such that*

(i) The morphisms $U_{n+1} \longrightarrow U_n$ are (left and right) flat.

(i) $U \cong \varprojlim_n U_n$.

The following condition will also be necessary:

Definition 2.4.9. *A pair (U, H) is called small if:*

(a) U is an affinoid subdomain of $\mathbf{X}$,

(b) H is an open compact subgroup of the stabilizer $G_U = \{g \in G : gU \subset U\}$ of U,

(c) $\mathcal{T}(U) = Der_K(\mathcal{O}(U))$ admits a H-stable free $\mathcal{A}$-Lie lattice for some H-stable formal model $\mathcal{A}$ of $\mathcal{O}(U)$.

Here is an example:

Example 2.4.10. *Let us consider the analytification $\mathbb{P}^{1,an}_{\mathbb{Q}_p}$ of the projective $\mathbb{Q}_p$-scheme $\mathbb{P}^1_{\mathbb{Q}_p}$. One has that $\mathbb{P}^{1,an}_{\mathbb{Q}_p} = U_0 \cup U_\infty$, where*

$$U_0 = Sp\left(\mathbb{Q}_p\langle \tfrac{t}{w}\rangle\right) \simeq \{x \in \mathbb{Q}_p, |x| \leq 1\} \text{ and } U_\infty = Sp\left(\mathbb{Q}_p\langle \tfrac{w}{t}\rangle\right) \simeq \{y \in \mathbb{Q}_p : |y| \leq 1\}.$$

The group $G = SL_2(\mathbb{Q}_p)$ acts on $\mathbb{P}^{1,an}_{\mathbb{Q}_p}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .x = \tfrac{ax+b}{cx+d} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} .y = \tfrac{c+dy}{a+by}.$$

with $a, b, c, d \in \mathbb{Q}_p, ad - bc = 1, x \in U_0, y \in U_\infty$.
Let I^+ be the Iwahori subgroup of G, which is defined as the preimage of the standard Borel subgroup of $SL_2(\mathbb{F}_p)$ in $SL_2(\mathbb{Z}_p) \subset G$. More precisely

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I^+ \Longleftrightarrow a, b, c, d \in \mathbb{Z}_p,\ ad - bc = 1,\ \bar{c} = 0 \in \mathbb{F}_p. \tag{2.7}$$

Then

* *The open affinoid subset U_0 is I^+-stable. Indeed, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I^+$. The condition (2.7) tells us that $a, d \in \mathbb{Z}_p^\times$ and $c \in p\mathbb{Z}_p$. So for $x \in U_0$, we have $cx + d \in p\mathbb{Z}_p + \mathbb{Z}_p^\times$ implying that $|cx + d| = 1$. Thus*

$$|ax+b| \leq \max\{|ax|, |b|\} \leq 1 = |cx+d|.$$

By consequence, $|\frac{ax+b}{cx+d}| \leq 1$*, so every element of* I^+ *stablizes* U_0*.*

- *The pair* (U_0, I^+) *is small.*
  First, we note that $I^+ \subset SL_2(\mathbb{Z}_p)$ *is an open compact subgroup of* $SL_2(\mathbb{Q}_p)$*. We also see that* I^+ *stabilizes the affine formal model* $\mathbb{Z}_p\langle x\rangle$ *of* $\mathcal{O}(U_0) = \mathbb{Q}_p\langle x\rangle$*. To see this, let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I^+$ *and* $x \in U_0$*, we compute*

$$\frac{ax+b}{cx+d} = \frac{ax+b}{d(d^{-1}cx+1)} = \frac{ax+b}{d}.\left(\sum_{i\geq 0}(-1)^i(d^{-1}cx)^i\right).$$

  Here $d \in \mathbb{Z}_p^\times$*, as* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I^+$*. Thus,* $\frac{ax+b}{cx+d} \in \mathbb{Z}_p\langle x\rangle$*. So for each* $g \in I^+$ *such that* $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I^+$ *and* $f(x) = \sum_{i\geq 0} a_i x^i \in \mathbb{Z}_p\langle x\rangle$*, we obtain that*

$$(gf)(x) = f(g^{-1}x) = \sum_{i\geq 0} a_i \left(\frac{ax+b}{cx+d}\right)^i \in \mathbb{Z}_p\langle x\rangle.$$

  This prove that $\mathbb{Z}_p\langle x\rangle$ *is a* I^+*-stable affine formal model of* $\mathcal{O}(U_0) = \mathbb{Q}_p\langle x\rangle$*.*

  Next, we note that $\mathcal{T}(U_0) = Der_{\mathbb{Q}_p}(\mathcal{O}(U_0)) = \mathbb{Q}_p\langle x\rangle[\partial_x]$ *and* $\mathbb{Z}_p\langle x\rangle[\partial_x]$ *is a free* $\mathbb{Z}_p\langle x\rangle$*- Lie lattice of* $\mathcal{T}(U_0)$*. Let* $g \in I^+$ *and* $f \in \mathbb{Z}_p\langle x\rangle$*, then* $(g.\partial_x)(f) = g\partial_x(g^{-1}f) \in \mathbb{Z}_p\langle x\rangle$ *(here* $\partial_x(g^{-1}f)$ *is a function of* $\mathbb{Z}_p\langle x\rangle$*, since* I^+ *stabilizes* $\mathbb{Z}_p\langle x\rangle$*). This proves that* $\mathbb{Z}_p\langle x\rangle[\partial_x]$ *is* I^+*-stable and* (U_0, I^+) *is small.*

Similarly, one has that the subgroup $I^- := wI^+w$ *with* $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Q}_p)$ *stabilizes* U_∞ *and the pair* (U_∞, I^-) *is small.*

Definition 2.4.11. *Let* $\mathcal{A}$ *be a* G*-stable affine formal model in* $\mathcal{O}(\mathbf{X})$ *and* $\mathcal{L}$ *be a* G*-stable* $\mathcal{A}$*-Lie lattice in* $\mathcal{T}(\mathbf{X})$*. A chain* $(J_n)_{n\in\mathbb{N}}$ *of open normal subgroups of* G *is called a good chain for* $\mathcal{L}$ *if each pair* $(\pi^n\mathcal{L}, J_n)$ *is an* $\mathcal{A}$*- trivialising pair for all* $n \geq 0$*.*

Now if $(\mathbf{X}, G)$ is small, then the fact that $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is a Fréchet-Stein K-algebra is guaranteed and is described as follows. Note that when $\mathcal{L}$ is free as a $\mathcal{A}$-module, then the ring $\widehat{U(\mathcal{L})_K}$ is noetherian [4, Corollary 4.1.10] (in fact, only the smoothness of $\mathcal{L}$ is required here). Furthermore:

Theorem 2.4.12. *[4, Lemma 3.3.4, Theorem 3.4.8] Suppose that* $(\mathbf{X}, G)$ *is small. Then there exists a* G*-stable affine formal model* $\mathcal{A}$ *of* $\mathcal{O}(\mathbf{X})$ *and* G*-stable free* $\mathcal{A}$*-Lie lattice* $\mathcal{L}$ *such that for every good chain* (J_n) *for* $\mathcal{L}$*, there is a canoical isomorphism of* K*-algebras*

$$\widehat{\mathcal{D}}(\mathbf{X}, G) \simeq \varprojlim_n \widehat{U(\pi^n\mathcal{L})_K} \rtimes_{J_n} G,$$

where the family $\{\widehat{U(\pi^n\mathcal{L})_K} \rtimes_{J_n} G\}_n$ *of noetherian* K*-Banach algebras gives a Fréchet-Stein structure on* $\widehat{\mathcal{D}}(\mathbf{X}, G)$*.*

Remark 2.4.13. *Let $\mathcal{D}(\mathbf{X}) = U(\mathcal{O}(\mathbf{X})) = U(\mathcal{L}) \otimes_{\mathcal{R}} K$ be the ring of (global) differential operators of finite order on $\mathbf{X}$. It follows that there is a canonical group homomorphism*

$$\gamma : G \longrightarrow (\widehat{\mathcal{D}}(\mathbf{X}, G))^{\times}$$

and a canonical K-algebra homomorphism

$$\iota : \mathcal{D}(\mathbf{X}) \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, G)$$

which is defined as the inverse limit of

$$\gamma_n : G \longrightarrow \widehat{U(\pi^n \mathcal{L})}_K \rtimes_{J_n} G$$

and of

$$\iota_n : \mathcal{D}(\mathbf{X}) \cong U(\pi^n \mathcal{L}) \otimes_{\mathcal{R}} K \longrightarrow \widehat{U(\pi^n \mathcal{L})}_K \rtimes_{J_n} G$$

respectively. These morphism define a morphism

$$\iota \rtimes \gamma : \mathcal{D}(\mathbf{X}) \rtimes G \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, G)$$

Remark 2.4.14. *When G is trivial, we obtain the Fréchet-Stein K-algebra*

$$\widehat{\mathcal{D}}(\mathbf{X}) = \varprojlim_n \widehat{U(\pi^n \mathcal{L})_K}$$

which is introduced in [6].

Notation: Let $\mathbf{X}$ be a smooth affinoid variety. Write $\mathcal{T} = Der_K(\mathcal{O}_{\mathbf{X}})$. We denote $\mathbf{X}_w(\mathcal{T})$ the set of all affinoid subdomains of $\mathbf{X}$ such that $\mathcal{T}(U)$ admits a free $\mathcal{A}$-Lie lattice for some affine formal model $\mathcal{A}$ in $\mathcal{O}(U)$.

The correspondence $\mathbf{U} \in \mathbf{X}_w(\mathcal{T}) \longmapsto \widehat{\mathcal{D}}(\mathbf{U}, H)$, with (U, H) small, does not give rise to a sheaf of K-algebras on the smooth affinoid variety $\mathbf{X}$ (except for G trivial, we then obtain the sheaf $\widehat{\mathcal{D}}_{\mathbf{X}}$ of infinite order differential operators on $\mathbf{X}$, which is defined in [6]). However, it may define a presheaf on certain Grothendieck topologies which are generally coarser than the (strong) Grothendieck topology on $\mathbf{X}$. In order to see this later, we first recall from [4] some important classes of affinoid subdomains of $\mathbf{X}$.
Let U be an affinoid subdomain of $\mathbf{X}$ together with the natural morphism of K-algebras $r_U^{\mathbf{X}}$: $\mathcal{O}(\mathbf{X}) \longrightarrow \mathcal{O}(U)$. Fix an affine formal model $\mathcal{A}$ of $\mathcal{O}(\mathbf{X})$ and an $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $\mathcal{T}(\mathbf{X})$.

Definition 2.4.15. *(i) An affine formal model $\mathcal{B}$ in $\mathcal{O}(U)$ is called $\mathcal{L}$-stable if $r_U^{\mathbf{X}}(\mathcal{A}) \subset \mathcal{B}$ and the action of $\mathcal{L}$ on $\mathcal{A}$ lifts to $\mathcal{B}$. If U admits an $\mathcal{L}$-stable affine formal model, then U is said to be $\mathcal{L}$-admissible.*

(ii) Suppose that U is rational. Then U is $\mathcal{L}$-accessible in n-steps if $U = \mathbf{X}$ for $n = 0$ and for $n > 0$, there is a chain $U \subset Z \subset \mathbf{X}$ such that

- *$Z \subset X$ is $\mathcal{L}$-accessible in $(n-1)$-steps,*
- *$U = Z(f)$ or $Z(1/f)$ for some non-zero $f \in \mathcal{O}(Z)$,*
- *there is a $\mathcal{L}$-stable affine formal model $\mathcal{C} \subset \mathcal{O}(Z)$ such that $\mathcal{L}.f \subset \pi\mathcal{C}$.*

(iii) An affinoid subdomain (not necessary rational) U of $\mathbf{X}$ is called $\mathcal{L}$-accessible if it is $\mathcal{L}$-admissible and there is a finite covering $U = \cup_{i=1}^r U_i$, where each U_i is a $\mathcal{L}$-accessible rational subdomain of $\mathbf{X}$.

We denote by $\mathbf{X}_w(\mathcal{L}, G)$ and $\mathbf{X}_{ac}(\mathcal{L}, G)$ the sets of G-stable affinoid subdomains of $\mathbf{X}$ which are also $\mathcal{L}$-admissible and $\mathcal{L}$-accessible respectively (note that $\mathbf{X}_{ac}(\mathcal{L}, G) \subset \mathbf{X}_w(\mathcal{L}, G)$). These sets, together with the trivial notion of coverings, are Grothendieck topologies on $\mathbf{X}$. If N is a subgroup of G such that $(\mathcal{L}, N)$ is an $\mathcal{A}$-trivialising pair, then following [4, Section 4], we may construct the presheaf $\widehat{\mathcal{U}(\mathcal{L})}_K \rtimes_N G$ on $\mathbf{X}_w(\mathcal{L}, G)$ as follows.

Definition 2.4.16. *Let $\mathbf{U} \in X_w(\mathcal{L}, G)$. Then for any choice of G-stable $\mathcal{L}$-stable affine formal model $\mathcal{B}$ of $\mathcal{O}(U)$, we set:*

$$(\widehat{\mathcal{U}(\mathcal{L})}_K \rtimes_N G)(U) := \widehat{U(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L})}_K \rtimes_N G.$$

It is proved ([4, Proposition 4.3.9]) that this definition is independent of the choice of $\mathcal{B}$ and is a sheaf on the Grothendieck topology $\mathbf{X}_{ac}(\mathcal{L}, G)$. Furthermore

Proposition 2.4.17. *([4, Theorem 4.3.14]) If $\mathcal{L}$ is smooth as an $\mathcal{A}$-module and $\mathbf{U} \in \mathbf{X}_{ac}(\mathcal{L}, G)$ is $\mathcal{L}$-accessible, then the (noetherian) ring $(\widehat{\mathcal{U}(\mathcal{L})}_K \rtimes_N G)(U)$ is (left and right) flat as a $\widehat{U(\mathcal{L})}_K \rtimes_N G$-module.*

This nice property will be important in the next sections of this dissertation. It is also worth pointing out that given an affinoid subdomain U of $\mathbf{X}$, we may rescale the Lie lattice $\mathcal{L}$, which means that we may replace $\mathcal{L}$ by some $\pi^n\mathcal{L}$ for n sufficiently large, so that U becomes a $\mathcal{L}$-accessible subdomain of $\mathbf{X}$.

2.4.3 Localisation of coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules and the category $\mathcal{C}_{\mathbf{X}/G}$

First of all, we collect here some basic notations and properties related to coadmissible modules over Fréchet-Stein algebras.
Let $U \cong \varprojlim_K U_n$ be a Fréchet-Stein K-algebra.

Definition 2.4.18. *A left (resp. right) U-module M is called **coadmissible** if $M = \varprojlim_n M_n$ satisfying the following conditions:*

(i) For all n, M_n is a finitely generated left (resp. right) U_n-module.

(ii) The natural morphism $U_n \otimes_{U_{n+1}} M_{n+1} \longrightarrow M_n$ is an isomorphism of U_n-modules for all n.

We denote $\mathcal{C}_U$ (resp. $\mathcal{C}_U^r$) the category of coadmissible left (resp. right) U-modules. Remark that $\mathcal{C}_U$ is an abelian subcategory of $Mod(U)$ which is stable under extensions (hence is a Serre subcategory). The same assertion holds for $\mathcal{C}_U^r$. Below we recall a result which will be used several times in the sequel:

Proposition 2.4.19. *([25, Lemma 8.4]) Let M be a coadmissible U-module. Then for every $i \geq 0$, the right U-module $Ext_U^i(M, U)$ is coadmissible and we have the following isomorphism of right U-modules:*

$$Ext_U^i(M, U) \xrightarrow{\sim} \varprojlim_n Ext_{U_n}^i(U_n \otimes_U M, U).$$

Proposition 2.4.20. *([6, Lemma 7.3]*
Let $U \cong \varprojlim_n U_n$ and $V \cong \varprojlim_n V_n$ be Fréchet-Stein K-algebras. Suppose that $U \longrightarrow V$ is a continuous homomorphism. Then for any coadmissible left U-module $M = \varprojlim_n M_n$ and coadmissible right U-module $N = \varprojlim_n N_n$, we have

$$V \widehat{\otimes}_U M := \varprojlim_n V_n \otimes_U M \cong V_n \otimes_{U_n} M_n$$

and

$$N \widehat{\otimes}_U V := \varprojlim_n N \otimes_U V_n \cong N_n \otimes_{U_n} V_n$$

are coadmissible left and right V-modules and they define completed tensor products of M, N and V over U, respectively.

Let $\mathbf{X}$ be a smooth affinoid K- variety and G be a compact p-adic Lie group acting continuously on $\mathbf{X}$ such that $(\mathbf{X}, G)$ is small. Since $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is a Fréchet-Stein algebra, there is a category $\mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ (resp. $\mathcal{C}^r_{\widehat{\mathcal{D}}(\mathbf{X},G)}$) of coadmissible left (resp. right) $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules. It is possible to localize coadmissible (left or right) $\widehat{\mathcal{D}}(\mathbf{X}, G)$- modules to obtain G-equivariant sheaves on $\mathbf{X}$ ([4, 3.5]). More concretely, let $M \in \mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ be a coadmissible left $\widehat{\mathcal{D}}(\mathbf{X}, G)$- module, we define a presheaf on the set $\mathbf{X}_w(\mathcal{T})$ of affinoid subdomains U of $\mathbf{X}$ such that $\mathcal{T}(U)$ admits a free $\mathcal{A}$-Lie lattice for some affine formal model $\mathcal{A}$ in $\mathcal{O}(U)$. Recall ([6, Lemma 9.3]) that $\mathbf{X}_w(\mathcal{T})$ is a basis for the Grothendieck topology on $\mathbf{X}$. For each $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, we set

$$M(\mathbf{U}, H) := \widehat{\mathcal{D}}(\mathbf{U}, H) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)} M,$$

By definition, this is a coadmissible (left) $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module. When H runs over the set of open subgroups of G such that (U, H) is small, all $M(\mathbf{U}, H)$ are in bijection and we may form the limit

$$\mathcal{P}_{\mathbf{X}}(M)(U) := \varprojlim_H M(\mathbf{U}, H).$$

Note that the map $\mathcal{P}_{\mathbf{X}}(M) : U \in \mathbf{X}_w(\mathcal{T}) \longmapsto \mathcal{P}_{\mathbf{X}}(M)(U)$ is a presheaf on $\mathbf{X}_w(\mathcal{T})$. The G-action on $\mathcal{P}_{\mathbf{X}}(M)$ is defined as follows. Let $g \in G$, then there is a continuous isomorphism of K- Fréchet algebras

$$\widehat{g}_{U,H} : \widehat{\mathcal{D}}(\mathbf{U}, H) \longrightarrow \widehat{\mathcal{D}}(gU, gHg^{-1}).$$

This isomorphism, together with the group homomorphism γ in Remark 2.2.3, determines the following isomorphism:

$$\begin{aligned} g^M_{U,H} : M(\mathbf{U}, H) &\longrightarrow M(gU, gHg^{-1}) \\ a \widehat{\otimes} m &\longmapsto \widehat{g}_{U,H}(a) \widehat{\otimes} \gamma(g) m, \end{aligned}$$

which is linear relatively to $\widehat{g}_{U,H}$. We then see that there is a G-equivariant structure on $\mathcal{P}_X(M)$ which is locally determined by the inverse limit of the maps $g^M_{U,H}$ when H runs over all the open compact subgroups H of G such that (U, H) is small. Furthermore, one has the following theorem:

Theorem 2.4.21. *([4, Theorem 3.5.8, Theorem 3.5.11]*
Let M be a coadmissible left $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module. Then $\mathcal{P}_{\mathbf{X}}(M)$ is a G-equivariant sheaf of $\mathcal{D}_{\mathbf{X}}$-modules on $\mathbf{X}_w(\mathcal{T})$, where $\mathcal{D}_{\mathbf{X}}$ is the sheaf of algebraic differential operators on $\mathbf{X}$ of finite order.
In particular, $\mathcal{P}_{\mathbf{X}}(M)$ can be extended to a unique sheaf on $\mathbf{X}$, which is denoted by $Loc_{\mathbf{X}}(M)$.

The functor $Loc_{\mathbf{X}}(-)$ on $\mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ is similar to the localisation functor in the classical theory of $\mathcal{D}_{\mathbf{X}}$-modules on complex varieties (see, for example, [12]). It is proved that $Loc_{\mathbf{X}}(-)$ is indeed an equivalence of categories between $\mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ and the category $\mathcal{C}_{\mathbf{X}/G}$ of coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules, which will be defined below:

Definition 2.4.22. *[4, Definition 3.6.7]*
Let $\mathbf{X}$ be a smooth rigid analytic variety and G be a p-adic Lie group acting continuously on $\mathbf{X}$.

*(a) A G-equivariant left $\mathcal{D}_{\mathbf{X}}$-module $\mathcal{M}$ on $\mathbf{X}$ is called **locally Fréchet** if for each $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, $\mathcal{M}(\mathbf{U})$ is equipped with a K-Fréchet topology and the maps $g^{\mathcal{M}}(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \longrightarrow \mathcal{M}(gU)$ are continuous for any $U \in \mathbf{X}_w(\mathcal{T})$ and $g \in G$. Morphisms of G-equivariant locally Fréchet $\mathcal{D}$-modules are morphisms of G-equivariant $\mathcal{D}$-modules whose local sections are continuous with respect to the Fréchet topologies on the source and the target. The category of G-equivariant locally Fréchet left $\mathcal{D}_{\mathbf{X}}$-modules is denoted by $Frech(G - \mathcal{D})$.*

(b) A G-equivariant locally Fréchet $\mathcal{D}_{\mathbf{X}}$-module $\mathcal{M}$ is called coadmissible if there exists a $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$ of $\mathbf{X}$ satisfying that for every $\mathbf{U} \in \mathcal{U}$, there is an open compact subgroup H of G stabilising U and a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module M such that one has an isomorphism

$$Loc_U(M) \simeq \mathcal{M}\mid_U$$

of H-equivariant locally Fréchet $\mathcal{D}_U$-modules.
The category of coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules is denoted by $\mathcal{C}_{\mathbf{X}/G}$. This is a full subcategory of $Frech(G - \mathcal{D})$.

Theorem 2.4.23. *[4, Theorem 3.6.11] Suppose that $(\mathbf{X}, G)$ is small. Then the functor*

$$Loc_{\mathbf{X}} : \mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)} \longrightarrow \mathcal{C}_{\mathbf{X}/G}$$

is an equivalence of categories.

Note that the category $\mathcal{C}^r_{\mathbf{X}/G}$ of coadmissible G-equivariant right $\mathcal{D}$-modules can also be defined similarly and the above theorem still holds for the category $\mathcal{C}^r_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ whenever the functor ${}^rLoc_{\mathbf{X}}(-)$ on $\mathcal{C}^r_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ is defined. More precisely, let M be a coadmissible right $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module. For each open affinoid subset $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, choose an open compact subgroup $H \leq G$ such that (U, H) is small. Then similarly as above, we define

$${}^r\mathcal{P}_{\mathbf{X}}(M)(U) := \varprojlim_{H} (M \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)} \widehat{\mathcal{D}}(\mathbf{U}, H)),$$

where the inverse limit is taken over the set of open compact subgroups H of G such that (U, H) is small.
By using the same arguments as in [4], we can see that ${}^r\mathcal{P}_{\mathbf{X}}(M)$ extends to a G-equivariant coadmissble right $\mathcal{D}_{\mathbf{X}}$-module, which is denoted by ${}^rLoc_{\mathbf{X}}(M)$. The group G acts (locally) on ${}^rLoc_{\mathbf{X}}(M)$ as follows: if $g \in G$ and (U, H) is small, then g produces an isomorphism of K-modules

$$\begin{aligned} g^M_{U,H} :\ M \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)} \widehat{\mathcal{D}}(\mathbf{U}, H) &\xrightarrow{\sim} M \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},gHg^{-1})} \widehat{\mathcal{D}}(gU, gHg^{-1}) \\ m \widehat{\otimes} a &\longmapsto m\gamma(g^{-1}) \widehat{\otimes} \widehat{g}^D(a). \end{aligned}$$

Theorem 2.4.24. *If $(\mathbf{X}, G)$ is small, then the localisation functor ${}^rLoc_{\mathbf{X}}(-)$ is an equivalence of categories between the category of coadmissible right $\widehat{\mathcal{D}}(\mathbf{X}, G)$ modules to the category of G equivariant coadmissible right $\mathcal{D}_X$-modules.*

Proof. The proof of [4, Theorem 3.6.11] remains true when applied to the functor ${}^rLoc_{\mathbf{X}}(-)$. ☐

2.4.4 Side-changing operators

This section is devoted to introducing the side-changing functors. The construction of these functors for coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$ modules is contained in [1]. We are allowed to state here some important results without giving any explicit proofs.

Recall that in the classical theory of $\mathcal{D}$-modules ([12]), when we work on a smooth complex variety X of dimension d, the functors

$$\Omega_X \otimes_{\mathcal{O}_X} - \ \text{ and } \ \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, -),$$

were $\Omega_X =: \mathcal{H}om_{\mathcal{O}_X}(\wedge^d_{\mathcal{O}_X}\mathcal{T}_X, \mathcal{O}_X)$ is the canonical sheaf on X, are mutually inverse equivalences between the category of (coherent) left $\mathcal{D}_X$-modules and the category of (coherent) right $\mathcal{D}_X$-modules.In the setting of the theory of equivariant $\mathcal{D}$-modules on rigid analytic varieties, we also want to prove that these functors remain equivalences of categories between left and right coadmissible equivariant modules.

We first suppose that $\mathbf{X} = Sp(A)$ is a smooth affinoid variety of dimension d and G is a compact p-adic group which acts continuously on $\mathbf{X}$ such that $(\mathbf{X}, G)$ is small. Write $L := \mathcal{T}(\mathbf{X})$ and suppose in addition that L admits a G-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ for some affine formal model $\mathcal{A}$ in A. The action of G on $\mathcal{A}$ defines naturally an action on the right $\mathcal{A}$-module $\Omega_{\mathcal{L}} = Hom_{\mathcal{A}}(\bigwedge^d_{\mathcal{A}}\mathcal{L}, \mathcal{A})$ as follows. For $\omega \in \Omega_{\mathcal{L}}$ and $g \in G$, then $\omega.g \in \Omega_{\mathcal{L}}$ is defined by

$$(\omega.g)(x_1 \wedge ... \wedge x_d) = g^{-1}.\left(\omega(gx_1 \wedge ....gx_d)\right). \tag{2.8}$$

There is a structure of right $U(\mathcal{L})$-module on $\Omega_{\mathcal{L}}$ given by

$$(\omega.x)(x_1 \wedge .... \wedge x_d) = -x(\omega(x_1 \wedge ... \wedge x_d)) + \sum_{i=1}^{d}(-1)^i \omega(x_1 \wedge ... \wedge [x, x_i] \wedge ... \wedge x_d) \tag{2.9}$$

where $\omega \in \Omega_{\mathcal{L}}$ and $x, x_1, ..., x_d \in \mathcal{L}$.
If M is a left $\mathcal{U}(\mathcal{L}) \rtimes G$-module. Then there is a structure of right $U(\mathcal{L}) \rtimes G$-module on $\Omega_{\mathcal{L}} \otimes_{\mathcal{A}} M$ which is defined by

$$(\omega \otimes m).x = \omega x \otimes m - \omega \otimes xm \tag{2.10}$$

and

$$(\omega \otimes m).g = \omega g \otimes g^{-1}m. \tag{2.11}$$

for all $\omega \in \Omega, m \in M, x \in \mathcal{L}$ and $g \in G$. Similarly, if N is a right $\mathcal{U}(\mathcal{L}) \rtimes G$-module, then $Hom_{\mathcal{A}}(\Omega_{\mathcal{L}}, N)$ is a left $U(\mathcal{L}) \rtimes G$-module determined by the following rules:

$$(x.f)(\omega) = f(\omega x) - f(\omega)x. \tag{2.12}$$

and

$$(g.f)(\omega) = f(\omega g)g^{-1}. \tag{2.13}$$

The action of G (2.8) and of $U(\mathcal{L})$ (2.9) on $\Omega_{\mathcal{L}}$ induce a structure of right $U(\mathcal{L}) \rtimes G$-module on $\Omega_{\mathcal{L}}$ ([1, Lemma 4.1.1]). This action extends naturally to a right action of $\widehat{U(\mathcal{L})} \rtimes G$ on $\Omega_{\mathcal{L}}$, since $\Omega_{\mathcal{L}}$ is finitely presented as an $\mathcal{A}$-module, so is π-adically complete. Furthermore, this $\widehat{U(\mathcal{L})} \rtimes G$-action factors through its quotient $\widehat{U(\mathcal{L})} \rtimes_H G$ for any choice of open normal subgroup H of G which is contained in $G_{\mathcal{L}}$. Therefore, $\Omega_{\mathcal{L}}$ is a right $\widehat{U(\mathcal{L})} \rtimes_H G$-module ([1, Lemma 4.1.6]). As a consequence, it follows that

$$\Omega(\mathbf{X}) := Hom_A(\bigwedge_A^d L, A) \cong \Omega_{\mathcal{L}} \otimes K$$

is a right-$\widehat{U(\mathcal{L})}_K \rtimes_H G$-module for every open normal subgroup $H \leq G_{\mathcal{L}}$ of G.
If M is a left $\widehat{U(\mathcal{L})}_K \rtimes_H G$-module and N be a right $\widehat{U(\mathcal{L})}_K \rtimes_H G$-module. Then the right (resp. left) $U(L) \rtimes G$-module structure on $\Omega(\mathbf{X}) \otimes_A M$ (resp. $Hom_A(\Omega(\mathbf{X}), N)$) induces a right (resp. left) $\widehat{U(\mathcal{L})} \rtimes_H G$-module structure on it. Furthermore

Theorem 2.4.25. *([1, Theorem 4.1.12] The functors $\Omega(\mathbf{X}) \otimes_A -$ and $Hom_A(\Omega(\mathbf{X}), -)$ are equivalence of categories between the categories of finitely generated left and right $\widehat{U(\mathcal{L})}_K \rtimes_H G$-modules.*

Now, let M be a coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module. Choose a good chain (J_n) for $\mathcal{L}$ such that

$$\widehat{\mathcal{D}}(\mathbf{X}, G) \cong \varprojlim_n \widehat{U(\pi^n \mathcal{L})}_K \rtimes_{J_n} G$$

Thus $M \cong \varprojlim_n M_n$, where $M_n := (\widehat{U(\pi^n \mathcal{L})}_K \rtimes_{J_n} G) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} M$. Since each $\Omega(\mathbf{X}) \otimes_A M_n$ is a right $\widehat{U(\pi^n \mathcal{L})}_K \rtimes_{J_n} G$-module for all n, it is showed that $\Omega(\mathbf{X}) \otimes_A M \cong \varprojlim_n \Omega(\mathbf{X}) \otimes_A M_n$ and that $\Omega(\mathbf{X}) \otimes_A M$ is a coadmissible right $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module. Similarly, if N is a coadmissible right $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module, then $Hom_A(\Omega(X), N)$ is also a coadmissible left $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module. In this way, the functors $\Omega(X) \otimes_A -$ and $Hom_A(\Omega(\mathbf{X}), -)$ are equivalences between $C_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ and $C^r_{\widehat{\mathcal{D}}(\mathbf{X},G)}$- the categories of coadmissible left and right $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules, respectively.

The result stays true in general and is contained in the following theorem. Let $\mathbf{X}$ be a smooth rigid analytic variety of dimension d and G be a p-adic Lie group which acts continuously on $\mathbf{X}$. Let

$$\Omega_{\mathbf{X}} = \mathcal{H}om_{\mathcal{O}_{\mathbf{X}}}(\wedge^d_{\mathcal{O}_{\mathbf{X}}} \mathcal{T}, \mathcal{O}_{\mathbf{X}})$$

denote the canonical sheaf on $\mathbf{X}$. This is an invertible sheaf of $\mathcal{O}_{\mathbf{X}}$-modules.

Theorem 2.4.26. *[1, Theorem 4.1.14, 4.1.15]*

(i) The functors $\Omega_{\mathbf{X}} \otimes_{\mathcal{O}_{\mathbf{X}}} -$ and $\mathcal{H}om_{\mathcal{O}_{\mathbf{X}}}(\Omega_{\mathbf{X}}, -)$ are mutually inverse equivalences of categories between $\mathcal{C}_{\mathbf{X}/G}$ and ${}^r\mathcal{C}_{\mathbf{X}/G}$.

(ii) If $(\mathbf{X}, G)$ is small. Then the functors $\Omega(\mathbf{X}) \otimes_{\mathcal{O}(\mathbf{X})} -$ and $Hom_{\mathcal{O}(\mathbf{X})}(\Omega(\mathbf{X}), -)$ are mutually inverse equivalences of categories between the category of coadmissible left $\widehat{\mathcal{D}}(\mathbf{X}, G)$- modules and the category of coadmissible right $\widehat{\mathcal{D}}(\mathbf{X}, G)$ modules. Furthermore, for any coadmissible left $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module M and coadmissible right $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module N, there are isomorphisms of coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules

$${}^r Loc(\Omega(\mathbf{X}) \otimes_{\mathcal{O}(\mathbf{X})} M) \simeq \Omega_{\mathbf{X}} \otimes_{\mathcal{O}_{\mathbf{X}}} Loc(M)$$

$$Loc(Hom_{\mathcal{O}(\mathbf{X})}(\Omega(\mathbf{X}), N)) \simeq \mathcal{H}om_{\mathcal{O}_{\mathbf{X}}}(\Omega_{\mathbf{X}}, {}^r Loc(N)).$$

Chapter 3

Dimension theory for coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules

3.1 Review on Auslander-Gorenstein rings

At this point we recall from [14, 16] the notion of an Auslander-Gorenstein ring. Let A be a ring. Then A is said to be an *Auslander-Gorenstein ring (or an AG ring)* if it is a two-sided noetherian ring and satisfies the following conditions:

$(AG1)$ For any noetherian left (or right) A-module M and any $i \geq 0$, one has $j_A(N) \geq i$ whenever N is a right (resp. left) submodule of $Ext^i_A(M, A)$, where

$$j_A(M) := \min\{i : Ext^i_A(M, A) \neq 0\}$$

denotes the grade of M.

$(AG2)$ A has finite left and right injective dimension.

Example 3.1.1. *The enveloping algebra $U(L)$ of a finite dimensional K-Lie algebra L is Auslander-Gorenstein of dimension at most $\dim_K L$. More generally, it is proved [2, Lemma 4.3] that if L is a $(K - A)$-Lie algebra of rank r with A Gorenstein (i.e A is of finite self-injective dimension), then $U(L)$ is Auslander-Gorenstein of dimension at most $\dim A + r$.*

The dimension of a finitely generated module over an AG ring is defined as follows:

Definition 3.1.2. *([14, Section x2]) let R be an AG ring of self-injective dimension n. For any finitely generated R-module M, the dimension of M is*

$$d(M) := n - j(M)$$

Motivated by [2, Section 5], in the next section we also want to formulate a dimension theory for coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules. In order to do it, we need to prove that $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is coadmissibly Auslander-Gorenstein in the sense of [2, Definition 5.1]. First, let us consider the following lemma, which is indeed a mild generalisation of [25, Lemma 8.8] to the non-noetherian case but it will play an important role in the next section.

Lemma 3.1.3. *Let $R_0 \longrightarrow R_1$ be a unital homomorphism of unital rings. (these rings are not supposed to be noetherian). Suppose that there are units $b_0 = 1, b_1, ..., b_m \in (R_1)^\times$ which form a basis of R_1 as a left R_0-module and which satisfy:*

(i) $b_i R_0 = R_0 b_i$ *for any* $1 \leq i \leq m$.

(ii) for any $0 \leq i, j \leq m$, *there is a natural integer* k *with* $0 \leq k \leq m$ *such that* $b_i b_j \in b_k R_0$.

(iii) For any $0 \leq i \leq m$, *there is a a natural integer* l *with* $0 \leq l \leq m$ *such that* $b_i^{-1} \in b_l R_0$.

Then for any (left or right) R_1-module M and (left or right) R_0-module N, we have an isomorphism of R_0-modules

$$Hom_{R_1}(M, R_1 \otimes_{R_0} N) \xrightarrow{\sim} Hom_{R_0}(M, N)$$
$$f \longmapsto p \circ f,$$

where $p : R_1 \longrightarrow R_0$ is the projection map onto the first summand in the decomposition

$$R_1 = \bigoplus_{i=0}^{m} b_i R_0 = \bigoplus_{i=0}^{m} R_0 b_i.$$

In particular, this induces an isomorphism of (right or left) R_0-modules.

$$Ext^i_{R_1}(M, R_1 \otimes_{R_0} N) \simeq Ext^i_{R_0}(M, N).$$

for any integer $i \geq 0$.

Proof. The proof is partly similar to [25, Lemma 8.8]. Note that p is R_0-linear on both sides. Indeed, if $a \in R_0$ and $\sum_{i=0}^{m} a_i b_i \in R_1$, one has

. $p(a.\sum_{i=0}^{m} a_i b_i) = p(\sum_{i=0}^{m} a a_i b_i) = a a_0 = a.p(\sum_{i=0}^{m} a_i b_i)$

. $p((\sum_{i=0}^{m} a_i b_i).a) = p(\sum_{i=0}^{m} a_i b_i a) = p(\sum_{i=0}^{m} a_i a_i^{'} b_i) = a_0 a_0^{'} = a_0 a = p(\sum_{i=0}^{m} a_i b_i).a,$

here $a_i' \in R_0$ such that $a = a_0'$ and $b_i a = a_i^{'} b_i \forall i \geq 1$, since $b_i R_0 = R_0 b_i$ from (i). Thus the morphism:

$$\tilde{p} : R_1 \otimes_{R_0} N \longrightarrow R_0 \otimes_{R_0} N \xrightarrow{\sim} N$$
$$b \otimes n \longmapsto p(b) \otimes n \longmapsto p(b)n$$

is R_0-linear. Now by using a free resolution $P^{\cdot}$ of the R_1-module M, which is also a free resolution of M as a R_0-module, we see that the map $\tilde{p}$ induces a map

$$Ext^i_{R_1}(M, R_1 \otimes_{R_0} N) = h^i(Hom_{R_1}(P^{\cdot}, R_1 \otimes_{R_0} N)) \longrightarrow h^i(Hom_{R_0}(P^{\cdot}, N)) = Ext^i_{R_0}(M, N).$$

Therefore, it suffices to show that for any $N \in Mod(R_0)$ and $M \in Mod(R_1)$, we have an isomorphism

$$Hom_{R_1}(M, R_1 \otimes_{R_0} N) \xrightarrow{\sim} Hom_{R_0}(M, N).$$

Take a presentation of M by free R_1-modules:

$$R_1^I \longrightarrow R_1^J \longrightarrow M \longrightarrow 0.$$

Since $Hom_{R_1}(-, N)$ is left exact, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & Hom_{R_1}(M, R_1 \otimes_{R_0} N) & \longrightarrow & Hom_{R_1}(R_1^J, R_1 \otimes_{R_0} N) & \longrightarrow & Hom_{R_1}(R_1^I, R_1 \otimes_{R_0} N) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & Hom_{R_0}(M, N) & \longrightarrow & Hom_{R_0}(R_1^J, N) & \longrightarrow & Hom_{R_0}(R_1^I, N) \end{array}$$

Hence it is enough to consider the case $M = R_1$ and to prove that

$$\Phi : Hom_{R_1}(R_1, R_1 \otimes_{R_0} N) \xrightarrow{\sim} Hom_{R_0}(R_1, N)$$
$$\psi \longmapsto \tilde{p} \circ \psi$$

This is well-defined since $\tilde{p}$ is R_0-linear.

(1). Φ is surjective
If $\phi : R_1 \longrightarrow N$ be an R_0-linear map, one defines:

$$\psi : R_1 \longrightarrow R_1 \otimes_{R_0} N$$
$$b \longmapsto \sum_{i=0}^{m} b_i \otimes \phi(b_i^{-1} b)$$

Then

- $\tilde{p} \circ \psi(b) = \tilde{p}(\sum_{i=0}^{m} b_i \otimes \phi(b_i^{-1} b)) = \sum_{i=0}^{m} p(b_i)\phi(b_i^{-1} b) = \phi(b)$, since $p(b_i) = 0$ for $i \neq 0$.
- ψ is R_1-linear. Indeed, if $b = \sum_{i=0}^{m} a_i b_i$ with $a_i \in R_0$ and $b' \in R_1$, one can compute:

$$\begin{aligned}
\psi(bb') &= \sum_i b_i \otimes \phi(b_i^{-1} bb') = \sum_j \sum_i b_i \otimes \phi(b_i^{-1} a_j b_j b') \\
&= \sum_j \sum_i b_i \otimes \phi(a_j^{'} b_i^{-1} b_j b') = \sum_j \sum_i b_i a_j^{'} \otimes \phi(b_i^{-1} b_j b') \\
&= \sum_j \sum_i a_j b_i \otimes \phi(b_i^{-1} b_j b') = \sum_j \left(\sum_i a_j b_j b_j^{-1} b_i \otimes \phi(b_i^{-1} b_j b') \right) \\
&= \sum_j a_j b_j \psi(b') = b\psi(b').
\end{aligned}$$

Here, thanks to (ii) and (iii), we have $\psi(b') = \sum_i b_i \otimes \phi(b_i^{-1} b') = \sum_i b_j b_i \otimes \phi(b_i^{-1} b_j b')$. Therefore ψ is R_1-linear. This implies that $\psi \in Hom_{R_1}(R_1, R_1 \otimes_{R_0} N)$ and Φ is surjective.

(2). Φ is injective.
First, let us prove that if $\psi : R_1 \longrightarrow R_1 \otimes_{R_0} N$ is an R_1-linear map, then

$$\psi(b) = \sum_{i=0}^{m} b_i \otimes (\tilde{p} \circ \psi)(b_i^{-1} b).$$

Indeed, suppose that $\psi(b) = \sum_i b_i \otimes n_i$, with $n_i \in N$ for all i. (recall that $R_1 \otimes_{R_0} N \simeq \bigoplus_i b_i R_0 \otimes_{R_0} N \simeq \bigoplus_i b_i \otimes N$), then

$$\psi(b_i^{-1} b) = b_i^{-1} \psi(b) = \sum_j b_i^{-1} b_j \otimes n_j.$$

Thus,

$$\sum_{i=0}^{m} b_i \otimes \tilde{p} \circ \psi(b_i^{-1} b) = \sum_{i=0}^{m} b_i \otimes \sum_{j=0}^{m} p(b_i^{-1} b_j) n_j = \sum_{i=0}^{m} b_i \otimes n_i = \psi(b).$$

Consequently, if $\Phi(\psi) = 0 \Longleftrightarrow \tilde{p} \circ \psi = 0 \rightarrow \psi(b) = 0$ for all b. This implies that Φ is injective.

□

Proposition 3.1.4. *Let R_0, R_1 be two rings which satisfy the assumptions in the above lemma. If a (left or right) R_1-module N is injective, then N is also injective as R_0-module. Moreover*

(i) $injdim(R_0) = injdim(R_1)$,

(ii) $Ext^i_{R_1}(N, R_1) \simeq Ext^i_{R_0}(N, R_0)$ and $j_{R_1}(N) = j_{R_0}(N)$,

(iii) If R_0, R_1 are noetherian and if R_0 is Auslander-Gorenstein, then so is R_1.

Proof. Suppose that N is an injective R_1-module. By assumption, R_1 is free over R_0 on both sides, so it is left and right flat over R_0. Moreover,

$$Hom_{R_0}(M, N) \simeq Hom_{R_1}(R_1 \otimes_{R_0} M, N)$$

for any $M \in Mod(R_0)$. By consequence, N is also injective as an R_0-module.
Now (ii) is a direct consequence of Lemma 3.1.3 while (iii) can be proved by using (i) and (ii), it remains to prove (i).
If $0 \longrightarrow R_0 \longrightarrow I^\cdot$ is an injective resolution of R_0 , then it follows from Lemma 3.1.3 that if M is an R_1-module, then $Hom_{R_1}(M, R_1 \otimes_{R_0} I^k) \simeq Hom_{R_0}(M, I^k)$ for any component I^k of the complex $I^\cdot$. Thus $R_1 \otimes_{R_0} I^k$ is an injective R_1-module for all k. This proves that $0 \longrightarrow R_1 \longrightarrow R_1 \otimes_{R_0} I^\cdot$ is an injective resolution of R_1 by R_1-modules. Therefore

$$injdim(R_1) \leq injdim(R_0).$$

It remains to prove that $injdim(R_0) \leq injdim(R_1)$. Suppose that $injdim(R_1) = n < \infty$, so we need to prove that $injdim(R_0) \leq n$. This is equivalent to

$$Ext^{n+1}_{R_0}(N, R_0) = 0 \text{ for any } N \in Mod(R_0).$$

Notice that

$$Ext^{n+1}_{R_0}(N, R_0) \otimes_{R_0} R_1 \simeq Ext^{n+1}_{R_1}(R_1 \otimes_{R_0} N, R_1).$$

Since $n = injdim(R_1)$, one has $Ext^{n+1}_{R_1}(R_1 \otimes_{R_0} N, R_1) = 0$ implying that $Ext^{n+1}_{R_0}(N, R_0) \otimes_{R_0} R_1 = 0$. On the other hand, R_1 is a free R_0-module on both sides, thus R_1 is faithfully flat over R_0 on both sides. As a result, $Ext^{n+1}_{R_0}(N, R_0) = 0$ which proves that $injdim(R_0) \leq n = injdim(R_1)$.

□

Now, as an application of the above lemma, let us consider the following example which will be important for the next section. Suppose that $\mathbf{X} = Sp(A)$ is a smooth affinoid K-variety for a K-affinoid algebra A and G is a compact p-adic Lie group which acts continuously on $\mathbf{X}$ such that $(\mathbf{X}, G)$ is small. We assume the following extra conditions:

* H is an open *normal* subgroup of G,
* $\mathcal{A}$ is a G-stable affine formal model in A,
* $(\mathcal{L}, J)$ is an $\mathcal{A}$-trivialising pair such that $J \leq H$.

Then Lemma 3.1.3 and Proposition 3.1.4 can be partially applied to the case where $R_1 = \widehat{U(\mathcal{L})}_K \rtimes_J G$ and $R_0 = \widehat{U(\mathcal{L})}_K \rtimes_J H$ as follows:

Lemma 3.1.5. *The natural morphism of rings $\widehat{U(\mathcal{L})}_K \rtimes_J H \longrightarrow \widehat{U(\mathcal{L})}_K \rtimes_J G$ satisfies the conditions (i), (ii), (iii) of Lemma 3.1.3. In particular, this induces a two-sided $\widehat{U(\mathcal{L})}_K \rtimes_J H$ linear map*

$$p^{\mathbf{X}}_{G,H,J} : \widehat{U(\mathcal{L})}_K \rtimes_J G \longrightarrow \widehat{U(\mathcal{L})}_K \rtimes_J H \tag{3.1}$$

Proof. Following [4, Lemma 2.2.6], the ring $\widehat{U(\mathcal{L})}_K \rtimes_J G$ is isomorphic to $(\widehat{U(\mathcal{L})}_K \rtimes_J H) \rtimes_H G$ and the later is isomorphic to the crossed product $(\widehat{U(\mathcal{L})}_K \rtimes_J H) * G/H$ ([4, Lemma 2.2.4]). On the other hand, since G is a compact p-adic Lie group and H (resp. J) is open in G, it follows that the group G/H is finite. Therefore, if we denote by $S = \{1 = g_1, g_2, ..., g_m\}$ the representatives of the right cosets of H in G, then $\widehat{U(\mathcal{L})}_K \rtimes_J G$ is freely generated over $\widehat{U(\mathcal{L})}_K \rtimes_J H$ by the image $\bar{S} = \{\bar{g_1}, ..., \bar{g}_m\}$ of S in $\widehat{U(\mathcal{L})}_K * G/J$ [17, Lemma 5.9(i)]. In particular, $\widehat{U(\mathcal{L})}_K \rtimes_J H$ is a subring of $\widehat{U(\mathcal{L})}_K \rtimes_J G$.
Now we check that the injective map $\widehat{U(\mathcal{L})}_K \rtimes_J H \longrightarrow \widehat{U(\mathcal{L})}_K \rtimes_J G$ satisfies the conditions $(i), (ii), (iii)$ in Lemma 3.1.3. Write $R_0 = \widehat{U(\mathcal{L})}_K \rtimes_J H$ and $R_1 = \widehat{U(\mathcal{L})}_K \rtimes_J G$. Then R_1 is freely generated on R_0 by $\bar{S}$. By definition of crossed product, one has

(i) $\bar{g}_i R_0 = R_0 \bar{g}_i$.

(ii) $\bar{g}_i \bar{g}_j R_0 = \bar{g_i g_j} R_0$. Furthermore, the set G/H is finite whose each element is represented by an element of S. This implies that there exists k, l such that $\bar{g}_i \bar{g}_j \in \bar{g}_k R_0$ and $\bar{g}_i^{-1} \in \bar{g}_l R_0$.

By consequence, this provides a two-sided $(\widehat{U(\mathcal{L})}_K \rtimes_J H)$-linear map

$$p^{\mathbf{X}}_{G,H,J} : \widehat{U(\mathcal{L})}_K \rtimes_J G \longrightarrow \widehat{U(\mathcal{L})}_K \rtimes_J H$$

as claimed. □

Remark 3.1.6. *If we take $H = \{1\}$ the trivial group, then the injection $\widehat{U(\mathcal{L})}_K \longrightarrow \widehat{U(\mathcal{L})}_K \rtimes_J G$ satisfies Lemma 3.1.3.*

Proposition 3.1.7. *Suppose that $(\mathbf{X}, G)$ is small and H be an open normal subgroup of G. The ring $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is freely generated over $\widehat{\mathcal{D}}(\mathbf{X}, H)$ with basis satisfying the conditions $(i), (ii), (iii)$ of Lemma 3.1.3.*

Proof. By taking the inverse limit of the morphisms $p^{X}_{G,H,J}$ in Lemma 3.1.5 when $(\mathcal{L}, J)$ runs over the set of all $\mathcal{A}$-trivialising pairs , we see that $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is freely generated as $\widehat{\mathcal{D}}(\mathbf{X}, H)$ module by the image $\tilde{S} = \{\tilde{g_1}, \tilde{g_2}, .., \tilde{g_m}\}$ of S in $\widehat{\mathcal{D}}(\mathbf{X}, G)$ which defines a two-sided $\widehat{\mathcal{D}}(\mathbf{X}, H)$-linear map

$$p^{\mathbf{X}}_{G,H} : \widehat{\mathcal{D}}(\mathbf{X}, G) \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, H) \tag{3.2}$$

$$\sum_{i=1}^{m} a_i \tilde{g}_i \longmapsto a_0.$$

□

Corollary 3.1.8. *(i) The maps $p^{\mathbf{X}}_{G,H}$ and $p^{\mathbf{X}}_{G,H,J}$ fit into a commutative diagram*

$$\begin{array}{ccc} \widehat{\mathcal{D}}(\mathbf{X}, G) & \xrightarrow{p^{X}_{G,H}} & \widehat{\mathcal{D}}(\mathbf{X}, H) \\ {\scriptstyle q_{G,J}}\downarrow & & \downarrow{\scriptstyle q_{H,J}} \\ \widehat{U(\mathcal{L})}_K \rtimes_J G & \xrightarrow{p^{X}_{G,H,J}} & \widehat{U(\mathcal{L})}_K \rtimes_J H, \end{array}$$

where $q_{G,J} : \widehat{\mathcal{D}}(\mathbf{X}, G) \longrightarrow \widehat{U(\mathcal{L})}_K \rtimes_J G$ *and* $q_{H,J} : \widehat{\mathcal{D}}(\mathbf{X}, H) \longrightarrow \widehat{U(\mathcal{L})}_K \rtimes_J H$ *denote the canonical maps induced from the definition of* $\widehat{\mathcal{D}}(\mathbf{X}, G)$ *and* $\widehat{\mathcal{D}}(\mathbf{X}, H)$ *respectively.*

(ii) If $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ *is such that* $(\mathbf{U}, G)$ *is small, then the diagram*

$$\begin{array}{ccc} \widehat{\mathcal{D}}(\mathbf{X}, G) & \xrightarrow{p^X_{G,H}} & \widehat{\mathcal{D}}(\mathbf{X}, H) \\ {\scriptstyle r^U_G}\downarrow & & \downarrow{\scriptstyle r^U_H} \\ \widehat{\mathcal{D}}(\mathbf{U}, G) & \xrightarrow{p^U_{G,H}} & \widehat{\mathcal{D}}(\mathbf{U}, H) \end{array}$$

is commutative.

Proof. The statement (i) is evident from definition. To show (ii), let us fix a G-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $\mathcal{T}(\mathbf{X})$ for some G-stable affine formal model $\mathcal{A}$ of A. By rescaling $\mathcal{L}$ if necessary, we may assume that U is $\mathcal{L}$-admissible [6, Lemma 7.6]. Under this assumption, [4, Proposition 4.3.6] showed that $\mathcal{L}' := \mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$ is a G-stable $\mathcal{B}$-Lie lattice in $\mathcal{T}(U)$ for any choice of a G stable $\mathcal{L}$-stable affine formal model $\mathcal{B}$ in $\mathcal{O}(U)$. This is even free as $\mathcal{B}$-module. Let $J \leq G_{\mathcal{L}}$ be an open normal subgroup of G such that $(\mathcal{L}, J)$ and $(\mathcal{L}', J)$ are trivialising pairs (this is thanks to [4, Proposition 4.3.6]). By definition, it is enough to show that the diagram

$$\begin{array}{ccc} \widehat{U(\mathcal{L})}_K \rtimes_J G & \xrightarrow{p^X_{G,H,J}} & \widehat{U(\mathcal{L})}_K \rtimes_J H \\ {\scriptstyle r^U_{G,J}}\downarrow & & \downarrow{\scriptstyle r^U_{H,J}} \\ \widehat{U(\mathcal{L}')}_K \rtimes_J G & \xrightarrow{p^U_{G,H,J}} & \widehat{U(\mathcal{L}')}_K \rtimes_J H \end{array}$$

is commutative.

Note that J is of finite index in G and in H, so that we can choose a set of representatives $1 = g_1, g_2, ..., g_m, ..., g_n$ $(m \leq n)$ of G modulo J such that $G/J = \{\bar{g_1}, ..., \bar{g_n}\}$ and $H/J = \{\bar{g_1}, \bar{g_2}, ...\bar{g_m}\}$. Therefore

$$\widehat{U(\mathcal{L})}_K \rtimes_J G \simeq \widehat{U(\mathcal{L})}_K * G/J = \{\textstyle\sum_{i=1}^n a_i\bar{g_i} : \ a_i \in \widehat{U(\mathcal{L})}_K\}$$

and

$$\widehat{U(\mathcal{L})}_K \rtimes_J H \simeq \widehat{U(\mathcal{L})}_K * H/J = \{\textstyle\sum_{i=1}^m a_i\bar{g_i} : \ a_i \in \widehat{U(\mathcal{L})}_K\}.$$

Notice that here we identified each $\bar{g_i} \in G/J$ with its image in $\widehat{U(\mathcal{L})}_K * G/J$. Furthermore, these formulas still hold when we replace $\mathcal{L}$ by $\mathcal{L}'$. Thus

$$r^U_{H,J} \circ p^X_{G,H,J}(\sum_{i=1}^n a_i\bar{g_i}) = r^U_{H,J}(\sum_{i=1}^m a_i\bar{g_i}) = \sum_{i=1}^m \tilde{a}_i\bar{g_i}$$

and

$$p^U_{G,H,J} \circ r^U_{G,J}(\sum_{i=1}^n a_i\bar{g_i}) = p^U_{G,H,J}(\sum_{i=1}^n \tilde{a}_i\bar{g_i}) = \sum_{i=1}^m \tilde{a}_i\bar{g_i}.$$

Here for each i, $\tilde{a}_i$ denotes the image of a_i in $\widehat{U(\mathcal{L}')}_K$ via the canonical morphism $\widehat{U(\mathcal{L})}_K \longrightarrow \widehat{U(\mathcal{L}')}_K$. This proves that the diagram is commutative. □

We end this section by giving an important result:

Corollary 3.1.9. *Suppose that $(\mathbf{X}, G)$ is small with $\dim X = d$ and that the $\mathcal{A}$-Lie lattice $\mathcal{L}$ is smooth as an $\mathcal{A}$-module. Then there exist $m \geq 0$ such that the ring $\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G$ is an Auslander-Gorenstein ring of dimension at most $2d$ for any $n \geq m$ and for any open normal subgroup J_n of G which is contained in $G_{\pi^n\mathcal{L}}$.*

Proof. Following [2, Theorem 4.3], there exists $m \geq 0$ such that the ring $\widehat{U(\pi^n\mathcal{L})}_K$ is Auslander-Gorenstein of dimension at most $2d$ for all $n \geq m$. Thanks to Proposition 3.1.4 and Remark 3.1.6 it follows that $\widehat{U(\pi^n\mathcal{L})}_K \rtimes_J G$ is Auslander-Gorenstein of dimension at most $2d$. □

3.2 Dimension theory for coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules

Recall from [2, Section 5.1] that a two-sided Fréchet-Stein algebra $U \simeq \varprojlim_n U_n$ is called **coadmissibly Auslander-Gorenstein** (or c-Auslander-Gorenstein) of dimension at most d if each U_n is an Auslander-Gorenstein ring with self-injective dimension at most d for a non negative integer d.

Theorem 3.2.1. *Let $\mathbf{X} = Sp(A)$ be a smooth affinoid variety of dimension d and G be a compact p-adic Lie group acting continuously on $\mathbf{X}$ such that $(\mathbf{X}, G)$ is small. Then the Fréchet-Stein K-algebra $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is coadmissibly Auslander-Gorenstein of dimension at most $2d$.*

Proof. We may choose a G-stable affine formal model $\mathcal{A}$ in A and a G-stable *free* $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $L = Der_K(A)$ and a good chain (J_n) for $\mathcal{L}$ such that

$$\widehat{\mathcal{D}}(\mathbf{X}, G) \simeq \varprojlim_n \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G.$$

By Corollary 3.1.9, there exists $m \geq 0$ such that the ring $\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G$ is Auslander-Gorenstein of dimension at most $2d$ for each $n \geq m$, so that the theorem follows. □

Definition 3.2.2. *Let M be a non-zero (left or right) coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module. The dimension of M is defined by:*

$$d_G(M) := 2d - j_G(M),$$

where $j_G(M) = min\{i \mid Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G)) \neq 0\}$ is the grade of M as a $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module.

Convention: If M is a zero, we set $d_G(M) = 0$.

Remark 3.2.3. *(i) Choose a G-stable affine model $\mathcal{A}$ and a G-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ as in Theorem 3.2.1. Write $D_n := \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G$. Then for any $M \in \mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)}$, one has that (Proposition 2.4.19):*

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G)) \cong \varprojlim_n Ext^i_{D_n}(D_n \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} M, D_n).$$

It follows that there exists n sufficiently large such that $j_G(M) = j_{D_n}(D_n \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} M) \leq 2d$. By consequence $0 \leq d_G(M) \leq 2d$.

(ii) If H be an open subgroup of G, then there exists an open normal subgroup N of G which is contained in H ([4], Lemma 3.2.1). Thus N is of finite index in G. Moreover

$$\widehat{\mathcal{D}}(\mathbf{X}, G) \simeq \widehat{\mathcal{D}}(\mathbf{X}, N) \rtimes_N G \simeq \widehat{\mathcal{D}}(\mathbf{X}, N) * G/N.$$

Then the $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module M is also coadmissible as $\widehat{\mathcal{D}}(\mathbf{X}, N)$-module. Therefore $d_G(M) = d_N(M)$ by Proposition 3.1.4(ii). The same assertion holds for H, so that

$$d_G(M) = d_H(M) = d_N(M).$$

For this reason, we will write $d(M)$ instead of $d_G(M)$ for simplicity.

Proposition 3.2.4. *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be an exact sequence of coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules. Then

$$d(M_2) = \max\{d(M_1), d(M_3)\}.$$

Proof. Suppose that

$$\widehat{\mathcal{D}}(\mathbf{X}, G) \cong \varprojlim_n \widehat{U(\pi^n \mathcal{L})}_K \rtimes_{J_n} G$$

for a G-stable free Lie lattice $\mathcal{L}$ of $Der_K(\mathcal{O}(\mathbf{X}))$ and a good chain (J_n) for $\mathcal{L}$. Write $\widehat{D} := \widehat{\mathcal{D}}(\mathbf{X}, G)$ and $D_n := \widehat{U(\pi^n \mathcal{L})}_K \rtimes_{J_n} G$. Note that there exists an integer m such that for every i and $n \geq m$, one has that (Remark 3.2.3(i)):

$$j_{\widehat{D}}(M_i) = j_{D_n}(D_n \otimes_{\widehat{D}} M_i)$$

Since $\widehat{D} \longrightarrow D_n$ is a flat morphism ([25, Remark 3.2]), it follows that the sequence

$$0 \longrightarrow D_n \otimes_{\widehat{D}} M_1 \longrightarrow D_n \otimes_{\widehat{D}} M_2 \longrightarrow D_n \otimes_{\widehat{D}} M_3 \longrightarrow 0$$

is exact. Now applying [16, Proposition 4.5(ii)] gives the result. □

Example 3.2.5. *The $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is of dimension $2d$. Indeed*

$$Hom_{\widehat{\mathcal{D}}(\mathbf{X},G)}(\widehat{\mathcal{D}}(\mathbf{X}, G), \widehat{\mathcal{D}}(\mathbf{X}, G)) \cong \widehat{\mathcal{D}}(\mathbf{X}, G).$$

Hence $j(\widehat{\mathcal{D}}(\mathbf{X}, G)) = 0$, so that $d(\widehat{\mathcal{D}}(\mathbf{X}, G)) = 2d$.

A non-trivial example is given by the following proposition:

Proposition 3.2.6. *Let $\mathbf{X}$ be a smooth affinoid variety of dimension d and $P \in \mathcal{D}(\mathbf{X})$ be a regular differential operator (P is not a zero divisor of $\mathcal{D}(\mathbf{X})$). Then the coadmissible left $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module*

$$M = \widehat{\mathcal{D}}(\mathbf{X}, G)/\widehat{\mathcal{D}}(\mathbf{X}, G)P$$

is of dimension $d(M) \leq 2d - 1$.

Proof. Write $D := \mathcal{D}(\mathbf{X})$ and $\widehat{D} := \widehat{\mathcal{D}}(\mathbf{X}, G)$. Choose a G-stable free $\mathcal{A}$-Lie lattice of $Der_K(\mathcal{O}(\mathbf{X}))$ for somme G-stable affine formal model $\mathcal{A}$ in $\mathcal{O}(\mathbf{X})$. Then

$$\widehat{D} \cong \varprojlim_n \widehat{U(\pi^n \mathcal{L})}_K \rtimes_{J_n} G$$

is a Fréchet-Stein structure on $\widehat{D}$. Write $D_n := \widehat{U(\pi^n \mathcal{L})}_K \rtimes_{J_n} G$, then

$$M \cong \varprojlim_n D_n/D_n P.$$

Thus there is a $n \geq 0$ such that $d(M) = d(D_n/D_nP)$. Furthermore, one has that

$$D_n/D_nP \cong D_n \otimes_D D/DP.$$

The ring D_n is flat as a right D-module. It follows that:

$$Ext^i_D(D/DP, D) \otimes_D D_n \cong Ext^i_{D_n}(D_n \otimes_D D/DP, D_n).$$

As a consequence, we obtain the inequality $d_{D_n}(D_n/D_nP) \leq d_D(D/DP)$. Now, since P is regular in D, the dimension of the left D-module D/DP can not be $2d$ (otherwise one has that $j_D(D/DP) = 0$, so $Hom_D(D/DP, D) = \{Q \in D : QP = 0\} \neq 0$, contradiction). So the proposition follows. ☐

3.3 Left-right comparison

Let $\mathbf{X}$ be an affinoid variety and G a p-adic Lie group acting continuously on $\mathbf{X}$ and such that $(\mathbf{X}, G)$ is small. Recall that the functors

$$\begin{aligned} \Omega(\mathbf{X}) \otimes_{\mathcal{O}(\mathbf{X})} - : \mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)} &\longrightarrow \mathcal{C}^r_{\widehat{\mathcal{D}}(\mathbf{X},G)} \\ M &\longmapsto \Omega(\mathbf{X}) \otimes_{\mathcal{O}(\mathbf{X})} M \end{aligned}$$

and

$$\begin{aligned} Hom_{\mathcal{O}(\mathbf{X})}(\Omega(\mathbf{X}), -) : \mathcal{C}^r_{\widehat{\mathcal{D}}(\mathbf{X},G)} &\longrightarrow \mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)} \\ N &\longmapsto Hom_{\mathcal{O}(\mathbf{X})}(\Omega(\mathbf{X}), N) \end{aligned}$$

are mutually inverse equivalences between the categories of left and right coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules. Having these side-changing operators for coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules at hand, we can now state the following proposition, which is about preservation of dimension of coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules under these functors.

Proposition 3.3.1. *Let M be a coadmissible left $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module. Then there is an isomorphism of left $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules*

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(\Omega(\mathbf{X}) \otimes_{\mathcal{O}(\mathbf{X})} M, \widehat{\mathcal{D}}(\mathbf{X}, G)) \simeq Hom_{\mathcal{O}(\mathbf{X})}(\Omega(\mathbf{X}), Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G))).$$

In particular, $d(M) = d(\Omega(\mathbf{X}) \otimes_{\mathcal{O}(\mathbf{X})} M)$.

Proof. The proof uses the same arguments as in [2, Lemma 5.2]. Write $A = \mathcal{O}(\mathbf{X})$, $\Omega := \Omega(\mathbf{X})$, $\widehat{D} := \widehat{\mathcal{D}}(\mathbf{X}, G)$. Then the left hand side is exactly the i-th cohomology of the complex $RHom_{\widehat{D}}(\Omega \otimes^{\mathbb{L}}_A M, \widehat{D})$, as Ω is a projective A-module. Now, the right hand side is the i-th cohomology of $RHom_A(\Omega, RHom_{\widehat{D}}(M, \widehat{D}))$. Now, using the derived tensor-Hom adjunction gives the first part of the Proposition.
For the second part, note that since Ω is a finitely generated projective A-module, one has

$$Hom_A(\Omega, Ext^i_{\widehat{D}}(M, \widehat{D})) \cong \Omega^* \otimes_A Ext^i_{\widehat{D}}(M, \widehat{\mathcal{D}})),$$

where $\Omega^* = Hom_A(\Omega, A)$ is its dual. Thus, if $Hom_A(\Omega, Ext^i_{\widehat{D}}(M, \widehat{D})) = 0$, then

$$Ext^i_{\widehat{D}}(M, \widehat{D}) \cong (\Omega \otimes_A \Omega^*) \otimes_A Ext^i_{\widehat{D}}(M, \widehat{D})) \cong \Omega \otimes_A Hom_A(\Omega, Ext^i_{\widehat{D}}(M, \widehat{D})) = 0.$$

Here, $\Omega \otimes_A \Omega^* \cong A$, as Ω is an invertible A-module. By consequence, $Ext^i_{\widehat{D}}(M, \widehat{D}) = 0$ if and only if $Hom_A(\Omega, Ext^i_{\widehat{D}}(M, \widehat{D})) = 0$ and hence $d(M) = d(\Omega(\mathbf{X}) \otimes M)$. ☐

Chapter 4

Dimension theory for coadmissible equivariant $\mathcal{D}$-modules

4.1 Modules over the sheaf of rings $\mathcal{Q}$

This section may be considered as a stepping stone to defining the 'Ext functors' E^i in the next section.

Let $\mathbf{X}$ be a smooth affinoid variety of dimension d and G be a compact p-adic Lie group acting continuously on $\mathbf{X}$. Fix a G-stable affine formal model $\mathcal{A}$ in $A = \mathcal{O}(\mathbf{X})$, a G-stable $\mathcal{A}$-Lie lattice $\mathcal{L}$ of $\mathcal{T}(\mathbf{X}) = Der_K(A)$ and an open normal subgroup J of G which is contained in $G_{\mathcal{L}}$ (which means that $(\mathcal{L}, J)$ is a $\mathcal{A}$-trivialising pair).

Notation: Throughout this section, we will be working under the following notations and assumptions:

- $\mathcal{L}$ is a smooth $\mathcal{A}$-module, which means that $\mathcal{L}$ is projective and finitely generated over $\mathcal{A}$.
- When H is an open subgroup of G, $\mathbf{X}_w(\mathcal{T})/H$ denotes the set of all open affinoid subsets $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ such that $(\mathbf{U}, H)$ is small. If $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})/H$, then H is called an **U-small** subgroup of G.
- $\mathbf{X}_w(\mathcal{L}, G)$ denotes the set of G-stable $\mathcal{L}$-admissible affinoid subdomains of $\mathbf{X}$,
- $\mathbf{X}_{ac}(\mathcal{L}, G)$ denotes the set of G-stable $\mathcal{L}$-accessible affinoid subdomains of $\mathbf{X}$.

Note that $\mathbf{X}_{ac}(\mathcal{L}, G) \subset \mathbf{X}_w(\mathcal{L}, G)$, together with the usual notion of coverings, are Grothendieck topologies on $\mathbf{X}$. Recall from Definition 2.4.16 the presheaf of rings on $\mathbf{X}_w(\mathcal{L}, G)$

$$\mathcal{Q}(-, G) := \widehat{\mathcal{U}(\mathcal{L})}_K \rtimes_J G$$

It is proved ([4, Corollary 4.3.12]) that $\mathcal{Q}(-, G)$ is a sheaf on the Grothendieck topology $\mathbf{X}_w(\mathcal{L}, G)$. Note that if $H \leq G$ is an open compact subgroup of G and J is contained in $G_{\mathcal{L}} \cap H$, then $\mathcal{Q}(-, H)$ is also a sheaf on the Grothendieck topology $\mathbf{X}_w(\mathcal{L}, H)$ containing all the H-stable $\mathcal{L}$-admissible affinoids subdomains of $\mathbf{X}$. In the sequel, if there is no ambiguity, we denote $\mathcal{Q}(-, G)$ simply by $\mathcal{Q}$ whenever the groups G and J are given. The fact that $\mathcal{Q}$ is a sheaf on $\mathbf{X}_{ac}(\mathbf{X}, G)$ allows us to give the following definitions:

Definition 4.1.1. *Let M be a finitely generated $\mathcal{Q}(\mathbf{X})$-module. Then there is a presheaf $Loc_{\mathcal{Q}}(M)$ on $\mathbf{X}_{ac}(\mathcal{L}, G)$ associated to M which is defined as follows:*

$$Loc_{\mathcal{Q}}(M)(\mathbf{Y}) := \mathcal{Q}(\mathbf{Y}) \otimes_{\mathcal{Q}(\mathbf{X})} M$$

for all $\mathbf{Y} \in \mathbf{X}_{ac}(\mathcal{L}, G)$.

Following [3, Corollary 4.3.19], under the extra assumption on $\mathcal{L}$ that $[\mathcal{L}, \mathcal{L}] \subset \pi\mathcal{L}$ and $\mathcal{L}.A \subset \pi\mathcal{A}$, then $Loc_{\mathcal{Q}}(M)$ is a sheaf of $\mathcal{Q}$-modules on $\mathbf{X}_{ac}(\mathcal{L}, G)$ for every finitely generated $\mathcal{Q}(\mathbf{X})$-module M.

Definition 4.1.2. *Let $\mathcal{U}$ be a $\mathbf{X}_{ac}(\mathcal{L}, G)$-covering of $\mathbf{X}$. Then a $\mathcal{Q}$-module $\mathcal{M}$ on $\mathbf{X}_{ac}(\mathcal{L}, G)$ is said to be $\mathcal{U}$-coherent if for any $\mathbf{Y} \in \mathcal{U}$, there exists a finitely generated $\mathcal{Q}(Y)$-module M such that*

$$Loc_{\mathcal{Q}|_{\mathcal{Y}}}(M) \cong \mathcal{M}|_{\mathcal{Y}},$$

where $\mathcal{Y} := \mathbf{X}_{ac}(\mathcal{L}, G) \cap \mathbf{Y}_w$.

It is proved in [4, Theorem 4.3.21] that if $[\mathcal{L}, \mathcal{L}] \subset \pi\mathcal{L}$, $\mathcal{L}.\mathcal{A} \subset \pi\mathcal{A}$, then for any $\mathcal{U}$-coherent sheaf of $\mathcal{Q}$-modules $\mathcal{M}$, $\mathcal{M}(\mathbf{X})$ is a finitely generated $\mathcal{Q}(\mathbf{X})$-module and we have an isomorphism of $\mathcal{Q}$-modules

$$Loc_{\mathcal{Q}}(\mathcal{M}(\mathbf{X})) \xrightarrow{\sim} \mathcal{M}.$$

In the following, we fix:

- $*$ $\mathcal{U}$ is a $\mathbf{X}_{ac}(\mathcal{L}, G)$-covering of $\mathbf{X}$.
- $*$ $\mathcal{M}$ is a $\mathcal{U}$-coherent sheaf of $\mathcal{Q}$-modules on $\mathbf{X}_{ac}(\mathcal{L}, G)$.

Proposition 4.1.3. *Let H be a normal open subgroup of G. There is an isomorphism of right $\mathcal{Q}(\mathbf{X}, H)$-modules*

$$p^i_{G,H}(\mathbf{X}) : \ Ext^i_{\mathcal{Q}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}), \mathcal{Q}(\mathbf{X}, G)) \xrightarrow{\sim} Ext^i_{\mathcal{Q}(\mathbf{X},H)}(\mathcal{M}(\mathbf{X}), \mathcal{Q}(\mathbf{X}, H)).$$

Furthermore, if $H' \leq H$ is another open normal subgroup of G, then one has

$$p^i_{H,H'}(\mathbf{X}) \circ p^i_{G,H}(\mathbf{X}) = p^i_{G,H'}(\mathbf{X}).$$

Proof. Write $M := \mathcal{M}(\mathbf{X})$. The first part of the proposition is in fact a consequence of Lemma 3.1.3 and Lemma 3.1.5. Recall that when $i = 0$, then

$$p_{G,H}(f) := p^0_{G,H}(\mathbf{X})(f) = p^{\mathbf{X}}_{G,H} \circ f,$$

for $f \in Hom_{\mathcal{Q}(\mathbf{X},G)}(M, \mathcal{Q}(\mathbf{X}, G))$, where $p^X_{G,H}$ is the projection map $\mathcal{Q}(\mathbf{X}, G) \longrightarrow \mathcal{Q}(\mathbf{X}, H)$ which is defined in Lemma 3.1.3. For the second part, if $H' \leq H$ are open normal subgroups of G, then both H and H' are of finite index in G and H' is of finite index in H (since G is compact). Hence we can choose a $\mathcal{Q}(\mathbf{X}, H')$-basis $\{1 = g_1, g_2, ..., g_m, ..., g_n\}$ of $\mathcal{Q}(\mathbf{X}, G)$ such that $\{g_1, ..., g_m\}$ is a basis of $\mathcal{Q}(\mathbf{X}, H)$ as a $\mathcal{Q}(\mathbf{X}, H')$-module. Then by definition

$$p_{G,H'}(a_1g_1 + a_2g_2 + ... + a_mg_m + ... + a_ng_n) = a_1$$

and

$$p_{H,H'} \circ p_{G,H}(a_1g_1 + a_2g_2 + ... + a_mg_m + ... + a_ng_n) = p_{H,H'}(a_1g_1 + a_2g_2 + ... + a_mg_m) = a_1$$

This implies $p_{G,H'} = p_{H,H'} \circ p_{G,H}$. Therefore $p_{H,H'}(X) \circ p_{G,H}(X) = p_{G,H'}(X)$, which means that the assertion is true for $i = 0$. For $i > 0$, it follows from the definition of $p^i_{G,H}(X)$ that after taking a resolution of M by free $\mathcal{Q}(\mathbf{X}, G)$-modules of finite rank, the case $i > 0$ reduces to the case $i = 0$. □

Lemma 4.1.4. *Let $\varphi : A \longrightarrow B$ be a flat morphism of rings and M be a finitely presented A-module. There is an isomorphism of right B-modules*

$$Ext^i_A(M, A) \otimes_A B \longrightarrow Ext^i_B(B \otimes_A M, B).$$

Proof. Let $P^\cdot$ be a resolution of M by free A-modules of finite rank. Since B is flat over A, one has that $B \otimes_A P^\cdot$ is also a resolution of $B \otimes_A M$ by free B-modules. So it is enough to consider the case where $i = 0$. For this we define

$$\begin{aligned} Hom_A(M, A) \otimes_A B &\longrightarrow Hom_B(B \otimes_A M, B) \\ f \otimes a &\longmapsto f_a. \end{aligned}$$

Here, $f_a \in Hom_B(B \otimes_A M, B)$ is defined as follows: $f_a(b \otimes m) := b\varphi(f(m))a \in B$ for any $b \in B, m \in M$. This map is an isomorphism when $M = A$ and also for general M since we can apply the Five lemma using the fact that M is finitely presented as an A-module. □

Proposition 4.1.5. *Let $\mathbf{U} \in \mathbf{X}_{ac}(\mathcal{L}, G)$. There is a morphism of right $\mathcal{Q}(\mathbf{X}, G)$-modules*

$$\tau^i_{\mathbf{X},\mathbf{U},G} : \; Ext^i_{\mathcal{Q}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}), \mathcal{Q}(\mathbf{X}, G)) \to Ext^i_{\mathcal{Q}(\mathbf{U},G)}(\mathcal{M}(\mathbf{U}), \mathcal{Q}(\mathbf{U}, G)).$$

Proof. Denote $M := \mathcal{M}(\mathbf{X})$. Then

$$\mathcal{M}(\mathbf{U}) \cong \mathcal{Q}(\mathbf{U}, G) \otimes_{\mathcal{Q}(\mathbf{X},G)} M.$$

Since U is $\mathcal{L}$-accessible, the morphism

$$\mathcal{Q}(\mathbf{X}, G) \longrightarrow \mathcal{Q}(\mathbf{U}, G)$$

is flat (Proposition 2.2.21). Now applying Lemma 4.1.4 gives

$$Ext^i_{\mathcal{Q}(\mathbf{U},G)}(\mathcal{M}(\mathbf{U}), \mathcal{Q}(\mathbf{U}, G)) \cong Ext^i_{\mathcal{Q}(\mathbf{X},G)}(M, \mathcal{Q}(\mathbf{X}, G)) \otimes_{\mathcal{Q}(\mathbf{X},G)} \mathcal{Q}(\mathbf{U}, G).$$

By consequence, we obtain the natural morphism of right $\mathcal{Q}(\mathbf{X}, G)$-modules:

$$\tau^i_{\mathbf{X},\mathbf{U},G} : \; Ext^i_{\mathcal{Q}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}), \mathcal{Q}(\mathbf{X}, G)) \to Ext^i_{\mathcal{Q}(\mathbf{U},G)}(\mathcal{M}(\mathbf{U}), \mathcal{Q}(\mathbf{U}, G)).$$

□

Proposition 4.1.6. *Let H be a normal open subgroup of G and $\mathbf{U} \in \mathbf{X}_{ac}(\mathcal{L}, G)$. Then the following diagram is commutative:*

$$\begin{array}{ccc} Ext^i_{\mathcal{Q}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}), \mathcal{Q}(\mathbf{X}, G)) & \xrightarrow{p^i_{G,H}(\mathbf{X})} & Ext^i_{\mathcal{Q}(\mathbf{X},H)}(\mathcal{M}(\mathbf{X}), \mathcal{Q}(\mathbf{X}, H)) \\ \tau^i_{\mathbf{X},\mathbf{U},G}\downarrow & & \downarrow\tau^i_{\mathbf{X},\mathbf{U},H} \\ Ext^i_{\mathcal{Q}(\mathbf{U},G)}(\mathcal{M}(\mathbf{U}), \mathcal{Q}(\mathbf{U}, G)) & \xrightarrow{p^i_{G,H}(\mathbf{U})} & Ext^i_{\mathcal{Q}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \mathcal{Q}(\mathbf{U}, H)) \end{array} \tag{4.1}$$

Proof. Write $M := \mathcal{M}(\mathbf{X})$. Then $Loc_{\mathcal{Q}}(M) \cong \mathcal{M}$. It follows that

$$\mathcal{M}(\mathbf{U}) \cong \mathcal{Q}(\mathbf{U}, G) \otimes_{\mathcal{Q}(\mathbf{X},G)} M \cong \mathcal{Q}(\mathbf{U}, H) \otimes_{\mathcal{Q}(\mathbf{X},H)} M.$$

Now take a resolution $P^{\cdot}$ of M by free $\mathcal{Q}(\mathbf{X}, G)$-modules of finite rank. Since $\mathbf{U} \in \mathbf{X}_{ac}(\mathcal{L}, G)$ is supposed to be $\mathcal{L}$-accessible, the ring $\mathcal{Q}(\mathbf{U}, G)$ is flat over $\mathcal{Q}(\mathbf{X}, G)$ (Proposition 2.4.17). This implies that $\mathcal{Q}(\mathbf{U}, G) \otimes_{\mathcal{Q}(\mathbf{X},G)} P^{\cdot}$ is also a free resolution of $\mathcal{Q}(\mathbf{U}, G) \otimes_{\mathcal{Q}(\mathbf{X},G)} M \cong \mathcal{M}(\mathbf{U})$. Hence it reduces to prove that for any $\mathcal{Q}(\mathbf{X}, G)$-module P, the following diagram is commutative:

$$\begin{array}{ccc} Hom_{\mathcal{Q}(\mathbf{X},G)}(P, \mathcal{Q}(\mathbf{X}, G)) & \xrightarrow{p_{H,H}(\mathbf{X})} & Hom_{\mathcal{Q}(\mathbf{X},H)}(P, \mathcal{Q}(\mathbf{X}, H)) \\ \downarrow & & \downarrow \\ Hom_{\mathcal{Q}(\mathbf{U},G)}(\mathcal{Q}(\mathbf{U}, G) \otimes_{\mathcal{Q}(\mathbf{X},G)} P, \mathcal{Q}(\mathbf{U}, G)) & \xrightarrow{p_{G,H}(\mathbf{U})} & Hom_{\mathcal{Q}(\mathbf{U},H)}(\mathcal{Q}(\mathbf{U}, H) \otimes_{\mathcal{Q}(\mathbf{X},H)} P, \mathcal{Q}(\mathbf{U}, H)). \end{array}$$

This means that the diagram

$$\begin{array}{ccc} \mathcal{Q}(\mathbf{X}, G) & \xrightarrow{p^{\mathbf{X}}_{G,H}} & \mathcal{Q}(\mathbf{X}, H) \\ \downarrow & & \downarrow \\ \mathcal{Q}(\mathbf{U}, G) & \xrightarrow{p^{\mathbf{U}}_{G,H}} & \mathcal{Q}(\mathbf{U}, H) \end{array}$$

is commutative, which is already proven in Corollary 3.1.8(ii).

□

4.2 The 'Ext-functor' on the category $\mathcal{C}_{\mathbf{X}/G}$

Let $\mathbf{X}$ be a smooth rigid analytic space and G be a p-adic Lie group acting continuously on $\mathbf{X}$. For each non negative integer $i \in \mathbb{N}$, we will construct so-called "Ext-functors" E^i from coadmissible G-equivariant left $\mathcal{D}_{\mathbf{X}}$-modules to coadmissible G-equivariant right $\mathcal{D}_{\mathbf{X}}$-modules. Let $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$ be a coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-module. Then as usual, locally we want $E^i(\mathcal{M})(\mathbf{U})$ to be isomorphic to $Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, H))$ for every open affinoid subset $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and open subgroup $H \leq G$ such that $(\mathbf{U}, H)$ is small. As in [4], we want the fact that this definition is independent of the choice of the subgroup H. That is why we take into account the following proposition:

Proposition 4.2.1. *Suppose that $\mathbf{X}$ is a smooth affinoid variety and G is such that $(\mathbf{X}, G)$ is small and H is an open normal subgroup of G. Then for any left $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module M, there is an isomorphism of right $\widehat{\mathcal{D}}(\mathbf{X}, H)$-modules:*

$$\widehat{p}^i_{G,H}(\mathbf{X}) : \ Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G)) \xrightarrow{\sim} Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},H)}(M, \widehat{\mathcal{D}}(\mathbf{X}, H)).$$

Furthermore, if $H' \leq H$ is another open normal subgroup of G, then one has

$$\widehat{p}^i_{H,H'}(\mathbf{X}) \circ \widehat{p}^i_{G,H}(\mathbf{X}) = \widehat{p}^i_{G,H'}(\mathbf{X}).$$

Proof. Since Lemma 3.1.3 holds for the morphism of rings $\widehat{\mathcal{D}}(\mathbf{X}, H) \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, G)$ (Proposition 3.1.7), the proof of this proposition uses exactly the same arguments as in the proof of Proposition 4.1.3. We just write down here the definition of $\widehat{p}^i_{G,H}(\mathbf{X})$. Let $P^\cdot$ be a resolution of M by free $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules. Then $\widehat{p}^i_{G,H}(\mathbf{X})$ is determined by taking the $i-th$-cohomology of the following isomorphism of complexes:

$$Hom_{\widehat{\mathcal{D}}(\mathbf{X},G)}(P^\cdot, \widehat{\mathcal{D}}(\mathbf{X}, G)) \longrightarrow Hom_{\widehat{\mathcal{D}}(\mathbf{X},H)}(P^\cdot, \widehat{\mathcal{D}}(\mathbf{X}, H))$$
$$f^\cdot \longmapsto p^{\mathbf{X}}_{G,H} \circ f^\cdot$$

In particular, when $i = 0$ then for every $f \in Hom_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G))$, one has

$$\widehat{p}_{G,H}(\mathbf{X})(f) := \widehat{p}^0_{G,H}(\mathbf{X})(f) := p^{\mathbf{X}}_{G,H} \circ f.$$

Here we recall that $p^{\mathbf{X}}_{G,H}$ is the projection map

$$p^{\mathbf{X}}_{G,H} : \widehat{\mathcal{D}}(\mathbf{X}, G) \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, H)$$
$$\sum_{i=0}^{m} a_i \bar{g}_i \longmapsto a_0,$$

where $\bar{g}_0, ..., \bar{g}_m$ denote the images of the set of cosets G/H (which is finite) in $\widehat{\mathcal{D}}(\mathbf{X}, G)$. □

Let $(\mathbf{X}, G)$ be small as above and M be a coadmissible (left) $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module. Suppose that $H \le G$ is an open normal subgroup of G. Let us choose a G-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ for some G-stable affine formal model $\mathcal{A}$ in $\mathcal{O}(\mathbf{X})$ and a good chain (J_n) for this Lie lattice such that $J_n \le H$ for any n. Then we may form the sheaves of rings

$$\mathcal{Q}_n(-, G) = \widehat{\mathcal{U}(\pi^n\mathcal{L})}_K \rtimes_{J_n} G, \text{ and } \mathcal{Q}_n(-, H) = \widehat{\mathcal{U}(\pi^n\mathcal{L})}_K \rtimes_{J_n} H \tag{4.2}$$

on $\mathbf{X}_{ac}(\mathcal{L}, G)$ and $\mathbf{X}_{ac}(\mathcal{L}, H)$ respectively. Hence

$$\widehat{\mathcal{D}}(\mathbf{X}, G) \simeq \varprojlim_n \mathcal{Q}_n(\mathbf{X}, G) \text{ and } \widehat{\mathcal{D}}(\mathbf{X}, H) \simeq \varprojlim_n \mathcal{Q}_n(\mathbf{X}, H).$$

Thus the projection map (3.2) : $p^{\mathbf{X}}_{G,H} : \widehat{\mathcal{D}}(\mathbf{X}, G) \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, H)$ is defined as the inverse limit of the maps (3.1): $p^{\mathbf{X}}_{G,H,n} : \mathcal{Q}_n(\mathbf{X}, G) \longrightarrow \mathcal{Q}_n(\mathbf{X}, H)$. Suppose that $M \cong \varprojlim_n M_n$ with $M_n = \mathcal{Q}_n(\mathbf{X}, G) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} M$, which is finitely generated over $\mathcal{Q}_n(\mathbf{X}, G)$. Then following Proposition 4.1.3 for every n, there is also an isomorphism of $D_n(\mathbf{X}, H)$-modules

$$p^i_{G,H,n}(\mathbf{X}) : \ Ext^i_{\mathcal{Q}_n(\mathbf{X},G)}(M_n, \mathcal{Q}_n(\mathbf{X}, G)) \xrightarrow{\sim} Ext^i_{\mathcal{Q}_n(\mathbf{X},H)}(M_n, \mathcal{Q}_n(\mathbf{X}, H)).$$

We will see right below that $\widehat{p}^i_{G,H}(\mathbf{X})$ is in fact isomorphic to the inverse limit of the maps $p^i_{G,H,n}(\mathbf{X})$.

Lemma 4.2.2. *There is a commutative diagram*

$$\begin{array}{ccc} Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G)) & \xrightarrow{\widehat{p}^i_{G,H}(\mathbf{X})} & Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},H)}(M, \widehat{\mathcal{D}}(\mathbf{X}, H)) \\ \downarrow & & \downarrow \\ Ext^i_{\mathcal{Q}_n(\mathbf{X},G)}(M_n, \mathcal{Q}_n(\mathbf{X}, G)) & \xrightarrow{p^i_{G,H,n}(\mathbf{X})} & Ext^i_{\mathcal{Q}_n(\mathbf{X},H)}(M_n, \mathcal{Q}_n(\mathbf{X}, H)). \end{array}$$

In particular, this implies that $\widehat{p}^i_{G,H}(\mathbf{X})$ equals to the inverse limit of the maps $p^i_{G,H,n}(\mathbf{X})$.

Proof. Note that $\widehat{\mathcal{D}}(\mathbf{X}, G)$ (which is finitely freely generated as a $\widehat{\mathcal{D}}(\mathbf{X}, H)$-module) is a coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, H)$-module. It follows that M is coadmissbile as a $\widehat{\mathcal{D}}(\mathbf{X}, H)$-module, so that

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G)) \cong \varprojlim_n Ext^i_{\mathcal{Q}_n(\mathbf{X},G)}(M_n, \mathcal{Q}_n(\mathbf{X}, G))$$

and

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},H)}(M, \widehat{\mathcal{D}}(\mathbf{X}, H)) \cong \varprojlim_n Ext^i_{\mathcal{Q}_n(\mathbf{X},H)}(M_n, \mathcal{Q}_n(\mathbf{X}, H)).$$

[25, Lemma 8.4]. These isomorphisms give the definitions of the two vertical arrows of the diagram in the lemma.
For any $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module P (which is not necessary coadmissible), we have an isomorphism of $\mathcal{Q}_n(\mathbf{X}, H)$-modules

$$\mathcal{Q}_n(\mathbf{X}, G) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} P \simeq (\mathcal{Q}_n(\mathbf{X}, H) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},H)} \widehat{\mathcal{D}}(\mathbf{X}, G)) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} P \simeq \mathcal{Q}_n(\mathbf{X}, H) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},H)} P.$$

Now, let $P^{\cdot} \to M \to 0$ be a projective resolution of M by free $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules. Since $\widehat{\mathcal{D}}(\mathbf{X}, G)$ is free over $\widehat{\mathcal{D}}(\mathbf{X}, H)$ on both sides, $P^{\cdot}$ is also a projective resolution of M as a $\widehat{\mathcal{D}}(\mathbf{X}, H)$-module. Moreover, it is proved [25, Remark 3.2] that the canonical maps $\widehat{\mathcal{D}}(\mathbf{X}, G) \to \mathcal{Q}_n(\mathbf{X}, G)$ and $\widehat{\mathcal{D}}(\mathbf{X}, H) \to \mathcal{Q}_n(\mathbf{X}, H)$ are right flat, so that $\mathcal{Q}_n(\mathbf{X}, G) \otimes P$ and $\mathcal{Q}_n(\mathbf{X}, H) \otimes P$ are projective resolutions of $\mathcal{Q}_n(\mathbf{X}, G) \otimes M$ and $\mathcal{Q}_n(\mathbf{X}, H) \otimes M$, respectively. Thus, by definitions of $\widehat{p}^i_{G,H}(\mathbf{X})$ and $p^i_{G,H,n}(\mathbf{X})$ it suffices to show that for any $\widehat{\mathcal{D}}(\mathbf{X}, G)$ -module P, the diagram

$$\begin{array}{ccc}
Hom_{\widehat{\mathcal{D}}(\mathbf{X},G)}(P, \widehat{\mathcal{D}}(\mathbf{X}, G)) & \xrightarrow{p^{\mathbf{X}}_{G,H}\circ} & Hom_{\widehat{\mathcal{D}}(\mathbf{X},H)}(P, \widehat{\mathcal{D}}(\mathbf{X}, H)) \\
{\scriptstyle id\bar{\otimes}-}\downarrow & & \downarrow{\scriptstyle id\bar{\otimes}-} \\
Hom_{\mathcal{Q}_n(\mathbf{X},G)}(\mathcal{Q}_n(\mathbf{X}, G) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} P, \mathcal{Q}_n(\mathbf{X}, G)) & \xrightarrow{p^{\mathbf{X}}_{G,H,n}\circ} & Hom_{\mathcal{Q}_n(\mathbf{X},H)}(\mathcal{Q}_n(\mathbf{X}, H) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},H)} P, \mathcal{Q}_n(\mathbf{X}, H))
\end{array}$$

is commutative. (Note that, for every $f \in Hom_{\widehat{\mathcal{D}}(\mathbf{X},G)}(P, \widehat{\mathcal{D}}(\mathbf{X}, G))$, the map

$$id\bar{\otimes}f \in Hom_{\mathcal{Q}_n(\mathbf{X},G)}(\mathcal{Q}_n(\mathbf{X}, G) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} P, \mathcal{Q}_n(\mathbf{X}, G))$$

is defined by $(id\bar{\otimes}f)(a \otimes m) = af(m)$ with $a \in \mathcal{Q}_n(\mathbf{X}, G)$, $m \in P$). This reduces to show that the diagram

$$\begin{array}{ccc}
\widehat{\mathcal{D}}(\mathbf{X}, G) & \xrightarrow{p^{\mathbf{X}}_{G,H}} & \widehat{\mathcal{D}}(\mathbf{X}, H) \\
\downarrow & & \downarrow \\
\mathcal{Q}_n(\mathbf{X}, G) & \xrightarrow{p^{\mathbf{X}}_{G,H,n}} & \mathcal{Q}_n(\mathbf{X}, H)
\end{array}$$

is commutative. Now the proof can be done by applying Corollary 3.1.8(i). □

Now let $\mathbf{X}$ be a smooth rigid analytic space, let G be a p-adic Lie group which acts continuously on $\mathbf{X}$. Let $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$ be a coadmissible G-equivariant left $\mathcal{D}_{\mathbf{X}}$-module. Fix an open affinoid subset $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$. Recall that for any $\mathbf{U}$-small subgroup $H \leq G$, one has an isomorphism of coadmissible H-equivariant $\mathcal{D}_U$-modules:

$$\mathcal{M}|_U \simeq Loc_U^{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U})).$$

Definition 4.2.3. *If $(\mathbf{U}, H)$ is small, we define for all $i \geq 0$:*

$$E^i(\mathcal{M})(\mathbf{U}, H) := Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, H)).$$

This is, in fact, a coadmissible right $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module. Now Proposition 4.2.1 gives the following result:

Proposition 4.2.4. *Let $H' \leq H$ be $\mathbf{U}$-small open subgroups of G. There is an isomorphism of right $\mathcal{D}(U)$-modules:*

$$\widehat{p}^i_{H',H}(U): \ Ext^i_{\widehat{\mathcal{D}}(U,H')}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(U, H')) \xrightarrow{\sim} Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, H)).$$

The family $(E^i(\mathcal{M})(\mathbf{U}, H), \widehat{p}^i_{H',H}(U))$ forms an inverse system when H', H run over the (partially ordered) set of all $\mathbf{U}$-small subgroups of G.

Proof. Since $H' \leq G$ is open compact in H, there is an open normal subgroup N of H which is contained in H' ([4, Lemma 3.2.1]). Hence following Proposition 4.2.1, one has the following isomorphism:

$$\widehat{p}^i_{H',N}(U): \ Ext^i_{\widehat{\mathcal{D}}(U,H')}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(U, H')) \xrightarrow{\sim} Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},N)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, N)).$$

and

$$\widehat{p}^i_{H,N}(U): \ Ext^i_{\widehat{\mathcal{D}}(U,H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(U, H)) \xrightarrow{\sim} Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},N)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, N)).$$

Now, we define

$$\widehat{p}^i_{H',H}(U) := (\widehat{p}^i_{H,N}(U))^{-1} \circ \widehat{p}^i_{H',N}(U).$$

By definition $\widehat{p}^i_{H',H}(U)$ is an isomorphism of $\mathcal{D}(U)$-modules. Furthermore, this is independent from the choice of an open normal subgroup N of H. Indeed, if $N^{'} \leq N$ is an other normal subgroup of H, then N' is also normal in N, thus Proposition gives

$$\widehat{p}^i_{H',N'}(U) = \widehat{p}^i_{H',N}(U) \circ \widehat{p}^i_{N,N'}(U) \text{ and } \widehat{p}^i_{H,N'}(U) = \widehat{p}^i_{H,N}(U) \circ \widehat{p}^i_{N,N'}(U).$$

Consequently

$$\begin{aligned}(\widehat{p}^i_{H,N'}(U))^{-1} \circ \widehat{p}^i_{H',N'}(U) &= (\widehat{p}^i_{H,N}(U) \circ \widehat{p}^i_{N,N'}(U))^{-1} \circ \widehat{p}^i_{H',N}(U) \circ \widehat{p}^i_{N,N'}(U) \\ &= (\widehat{p}^i_{H,N}(U))^{-1} \circ \widehat{p}^i_{H',N}(U).\end{aligned}$$

□

Remark 4.2.5. *If H' is normal in G, then we may choose $N = H'$ in the proof of the above proposition. Thus*

$$\widehat{p}^i_{H',H}(U) = (\widehat{p}^i_{H,H'}(U))^{-1}.$$

Thanks to Proposition 4.2.4, we are ready to give the following definition:

Definition 4.2.6. *For every open affinoid subset $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$, we define:*

$$E^i(\mathcal{M})(U) := \varprojlim_H E^i(\mathcal{M})(\mathbf{U}, H) = \varprojlim_H Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, H)),$$

where the inverse limit is taken over the set of all $\mathbf{U}$-small subgroups H of G.

Remark 4.2.7. *$E^i(\mathcal{M})(U)$ obviously has a structure of right $\mathcal{D}(U)$-module. Furthermore, we obtain from Proposition 4.2.1 that the natural map*

$$E^i(\mathcal{M})(U) \longrightarrow E^i(\mathcal{M})(\mathbf{U}, H)$$

is a bijection for every $\mathbf{U}$-small subgroup H of G.

Lemma 4.2.8. *Let $U \cong \varprojlim_n U_n$, $V \cong \varprojlim_n V_n$ be Fréchet-Stein algebras and $U \longrightarrow V$ be a continuous morphism of Fréchet-Stein algebras. Suppose that for each n, the induced morphism of rings $U_n \longrightarrow V_n$ is flat. Then for any coadmissible U-module M, there is an isomorphism of right V-modules*

$$Ext^i_U(M, U)\widehat{\otimes}_U V \longrightarrow Ext^i_V(V\widehat{\otimes}_U M, V).$$

Proof. Since M is coadmissible as U-module, we have the following isomorphism:

$$M \cong \varprojlim_n U_n \otimes_U M = \varprojlim_n M_n$$

with $M_n := U_n \otimes_U M$ for every n. Hence $V\widehat{\otimes}_U M \cong \varprojlim_n V_n \otimes_{U_n} M_n$ and this implies that:

$$Ext^i_U(M, U)\widehat{\otimes}_U V \cong \varprojlim_n Ext^i_{U_n}(M_n, U_n)\otimes_{U_n} V_n$$

and

$$Ext^i_V(V\widehat{\otimes}_U M, V) \cong \varprojlim_n Ext^i_{V_n}(V_n\otimes_{U_n} M_n, V_n).$$

So it reduces to prove that for every n, there is an isomorphism of right V_n-modules

$$Ext^i_{U_n}(M_n, U_n)\otimes_{U_n} V_n \xrightarrow{\sim} Ext^i_{V_n}(V_n\otimes_{U_n} M_n, V_n).$$

Now apply Lemma 4.1.4 □

Proposition 4.2.9. *Suppose that $(\mathbf{U}, H)$ is small and $\mathbf{V} \subset \mathbf{U}$ is an open affinoid subset in $\mathbf{X}_w(\mathcal{T})/H$, then there is a morphism of right $\widehat{\mathcal{D}}(\mathbf{U}, H)$-modules*

$$\widehat{\tau}^i_{\mathbf{U},\mathbf{V},H} : \ E^i(\mathcal{M})(\mathbf{U}, H) \to E^i(\mathcal{M})(\mathbf{V}, H).$$

If $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$ are open subsets in $\mathbf{X}_w(\mathcal{T})/H$, then the diagram

$$\begin{array}{ccc} E^i(\mathcal{M})(\mathbf{U}, H) & \xrightarrow{\widehat{\tau}^i_{\mathbf{U},\mathbf{V},H}} & E^i(\mathcal{M})(\mathbf{V}, H) \\ & \searrow^{\widehat{\tau}^i_{\mathbf{U},\mathbf{W},H}} & \downarrow \widehat{\tau}^i_{\mathbf{V},\mathbf{W},H} \\ & & E^i(\mathcal{M})(\mathbf{W}, H) \end{array}$$

is commutative.

Proof. We choose a free $\mathcal{A}$-Lie lattice $\mathcal{L}$ of $\mathcal{T}(\mathbf{U})$ for some H-stable affine formal model $\mathcal{A}$ of $\mathcal{O}(\mathbf{U})$ and a good chain (J_n) for $\mathcal{L}$. By rescaling $\mathcal{L}$, we may assume that $\mathbf{V}$ is $\mathcal{L}$-accessible.
Recall the sheaves $\mathcal{Q}_n(-, H)$ on $\mathbf{U}_{ac}(\mathcal{L}, H)$. Under these assumptions, the morphism $\mathcal{Q}_n(\mathbf{U}, H) \longrightarrow \mathcal{Q}_n(\mathbf{V}, H)$ is flat. Thus we can apply Lemma 4.2.7 and obtain:

$$E^i(\mathcal{M})(\mathbf{U}, H)\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},H)} \widehat{\mathcal{D}}(\mathbf{V}, H) \simeq E^i(\mathcal{M})(\mathbf{V}, H).$$

This provides a natural map

$$\begin{aligned} E^i(\mathcal{M})(\mathbf{U},H) &\longrightarrow E^i(\mathcal{M})(\mathbf{U},H)\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},H)}\widehat{\mathcal{D}}(\mathbf{V},H) \simeq E^i(\mathcal{M})(\mathbf{V},H) \\ m &\longmapsto m\widehat{\otimes}1. \end{aligned}$$

If $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$ are open subsets in $\mathbf{X}_w(\mathcal{T})/H$, then following [6, Corollary 7.4]

$$\begin{aligned} E^i(\mathcal{M})(\mathbf{U},H)\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},H)}\widehat{\mathcal{D}}(\mathbf{W},H) &\simeq E^i(\mathcal{M})(\mathbf{U},H)\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},H)}\widehat{\mathcal{D}}(\mathbf{V},H)\widehat{\otimes}_{\widehat{\mathcal{D}}(V,H)}\widehat{\mathcal{D}}(\mathbf{W},H) \\ &\simeq E^i(\mathcal{M})(\mathbf{V},H)\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{V},H)}\widehat{\mathcal{D}}(\mathbf{W},H) \quad (\simeq E^i(\mathcal{M})(\mathbf{W},H)). \end{aligned}$$

Hence the commutative diagram follows. □

Proposition 4.2.10. *Let H be an open compact subgroup of G and $\mathbf{U},\mathbf{V} \in \mathbf{X}_w(\mathcal{T})/H$ such that $\mathbf{V} \subset \mathbf{U}$. Suppose that $N \leq H$ is another open compact subgroup of G. Then the following diagram is commutative:*

$$\begin{array}{ccc} E^i(\mathcal{M})(\mathbf{U},N) & \xrightarrow{\widehat{p}^i_{N,H}(\mathbf{U})} & E^i(\mathcal{M})(\mathbf{U},H) \\ \widehat{\tau}^i_{\mathbf{U},\mathbf{V},N}\big\downarrow & & \big\downarrow\widehat{\tau}^i_{\mathbf{U},\mathbf{V},H} \\ E^i(\mathcal{M})(\mathbf{V},N) & \xrightarrow{\widehat{p}^i_{N,H}(\mathbf{V})} & E^i(\mathcal{M})(\mathbf{V},H). \end{array} \tag{4.3}$$

Proof. Firstly, suppose that N is normal in H. Then following Remark 4.2.5

$$\widehat{p}^i_{N,H}(\mathbf{U}) = (\widehat{p}^i_{H,N}(\mathbf{U}))^{-1} \text{ and } \widehat{p}^i_{N,H}(\mathbf{U}) = (\widehat{p}^i_{H,N}(\mathbf{V}))^{-1}.$$

We need to prove that:

$$\widehat{\tau}^i_{\mathbf{U},\mathbf{V},N}\circ\widehat{p}^i_{H,N}(\mathbf{U}) = \widehat{p}^i_{H,N}(\mathbf{V})\circ\widehat{\tau}^i_{\mathbf{U},\mathbf{V},H}.$$

For this we choose a H-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $\mathcal{T}(\mathbf{U})$ for some H-stable affine formal model $\mathcal{A}$ of $\mathcal{O}(\mathbf{U})$ and a good chain (J_n) for $\mathcal{L}$ such that $J_n \leq N$ for any n. By rescaling $\mathcal{L}$ if necessary, we may suppose in addition that $\mathbf{V}$ is $\mathcal{L}$-accessible, which means that $\mathbf{V} \in \mathbf{U}_{ac}(\mathcal{L},H)$. Consider the sheaves of rings

$$\mathcal{Q}_n(-,H) = \widehat{\mathcal{U}(\pi^n\mathcal{L})}_K \rtimes_{J_n} H \text{ and } \mathcal{Q}_n(-,N) = \widehat{\mathbf{U}(\pi^n\mathcal{L})}_K \rtimes_{J_n} N.$$

on $\mathbf{U}_{ac}(\mathcal{L},H)$ and $\mathbf{U}_{ac}(\mathcal{L},N)$ respectively. Since $\mathbf{V} \in \mathbf{U}_{ac}(\mathcal{L},H)$, then

$$\begin{aligned} \widehat{\mathcal{D}}(\mathbf{U},H) = \varprojlim_n \mathcal{Q}_n(\mathbf{U},H) \text{ and } \widehat{\mathcal{D}}(\mathbf{U},N) = \varprojlim_n \mathcal{Q}_n(\mathbf{U},N), \\ \widehat{\mathcal{D}}(\mathbf{V},H) = \varprojlim_n \mathcal{Q}_n(\mathbf{V},H) \text{ and } \widehat{\mathcal{D}}(\mathbf{V},N) = \varprojlim_n \mathcal{Q}_n(\mathbf{V},N). \end{aligned}$$

Since all modules appearing in the diagram (4.3) are coadmissible, following Lemma 4.2.2, it suffices to prove that:

$$\begin{array}{ccc} Ext^i_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{Q}_n(\mathbf{U},H)\otimes\mathcal{M}(\mathbf{U}),\mathcal{Q}_n(\mathbf{U},H)) & \xrightarrow{p^i_{H,N,n}(\mathbf{U})} & Ext^i_{\mathcal{Q}_n(\mathbf{U},N)}(\mathcal{Q}_n(\mathbf{U},N)\otimes\mathcal{M}(\mathbf{U}),\mathcal{Q}_n(\mathbf{U},N)) \\ \tau^i_{\mathbf{U},\mathbf{V},H}\big\downarrow & & \big\downarrow\tau^i_{\mathbf{U},\mathbf{V},N} \\ Ext^i_{\mathcal{Q}_n(\mathbf{V},H)}(\mathcal{Q}_n(\mathbf{V},H)\otimes\mathcal{M}(\mathbf{V}),\mathcal{Q}_n(\mathbf{V},H)) & \xrightarrow{p^i_{H,N,n}(\mathbf{V})} & Ext^i_{\mathcal{Q}_n(\mathbf{V},N)}(\mathcal{Q}_n(\mathbf{V},N)\otimes\mathcal{M}(\mathbf{V}),\mathcal{Q}_n(\mathbf{V},N)) \end{array}$$

is commutative. Now, applying Proposition 4.1.6 gives the result for the case N is normal in H.

When N is not normal in H, there is an open normal subgroup N' of H in N (as H is compact and N is open). Then

$$\begin{aligned}\widehat{\tau}^i_{\mathbf{U},\mathbf{V},H} \circ \widehat{p}^i_{N,H}(\mathbf{U}) &= \widehat{\tau}^i_{\mathbf{U},\mathbf{V},H} \circ (\widehat{p}^i_{N',H}(\mathbf{U}) \circ (\widehat{p}^i_{N',N}(\mathbf{U}))^{-1}) \\ &= \widehat{p}^i_{N',H}(\mathbf{V}) \circ \widehat{\tau}^i_{\mathbf{U},\mathbf{V},N} \circ (\widehat{p}^i_{N',N}(\mathbf{U}))^{-1} \\ &= \widehat{p}^i_{N,H}(\mathbf{V}) \circ \widehat{p}^i_{N',N}(\mathbf{V}) \circ \widehat{\tau}^i_{\mathbf{U},\mathbf{V},N} \circ (\widehat{p}^i_{N',N}(\mathbf{U}))^{-1} \\ &= \widehat{p}^i_{N,H}(\mathbf{V}) \circ \widehat{\tau}^i_{\mathbf{U},\mathbf{V},N} \circ \widehat{p}^i_{N',N}(\mathbf{U}) \circ (\widehat{p}^i_{N',N}(\mathbf{U}))^{-1} \\ &= \widehat{p}^i_{N,H}(\mathbf{V}) \circ \widehat{\tau}^i_{\mathbf{U},\mathbf{V},N}.\end{aligned}$$

Hence the commutativity of (4.3) follows. □

Proposition 4.2.11. *For every $\mathbf{U}, \mathbf{V} \in \mathbf{X}_w(\mathcal{T})$ such that $\mathbf{V} \subset \mathbf{U}$, there is a right $\mathcal{D}(\mathbf{U})$-linear restriction map*

$$\tau^i_{\mathbf{U},\mathbf{V}} : E^i(\mathcal{M})(\mathbf{U}) \longrightarrow E^i(\mathcal{M})(\mathbf{V}).$$

Proof. Let N be a $\mathbf{V}$-small subgroup of G. Then there exists a $\mathbf{U}$-small subgroup H inside $N_{\mathbf{U}}$-the stabiliser of $\mathbf{U}$ in N-which is normal in N [4, Lemma 3.2.1]. By Proposition 4.2.9, one has a morphism of right $\mathcal{D}(\mathbf{U})$-modules

$$\widehat{\tau}^i_{\mathbf{U},\mathbf{V},H} : E^i(\mathcal{M})(\mathbf{U}, H) \longrightarrow E^i(\mathcal{M})(\mathbf{V}, H).$$

Then we can define a right $\mathcal{D}(U)$-linear morphism

$$E^i(\mathcal{M})(\mathbf{U}) \longrightarrow E^i(\mathcal{M})(\mathbf{V}, N)$$

as the composition

$$E^i(\mathcal{M})(\mathbf{U}) = \varprojlim_H E^i(\mathcal{M})(\mathbf{U}, H) \longrightarrow E^i(\mathcal{M})(\mathbf{U}, H) \xrightarrow{\widehat{\tau}^i_{\mathbf{U},\mathbf{V},H}} E^i(\mathcal{M})(V, H) \xrightarrow{\widehat{p}^i_{H,N}(\mathbf{V})} E^i(\mathcal{M})(\mathbf{V}, N).$$

If H' is another open $\mathbf{U}$-small subgroup of H in $N_{\mathbf{U}}$, then Proposition 4.2.4 and Proposition 4.2.10 ensure that this map is independent of the choice of H. It amounts to showing that if $N' \leq N$ is another $\mathbf{V}$-small subgroup in G, then the following diagram is commutative:

$$\begin{array}{ccc} E^i(\mathcal{M})(\mathbf{U}) & \longrightarrow & E^i(\mathcal{M})(\mathbf{V}, N') \\ & \searrow & \downarrow \widehat{p}^i_{N',N}(\mathbf{V}) \\ & & E^i(\mathcal{M})(\mathbf{V}, N). \end{array} \tag{4.4}$$

If we take $H' := N'_{\mathbf{U}} \cap H$, then H' is a $\mathbf{U}$-small subgroup of $N'_{\mathbf{U}}$. Again by Proposition 4.2.10 and Corollary 4.2.4, it follows that the diagram

$$\begin{array}{ccccc} E^i(\mathcal{M})(\mathbf{U}, H') & \xrightarrow{\widehat{\tau}^i_{\mathbf{U},\mathbf{V},H'}} & E^i(\mathcal{M})(\mathbf{V}, H') & \xrightarrow{\widehat{p}^i_{H',N'}(\mathbf{V})} & E^i(\mathcal{M})(\mathbf{V}, N') \\ \downarrow & & \downarrow & & \downarrow \\ E^i(\mathcal{M})(\mathbf{U}, H) & \xrightarrow{\widehat{\tau}^i_{\mathbf{U},\mathbf{V},H}} & E^i(\mathcal{M})(\mathbf{V}, H) & \xrightarrow{\widehat{p}^i_{H,N}(\mathbf{V})} & E^i(\mathcal{M})(\mathbf{V}, N) \end{array}$$

is commutative, so that the triangle (4.4) is commutative. Now, by the universal property of the inverse limit, this induces a right $\mathcal{D}(\mathbf{U})$-linear map

$$E^i(\mathcal{M})(\mathbf{U}) = \varprojlim_H E^i(\mathcal{M})(\mathbf{U}, H) \longrightarrow E^i(\mathcal{M})(\mathbf{V}) = \varprojlim_N E^i(\mathcal{M})(\mathbf{V}, N)$$

as claimed.

□

Remark 4.2.12. *Thanks to Proposition* 4.2.11*, we see that $E^i(\mathcal{M})$ is a presheaf of $\mathcal{D}_{\mathbf{X}}$-modules on the set $\mathbf{X}_w(\mathcal{T})$ (which forms a basis for the Grothendieck topology on $\mathbf{X}$). Furthermore, if $\mathbf{U} \in X_(\mathcal{T})$ is an open affinoid of $\mathbf{X}$, then one has that $E^i(\mathcal{M})|_{\mathbf{U}} = E^i(\mathcal{M}|_{\mathbf{U}})$.*

Let us now define a G-equivariant structure on the presheaf $E^i(\mathcal{M})$ of right $\mathcal{D}_{\mathbf{X}}$-modules on $\mathbf{X}_w(\mathcal{T})$. Let $g \in G$ and $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$. Recall that g defines a morphism

$$\begin{aligned} g = g^{\mathcal{O}}(\mathbf{U}) : \ \mathcal{O}(\mathbf{U}) &\longrightarrow \mathcal{O}(g\,\mathbf{U}) \\ f &\longmapsto g.f. \end{aligned}$$

Here, for any function $f \in \mathcal{O}(\mathbf{U})$, the function $g.f \in \mathcal{O}(g\,\mathbf{U})$ is defined as $(g.f)(y) := f(g^{-1}y)$, $\forall y \in g\,\mathbf{U}$.

This induces an isomorphism of K-Lie algebras

$$\begin{aligned} g^{\mathcal{T}} := g^{\mathcal{T}}(\mathbf{U}) : \ \mathcal{T}(\mathbf{U}) &\longrightarrow \mathcal{T}(g\,\mathbf{U}) \\ v &\longmapsto g \circ v \circ g^{-1} \end{aligned}$$

which is linear relative to $g^{\mathcal{O}}(\mathbf{U})$.
Let H be a $\mathbf{U}$-small subgroup of G. Suppose that $\mathcal{A}$ is a H-stable formal model in $\mathcal{O}(\mathbf{U})$ and $\mathcal{L}$ is a H-stable $\mathcal{A}$-Lie lattice in $\mathcal{T}(\mathbf{U})$.

Lemma 4.2.13. *(i) $g(\mathcal{A})$ is a gHg^{-1}-stable formal model of $\mathcal{O}(g\,\mathbf{U})$ and $g^{\mathcal{T}}(\mathcal{L})$ is a gHg^{-1}-stable $g(\mathcal{A})$-Lie lattice in $\mathcal{T}(g\,\mathbf{U})$. If $\mathcal{L}$ is smooth (resp. free) over $\mathcal{A}$, then $g^{\mathcal{T}}(\mathcal{L})$ is smooth (resp. free) over $g(\mathcal{A})$.*

(ii) If $\mathbf{V} \in \mathbf{X}_w(\mathcal{T})$ is an open affinoid subset in $\mathbf{U}$, then $\mathbf{V}$ being a $\mathcal{L}$-accessible subdomain of $\mathbf{U}$ implies that $g\,\mathbf{V}$ is a $g^{\mathcal{T}}(\mathcal{L})$-accessible subdomain of $g\,\mathbf{U}$.

Proof. (i) Let $g \in G$ and $f \in \mathcal{O}(\mathbf{U})$. Since the morphism $g : \mathcal{O}(\mathbf{U}) \longrightarrow \mathcal{O}(g\,\mathbf{U})$ is K-linear, then

- $Kg(\mathcal{A}) = g(K\mathcal{A}) = g(\mathcal{O}(\mathbf{U})) = \mathcal{O}(g\,\mathbf{U})$.
- if $h \in H$ then $ghg^{-1}(g(\mathcal{A})) = g(h\mathcal{A}) \subset g(\mathcal{A})$, so that $g(\mathcal{A})$ is gHg^{-1}-stable.

Similarly,

- $Kg^{\mathcal{T}}(\mathcal{L}) = g^{\mathcal{T}}(K\mathcal{L}) = \tilde{g}(\mathcal{T}(\mathbf{U})) = \mathcal{T}(g\,\mathbf{U})$.
- $(ghg^{-1})^{\mathcal{T}}(g^{\mathcal{T}}(\mathcal{L})) = (gh)^{\mathcal{T}}(\mathcal{L}) \subset g^{\mathcal{T}}(\mathcal{L})$. Hence $\mathcal{L}$ is a gHg^{-1}-stable Lie lattice in $\mathcal{T}(gU)$. It remains to prove that if $\mathcal{L}$ is smooth (resp. free) over $\mathcal{A}$, then $g^{\mathcal{T}}(\mathcal{L})$ is smooth (resp. free) over $g(\mathcal{A})$. But this is straighforward in view of the fact that we have the bijection

$$g^{\mathcal{T}}|_{\mathcal{L}} : \mathcal{L} \xrightarrow{\sim} g^{\mathcal{T}}(\mathcal{L})$$

which is linear with respect to the (iso)morphism of rings $g|_{\mathcal{A}} : \mathcal{A} \longrightarrow g(\mathcal{A})$.

(ii) Without loss of generality, we may suppose that $\mathbf{U} = \mathbf{X}$ and $\mathbf{V}$ is a rational subset of $\mathbf{X}$. We prove (ii) by induction on n. If V is $\mathcal{L}$-accessible in 0- step, that means $\mathbf{V} = \mathbf{X}$, then $\mathbf{V}$ is $g^{\mathcal{T}}(\mathcal{L})$ accessible in 0- step. Now, suppose that the statement is true for $n-1$. Let $\mathbf{V}$ be $\mathcal{L}$-accessible in n-steps. We may assume that there is a chain $\mathbf{V} \subset \mathbf{Z} \subset \mathbf{X}$ such that $\mathbf{Z}$ is $\mathcal{L}-$accessible in $n-1$-steps, $\mathbf{V} = \mathbf{Z}(f)$ for some non zero $f \in \mathcal{O}(\mathbf{Z})$ and there is a $\mathcal{L}$ stable formal model $\mathcal{C} \subset \mathcal{O}(\mathbf{Z})$ such that $\mathcal{L}.f \subset \pi\mathcal{C}$. Then

$$gV = \{gy : y \in \mathbf{V}\}$$

and

$$(gZ)(g.f) = \{gy : |(g.f)(gy)| \leq 1, \forall y \in Z\} = \{gy : |f(g^{-1}gy)| = |f(y)| \leq 1, \forall y \in V\}.$$

Hence $g\,\mathbf{V} = (g\,\mathbf{Z})(gf)$. By assumption $g\,\mathbf{Z} \subset \mathbf{X}$ is $g^{\mathcal{T}}(\mathcal{L})$-accessible in $n-1$-steps. Furthermore, by (i), $g(\mathcal{C})$ is a gHg^{-1}-stable formal model of $\mathcal{O}(g\,\mathbf{Z})$ and it is straightforward that $g^{\mathcal{T}}(\mathcal{L}).(gf) \subset \pi.(g(\mathcal{C}))$. This shows that $g\,\mathbf{U}$ is also $g\mathcal{L}$-accessible in n-steps.

□

Let $(\mathbf{U}, H)$ be small. Recall the isomorphism of $K-$algebras

$$\widehat{g}_{\mathbf{U},H} : \widehat{\mathcal{D}}(\mathbf{U}, H) \xrightarrow{\sim} \widehat{\mathcal{D}}(g\,\mathbf{U}, gHg^{-1}).$$

and the isomorphism

$$g^{\mathcal{M}}_{\mathbf{U},H} : \ \mathcal{M}(\mathbf{U}) \longrightarrow \mathcal{M}(g\,\mathbf{U})$$

which is linear with respect to $\widehat{g}_{\mathbf{U},H}$ (since $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$).

Proposition 4.2.14. *Suppose that $(\mathbf{U}, H)$ is small and $g \in G$. There exists a K-linear map*

$$g^{E^i(\mathcal{M})}_{\mathbf{U},H} \ : E^i(\mathcal{M})(\mathbf{U}, H) \longrightarrow E^i(\mathcal{M})(g\,\mathbf{U}, gHg^{-1})$$

such that for every $a \in \widehat{\mathcal{D}}(\mathbf{U}, H), m \in E^i(\mathcal{M})(\mathbf{U}, H))$, we have:

$$g^{E^i(\mathcal{M})}_{\mathbf{U},H}(ma) = g^{E^i(\mathcal{M})}_{\mathbf{U},H}(m).\widehat{g}_{\mathbf{U},H}(a). \tag{4.5}$$

Proof. Denote ${}^gH := gHg^{-1}$. We construct a map

$$g^{E^i(\mathcal{M})}_{\mathbf{U},H} : \ Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, H)) \longrightarrow Ext^i_{\widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH)}(\mathcal{M}(g\,\mathbf{U}), \widehat{\mathcal{D}}(g\,\mathbf{U}, {}^gH))$$

as follows: Let $P^{\cdot} \longrightarrow \mathcal{M}(\mathbf{U}) \longrightarrow 0$ be a free resolution of $\mathcal{M}(\mathbf{U})$ as a $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module. Then by regarding each term of the complex $P^{\cdot}$ as a $\widehat{\mathcal{D}}(g\,\mathbf{U}, gHg^{-1})$-module via the isomorphism of rings $\widehat{g}^{-1}_{\mathbf{U},H} : \ \widehat{\mathcal{D}}(g\,\mathbf{U}, {}^gH) \xrightarrow{\sim} \widehat{\mathcal{D}}(\mathbf{U}, H)$, we can also view $P^{\cdot}$ as a free resolution of $\mathcal{M}(g\,\mathbf{U}) \simeq \mathcal{M}(\mathbf{U})$ by $\widehat{\mathcal{D}}(g\,\mathbf{U}, {}^gH)$-modules and denote it by ${}^gP^{\cdot}$. Thus, the map $g^{E^i(\mathcal{M})}_{\mathbf{U},H}$ can be defined by applying the i-th cohomology functor to the morphism of complexes whose components are morphisms of the form:

$$\begin{aligned} \phi^g_{\mathbf{U},H} : \ Hom_{\widehat{\mathcal{D}}(\mathbf{U},H)}(P^k, \widehat{\mathcal{D}}(\mathbf{U}, H)) &\longrightarrow Hom_{\widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH)}({}^g(P^k), \widehat{\mathcal{D}}(g\,\mathbf{U}, {}^gH)) \\ f &\longmapsto \ \ \widehat{g}_{\mathbf{U},H} \circ f, \end{aligned}$$

where ${}^g(P^k)$ denotes the component P^k of the complex $P^{\cdot}$ viewed as a $\widehat{\mathcal{D}}(g\,\mathbf{U}, {}^gH)$-module via the morphism $\widehat{g}^{-1}_{\mathbf{U},H}$.

We need to check the following facts:

1. $\widehat{g}_{\mathbf{U},H} \circ f \in Hom_{\widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH)}({}^g(P^k), \widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH))$, which means that $\widehat{g}_{\mathbf{U},H} \circ f$ is $\widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH)$-linear. Indeed, if $b \in \widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH)$ and $m \in {}^g(P^k)$, then:

$$(\widehat{g}_{\mathbf{U},H} \circ f)(b.m) = \widehat{g}_{\mathbf{U},H}(f(\widehat{g^{-1}}_{\mathbf{U},H}(b)m) = \widehat{g}_{\mathbf{U},H}(\widehat{g^{-1}}_{\mathbf{U},H}(b)f(m)) = b(\widehat{g}_{\mathbf{U},H} \circ f)(m),$$

here the second equality follows from the fact that f is $\widehat{\mathcal{D}}(\mathbf{U},H)$-linear and the third one is based on the fact that $\widehat{g}_{\mathbf{U},H}$ is a morphism of K-algebras.

2. For any $a \in \widehat{\mathcal{D}}(\mathbf{U},H)$ and $f \in Hom_{\widehat{\mathcal{D}}(\mathbf{U},H)}(P^k, \widehat{\mathcal{D}}(\mathbf{U},H))$, we check that:

$$\phi^g_{\mathbf{U},H}(fa) = \phi^g_{\mathbf{U},H}(f)\widehat{g}_{\mathbf{U},H}(a).$$

Let $m \in {}^g(P)^k$. We compute:

$$\phi^g_{\mathbf{U},H}(fa)(m) = \widehat{g}_{\mathbf{U},H}(f(m)a) = \widehat{g}_{\mathbf{U},H}(f(m))\widehat{g}_{\mathbf{U},H}(a) = \phi^g_{\mathbf{U},H}(f)(m)\widehat{g}_{\mathbf{U},H}(a).$$

Finally, by definition of $g^{E^i(\mathcal{M})}_{\mathbf{U},H}$, this implies (4.5). □

Next, we study some properties of the morphisms $g^{E^i(\mathcal{M})}_{\mathbf{U},H}(g \in G)$. Let $\mathcal{L}$-be a H-stable free $\mathcal{A}$-Lie lattice of $\mathcal{T}(\mathbf{U})$ for some H-stable affine formal model $\mathcal{A}$ in $\mathcal{O}(\mathbf{U})$. Write $\mathcal{A}' := g(\mathcal{A})$ and $\mathcal{L}' := g^{\mathcal{T}}(\mathcal{L})$. Lemma 4.2.12 shows us that there is a bijection between the following $\mathbf{G}$-topologies:

$$\mathbf{U}_{ac}(\mathcal{L},H) \longrightarrow (g\,\mathbf{U})_{ac}(\mathcal{L}', gHg^{-1})$$
$$\mathbf{V} \longmapsto g\,\mathbf{V}\,.$$

Furthermore, if $J \leq G_{\mathcal{L}}$ is an open normal subgroup of G such that $(J,\mathcal{L})$ is an $\mathcal{A}$-trivialising pair in H, then $(gJg^{-1},\mathcal{L}')$ is also an $\mathcal{A}'$-trivialising pair in gHg^{-1}. Let $(J_n)_n$ be a good chain for $\mathcal{L}$ and recall the sheaves $\mathcal{Q}_n$ from (4.2). If $\mathbf{V} \in \mathbf{U}_{ac}(\mathcal{L},H)$, there is an isomorphism of K-algebras:

$$g^{\mathcal{Q}_n}_{\mathbf{V},H} : \mathcal{Q}_n(\mathbf{V},H) \longrightarrow \mathcal{Q}_n(g\,\mathbf{V}, gHg^{-1}).$$

These maps satisfy

$$\widehat{g}_{\mathbf{V},H} = \varprojlim_n g^{\mathcal{Q}_n}_{\mathbf{V},H}.$$

Let $\mathcal{M}$ be a coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-module. For each n, we define the following presheaves. Let $\mathbf{V} \in \mathbf{U}_w(\mathcal{L},H)$, then:

$$\mathcal{M}_n(\mathbf{V}) := \mathcal{Q}_n(\mathbf{V},H) \otimes_{\widehat{\mathcal{D}}(\mathbf{U},H)} \mathcal{M}(\mathbf{U}) \tag{4.6}$$

and

$$\mathcal{M}_n(g\,\mathbf{V}) := \mathcal{Q}_n(g\,\mathbf{V}, gHg^{-1}) \otimes_{\widehat{\mathcal{D}}(g\,\mathbf{U},gHg^{-1})} \mathcal{M}(g\,\mathbf{U}). \tag{4.7}$$

Note that they defined sheaves of modules on $\mathbf{U}_w(\mathcal{L},H)$ and on $(g\,\mathbf{U})_w(\mathcal{L}', gHg^{-1})$, respectively. If $\mathbf{V} \in \mathbf{U}_w(\mathcal{L},H)$, the isomorphism

$$g^{\mathcal{M}}_{\mathbf{V},H} : \mathcal{M}(\mathbf{V}) \longrightarrow \mathcal{M}(g\,\mathbf{V})$$

induces an isomorphism

$$g^{\mathcal{M}_n}_{\mathbf{V},H} : \ \mathcal{M}_n(\mathbf{V}) \longrightarrow \mathcal{M}_n(g\,\mathbf{V})$$
$$s \otimes m \longmapsto g_{\mathbf{V},H,n}(s) \otimes g^{\mathcal{M}}_{\mathbf{V},H}(m).$$

We have the following result:

Proposition 4.2.15. *Let $g \in G$. There is an isomorphism*

$$g^{E^i(\mathcal{M})}_{\mathbf{U},H,n} : \ Ext^i_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{M}_n(\mathbf{U}), \mathcal{Q}_n(\mathbf{U},H)) \longrightarrow Ext^i_{\mathcal{Q}_n(g\,\mathbf{U},{}^gH)}(\mathcal{M}_n(g\,\mathbf{U}), \mathcal{Q}_n(g\,\mathbf{U},{}^gH))$$

such that

1. *For any $s \in \mathcal{Q}_n(\mathbf{U},H)$ and $m \in Ext^i_{\mathcal{Q}_n(\mathcal{L},H)}(\mathcal{M}_n(\mathbf{U}), \mathcal{Q}_n(\mathbf{U},H))$, one has that:*
$$g^{E^i(\mathcal{M})}_{\mathbf{U},H,n}(ms) = g^{E^i(\mathcal{M})}_{\mathbf{U},H,n}(m).g^{\mathcal{Q}_n}_{\mathbf{U},H}(s).$$
2. *Let $\mathbf{V} \in \mathbf{U}_{ac}(\mathcal{L},H)$. Then the following diagram is commutative:*
$$\begin{array}{ccc} Ext^i_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{M}_n(\mathbf{U}), \mathcal{Q}_n(\mathbf{U},H)) & \xrightarrow{g^{E^i(\mathcal{M})}_{\mathbf{U},H,n}} & Ext^i_{\mathcal{Q}_n(g\,\mathbf{U},{}^gH)}(\mathcal{M}_n(g\,\mathbf{U}), \mathcal{Q}_n(g\,\mathbf{U},{}^gH)) \\ \downarrow \tau^i_{H,n} & & \downarrow \tau^i_{gH,n} \\ Ext^i_{\mathcal{Q}_n(\mathbf{V},H)}(\mathcal{M}_n(\mathbf{V}), \mathcal{Q}_n(\mathbf{V},H)) & \xrightarrow{g^{E^i(\mathcal{M})}_{\mathbf{V},H,n}} & Ext^i_{\mathcal{Q}_n(g\,\mathbf{V},{}^gH)}(\mathcal{M}_n(g\,\mathbf{V}), \mathcal{Q}_n(g\,\mathbf{V},{}^gH)) \end{array}.$$
Here $\tau^i_{H,n}$ and $\tau^i_{gH,n}$ are transition maps which are defined in Proposition 4.1.6.

Proof. (1.) We define $g^{E^i(\mathcal{M})}_{\mathbf{U},H,n}$ similarly as defining $g^{E^i(\mathcal{M})}_{\mathbf{U},H}$ in Proposition 4.2.14. Let $P^\cdot_n \longrightarrow \mathcal{M}_n(\mathbf{U}) \longrightarrow 0$ be a resolution of $\mathcal{M}_n(\mathbf{U})$ by free $\mathcal{Q}_n(\mathbf{U},H)$-modules. Then by considering each term of this resolution as a $\mathcal{Q}_n(g\,\mathbf{U},{}^gH)$-module via the isomorphism of K-algebras $g^{\mathcal{Q}_n}_{\mathbf{U},H} : \mathcal{Q}_n(\mathbf{U},H) \longrightarrow \mathcal{Q}_n(g\,\mathbf{U},{}^gH)$, we see that $P^\cdot_n$ is also a resolution of $\mathcal{M}_n(g\,\mathbf{U})$ by free $\mathcal{Q}_n(g\,\mathbf{U},{}^gH)$-modules. Let us denote this by ${}^gP^\cdot_n$. Then the morphism $g^{E^i(\mathcal{M})}_{\mathbf{U},H,n}$ is determined by taking the i-th cohomology of the following morphism of complexes:

$$\begin{aligned} Hom_{\mathcal{Q}_n(\mathbf{U},H)}(P^\cdot_n, \mathcal{Q}_n(\mathbf{U},H)) &\longrightarrow Hom_{\mathcal{Q}_n(g\,\mathbf{U},{}^gH)}({}^gP^\cdot_n, \mathcal{Q}_n(g\,\mathbf{U},{}^gH)) \\ f &\longmapsto g^{\mathcal{Q}_n}_{\mathbf{U},H} \circ f. \end{aligned}$$

Now the required property can be proved similarly as for $g^{E^i(\mathcal{M})}_{\mathbf{U},H}$.

(2.) Note that

$$\mathcal{M}_n(\mathbf{V}) = \mathcal{Q}_n(\mathbf{V},H) \otimes_{\widehat{\mathcal{D}}(\mathbf{V},H)} \mathcal{M}(\mathbf{V}) \cong \mathcal{Q}_n(\mathbf{V},H) \otimes_{\mathcal{Q}_n(\mathbf{U},H)} \mathcal{M}_n(\mathbf{U}).$$

$$\mathcal{M}_n(g\,\mathbf{V}) = \mathcal{Q}_n(g\,\mathbf{V},{}^gH) \otimes_{\widehat{\mathcal{D}}(g\,\mathbf{V},{}^gH)} \mathcal{M}(g\,\mathbf{V}) \cong \mathcal{Q}_n(g\,\mathbf{V},{}^gH) \otimes_{\mathcal{Q}_n(g\,\mathbf{U},{}^gH)} \mathcal{M}_n(g\,\mathbf{U}).$$

By taking a projective resolution of $\mathcal{M}_n(\mathbf{U})$ by free $\mathcal{Q}_n(\mathbf{U},H)$-modules together with the flatness of the morphisms $\mathcal{Q}_n(\mathbf{U},H) \longrightarrow \mathcal{Q}_n(\mathbf{V},H)$ and $\mathcal{Q}_n(g\,\mathbf{U},{}^gH) \longrightarrow \mathcal{Q}_n(g\,\mathbf{V},{}^gH)$ (Proposition 2.4.17), it reduces to show the assertion for $i = 0$, which means that the diagram

$$\begin{array}{ccc} Hom_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{M}_n(\mathbf{U}), \mathcal{Q}_n(\mathbf{U},H)) & \longrightarrow & Hom_{\mathcal{Q}_n(g\,\mathbf{U},{}^gH)}(\mathcal{M}_n(gU), \mathcal{Q}_n(g\,\mathbf{U},{}^gH)) \\ \downarrow & & \downarrow \\ Hom_{\mathcal{Q}_n(\mathbf{V},H)}(\mathcal{M}_n(\mathbf{V}), \mathcal{Q}_n(\mathbf{V},H)) & \longrightarrow & Hom_{\mathcal{Q}_n(g\,\mathbf{V},{}^gH)}(\mathcal{M}_n(g\,\mathbf{V}), \mathcal{Q}_n(g\,\mathbf{V},{}^gH)) \end{array}$$

is commutative.

Let $f : \mathcal{M}_n(\mathbf{U}) \longrightarrow \mathcal{Q}_n(\mathbf{U},H)$ be a $\mathcal{Q}_n(\mathbf{U},H)$-linear morphism and write r_n, r^g_n for the restrictions $\mathcal{Q}_n(\mathbf{U},H) \longrightarrow \mathcal{Q}_n(\mathbf{V},H)$ and $\mathcal{Q}_n(g\,\mathbf{U},{}^gH) \longrightarrow \mathcal{Q}_n(g\,\mathbf{V},{}^gH)$, respectively. For $a \in \mathcal{Q}_n(g\,\mathbf{V},{}^gH)$, $m \in \mathcal{M}_n(g\,\mathbf{U})$, we have:

$$\left(1\bar{\otimes}r_{n^g}\circ(g^{\mathcal{Q}_n}_{\mathbf{U},H}\circ f\circ(g^{\mathcal{M}_n}_{\mathbf{U},H})^{-1})\right)(a\otimes m)=a.r^g_n(g^{\mathcal{Q}_n}_{\mathbf{U},H}(f((g^{\mathcal{M}_n}_{\mathbf{U},H})^{-1}(m))))$$

and

$$\left(g^{\mathcal{Q}_n}_{\mathbf{V},H}\circ(1\bar{\otimes}r_n\circ f)\circ(g^{\mathcal{M}_n}_{\mathbf{V},H})^{-1}\right)(a\otimes m)=a.g^{\mathcal{Q}_n}_{\mathbf{V},H}(r_n(f((g^{\mathcal{M}_n}_{\mathbf{V},H})^{-1}(m)))).$$

So it reduces to prove that for any $b\in\mathcal{Q}_n(g\,\mathbf{V},{}^gH)$, one has that:

$$r^g_n(g^{\mathcal{Q}_n}_{\mathbf{U},H}((b))=g^{\mathcal{Q}_n}_{\mathbf{V},H}(r_n(b)),$$

which is a consequence of [4, Definition 3.4.9(c) and Proposition 3.4.10].

□

Notation: In the sequel, whenever $\mathbf{V},H$ are given and whenever there is no ambiguity, we simply write $\widehat{g}$ for $\widehat{g}_{\mathbf{V},H}$ and $g^{\mathcal{M}_n}$, $g^{\mathcal{Q}_n}$, $g_n^{E^i(\mathcal{M})}$... instead of $g^{\mathcal{M}_n}_{\mathbf{V},H}$, $g^{\mathcal{Q}_n}_{\mathbf{V},H}$, $g^{E^i(\mathcal{M})}_{\mathbf{V},H,n}$... respectively.

Proposition 4.2.16. *The following diagram is commutative:*

$$\begin{array}{ccc} Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}),\widehat{\mathcal{D}}(\mathbf{U},H)) & \xrightarrow{g^{E^i(\mathcal{M})}} & Ext^i_{\widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH)}(\mathcal{M}(g\,\mathbf{U}),\widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH)) \\ \downarrow & & \downarrow \\ Ext^i_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{M}_n(\mathbf{U}),\mathcal{Q}_n(\mathbf{U},H)) & \xrightarrow{g_n^{E^i(\mathcal{M})}} & Ext^i_{\mathcal{Q}_n(g\,\mathbf{U},{}^gH)}(\mathcal{M}_n(g\,\mathbf{U}),\mathcal{Q}_n(g\,\mathbf{U},{}^gH)). \end{array}$$

Proof. First, we note that the morphisms

$$q_{\mathbf{U},n}:\widehat{\mathcal{D}}(\mathbf{U},H)\longrightarrow\mathcal{Q}_n(\mathbf{U},H)\text{ and }q_{g\,\mathbf{U},n}:\widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH)\longrightarrow\mathcal{Q}_n(g\,\mathbf{U},{}^gH)$$

are flat. By using a resolution $P^{\cdot}\longrightarrow\mathcal{M}(\mathbf{U})\longrightarrow 0$ of $\mathcal{M}(\mathbf{U})$ by free $\widehat{\mathcal{D}}(\mathbf{U},H)$-modules, it reduces to prove the commutativity of the above diagram for the case where $i=0$. Let $f\in Hom_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}),\widehat{\mathcal{D}}(\mathbf{U},H))$, then by definition:

$$g^{E^0(\mathcal{M})}(f)=\widehat{g}\circ f\circ\left(g^{\mathcal{M}}\right)^{-1}.$$

Let $s\in\mathcal{Q}_n(g\,\mathbf{U},{}^gH)$ and $m\in\mathcal{M}_n(g\,\mathbf{U})$. It follows that

$$\left(id\bar{\otimes}q_{g\,\mathbf{U},n}(\widehat{g}\circ f\circ\left(g^{\mathcal{M}}\right)^{-1}\right)(s\otimes m)=s.q_{g\,\mathbf{U},n}(\widehat{g}(f((g^{\mathcal{M}})^{-1}(m))))$$

and

$$\left(g^{\mathcal{Q}_n}\circ(1\bar{\otimes}q_{\mathbf{U},n}\circ f)\circ(g^{\mathcal{M}}_n)^{-1}\right)(s\otimes m)=s.g^{\mathcal{Q}_n}(q_{\mathbf{U},n}(f((g^{\mathcal{M}})^{-1})(m)))).$$

Now the result follows from the commutativity of the following diagram:

$$\begin{array}{ccc} \widehat{\mathcal{D}}(\mathbf{U},H) & \xrightarrow{\widehat{g}} & \widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH) \\ {\scriptstyle q_{\mathbf{U},n}}\downarrow & & \downarrow{\scriptstyle q_{g\,\mathbf{U},n}} \\ \mathcal{Q}_n(\mathbf{U},H) & \xrightarrow{g^{\mathcal{Q}_n}} & \mathcal{Q}_n(g\,\mathbf{U},{}^gH) \end{array}$$

which is proved in Corollary 3.1.8(i).

□

Remark 4.2.17. *The above proposition shows that for any $g \in G$, $\mathbf{U} \in X_w(\mathcal{T})$ and $H \leq G$ such that $(\mathbf{U}, H)$ is mall, the following equality holds:*

$$g^{E^i(\mathcal{M})}_{\mathbf{U},H} = \varprojlim_n g^{E^i(\mathcal{M})}_{\mathbf{U},H,n}.$$

Proposition 4.2.18. *If $N \leq H$ and $\mathbf{V}$ is a N-stable subdomain of $\mathbf{U}$ in $\mathbf{X}_w(\mathcal{T})$, the diagram*

$$\begin{array}{ccc}
E^i(\mathcal{M})(\mathbf{V}, N) & \xrightarrow{g^{E^i(\mathcal{M})}_{\mathbf{V},N}} & E^i(\mathcal{M})(g\,\mathbf{V}, gNg^{-1}) \\
\Big\downarrow \widehat{p}^i_{N,H}(\mathbf{V}) & & \Big\downarrow \widehat{p}^i_{N,H}(\mathbf{V}) \\
E^i(\mathcal{M})(\mathbf{V}, H) & \xrightarrow{g^{E^i(\mathcal{M}))}_{\mathbf{V},N}} & E^i(\mathcal{M})(g\,\mathbf{V}, gHg^{-1}) \\
\Big\uparrow \widehat{\tau}^i_{\mathbf{U},\mathbf{V},H} & & \Big\uparrow \widehat{\tau}^i_{\mathbf{U},\mathbf{V},H} \\
E^i(\mathcal{M})(\mathbf{U}, H) & \xrightarrow{g^{E^i(\mathcal{M}))}_{\mathbf{U},H}} & E^i(\mathcal{M})(g\,\mathbf{U}, gHg^{-1})
\end{array}$$

is commutative.

Proof. We now prove the proposition for the case where N is normal in H, as the general case can be proved by choosing an open normal subgroup of H which is contained in N.

1. Let us prove the commutativity of the upper square. Take a projective resolution of $\mathcal{M}(\mathbf{V})$ by free modules in $Mod(\widehat{\mathcal{D}}(\mathbf{V}, H))$. It is enough to show that for any (left) $\widehat{\mathcal{D}}(\mathbf{V}, H)$-module P, the diagram

$$\begin{array}{ccc}
Hom_{\widehat{\mathcal{D}}(\mathbf{V},H)}(P, \widehat{\mathcal{D}}(\mathbf{V}, H)) & \xrightarrow{\phi^g_{\mathbf{V},H}} & Hom_{\widehat{\mathcal{D}}(g\,\mathbf{V},{}^gH)}({}^gP, \widehat{\mathcal{D}}(g\,\mathbf{V}, {}^gH)) \\
\widehat{p}_{H,N}(\mathbf{V}) \Big\downarrow & & \Big\downarrow \widehat{p}_{{}^gH,{}^gN}(g\,\mathbf{V}) \\
Hom_{\widehat{\mathcal{D}}(\mathbf{V},N)}(P, \widehat{\mathcal{D}}(\mathbf{V}, N)) & \xrightarrow{\phi^g_{\mathbf{V},N}} & Hom_{\widehat{\mathcal{D}}(g\,\mathbf{V},gNg^{-1})}({}^gP, \widehat{\mathcal{D}}(g\,\mathbf{V}, {}^gN))
\end{array}$$

is commutative. It means that if $f \in Hom_{\widehat{\mathcal{D}}(\mathbf{V},H)}(P, \widehat{\mathcal{D}}(\mathbf{V}, H))$, then one has :

$$\widehat{p}_{{}^gH,{}^gN}(g\,\mathbf{V})(\widehat{g}_{\mathbf{V},H} \circ f) = \widehat{g}_{\mathbf{V},N} \circ \widehat{p}_{H,N}(\mathbf{V})(f).$$

But this reduces to proving that the diagram

$$\begin{array}{ccc}
\widehat{\mathcal{D}}(\mathbf{V}, H) & \xrightarrow{\widehat{g}_{\mathbf{V},H}} & \widehat{\mathcal{D}}(g\,\mathbf{V}, {}^gH) \\
p^{\mathbf{V}}_{H,N} \Big\downarrow & & \Big\downarrow p^{\mathbf{V}}_{{}^gH,{}^gN} \\
\widehat{\mathcal{D}}(\mathbf{V}, N) & \xrightarrow{\widehat{g}_{\mathbf{V},N}} & \widehat{\mathcal{D}}(g\,\mathbf{V}, {}^gN)
\end{array} \tag{4.8}$$

is commutative. For this, choose a H-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ for some H-stable formal model $\mathcal{A}$ of $\mathcal{O}(\mathbf{V})$ and a good chain (J_n) for $\mathcal{L}$. Recall from Lemma 4.2.13(i) that $\mathcal{L}' = g^{\mathcal{T}}(\mathcal{L})$

is a gHg^{-1}-stable free $g(\mathcal{A})$-Lie lattice in $\mathcal{T}(g\,\mathbf{U})$. For a fixed natural integer $n \in \mathbb{N}$, we consider the following diagram:

$$\begin{array}{ccc} \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} H & \xrightarrow{g^{\mathcal{Q}_n}_{V,H}} & \widehat{U(\pi^n\mathcal{L}')}_K \rtimes_{gJ_ng^{-1}} gHg^{-1} \\ {\scriptstyle p^V_{H,N,n}}\downarrow & & \downarrow{\scriptstyle p^V_{^gH,^gN,n}} \\ \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} N & \xrightarrow{g^{\mathcal{Q}_n}_{V,N}} & \widehat{U(\pi^n\mathcal{L}')}_K \rtimes_{gJ_ng^{-1}} gNg^{-1}. \end{array}$$

Let $\{g_1 = 1, ..., g_m, ..., g_n\}$ be a set of representatives of cosets of G modulo J_n such that $\{\bar{g_1} = 1, \bar{g_2}, \bar{g_m}, .., \bar{g_n}\}$ is a basis of $\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} H$ and $\{\bar{g}_1, ..., \bar{g}_m\}$ is a basis of $\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} N$ over the ring $\widehat{U(\pi^n\mathcal{L})_K}$ as left modules. Then we get a basis of $\widehat{U(\pi^n\mathcal{L}')}_K \rtimes_{gJ_ng^{-1}} gHg^{-1}$ (respectively, of $\widehat{U(\pi^n\mathcal{L}')}_K \rtimes_{J_n} gNg^{-1}$) over the ring $\widehat{U(\pi^n\mathcal{L}')_K}$ induced by classes of $\{gg_1g^{-1}, ..., gg_mg^{-1}, ..., gg_ng^{-1}\}$ (respectively, of $\{gg_1g^{-1}, ..., gg_mg^{-1}\}$) modulo gJ_ng^{-1}. This implies, by definition of the projection maps $p^{\mathbf{V}}_{H,N,n}$ and $p^{g\,\mathbf{V}}_{^gH,^gN,n}$, that the above diagram is commutative for each n, which produces the commutivity of (4.8).

2. It remains to show the commutativity of the lower square. We still fix a H-stable free $\mathcal{A}$-Lie lattice of $\mathcal{T}(\mathbf{U})$, a good chain (J_n) for $\mathcal{L}$ and keep notations as above. Suppose in addition that $\mathbf{V}$ is an $\mathcal{L}$- accessible subdomain of $\mathbf{U}$ (by rescaling $\mathcal{L}$). Then $g\,\mathbf{V}$ is an $\mathcal{L}'$- accessible subdomain of $g\,\mathbf{U}$ by Lemma 4.1.12(*ii*). Now, since all morphisms of the lower square are linear maps between coadmissible modules, it is enough to show that the diagram

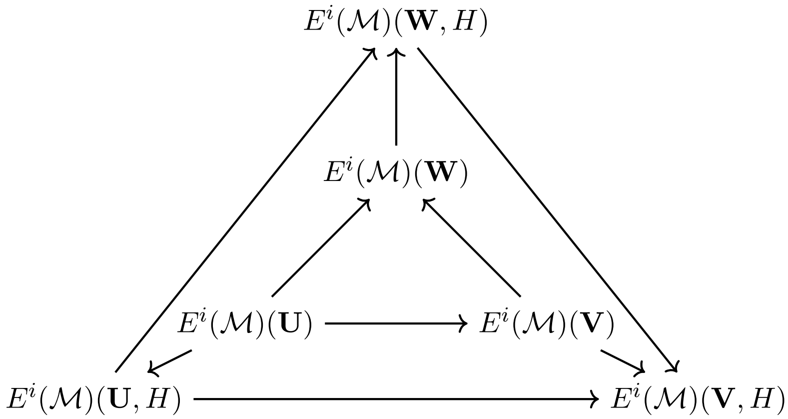
$$\begin{array}{ccc} Ext^i_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{M}_n(\mathbf{U}), \mathcal{Q}_n(\mathbf{U},H)) & \longrightarrow & Ext^i_{\mathcal{Q}_n(g\,\mathbf{U},^gN)}\mathcal{M}_n(g\,\mathbf{U}), \mathcal{Q}_n(gU, {}^gH)) \\ \downarrow & & \downarrow \\ Ext^i_{\mathcal{Q}_n(\mathbf{V},H)}(\mathcal{M}_n(\mathbf{V}), \mathcal{Q}_n(\mathbf{V},H)) & \longrightarrow & Ext^i_{\mathcal{Q}_n(g\,\mathbf{V},^gH)},(\mathcal{M}_n(g\,\mathbf{V}), \mathcal{Q}_n(g\,\mathbf{V}, {}^gH)) \end{array}$$

is commutative. This is indeed Proposition 4.2.15(2).

□

Theorem 4.2.19. *Let $\mathbf{X}$ be a smooth rigid analytic space and G be a p-adic Lie group acting continuously on $\mathbf{X}$. Let $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$, then for all $i \geq 0$, $E^i(\mathcal{M})$ is a G-equivariant presheaf of right $\mathcal{D}_{\mathbf{X}}$-modules on $\mathbf{X}_w(\mathcal{T})$.*

Proof. Let $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$ be affinoid subdomains of $\mathbf{X}$ in $\mathbf{X}_w(\mathcal{T})$. By [4, Lemma 3.4.7] there exists an open compact subgroup $H \leq G$ such that the pairs $(\mathbf{W}, H), (\mathbf{V}, H), (\mathbf{U}, H)$ are all small. Then we consider the following diagram:



The three quadrilaterals are commutative by definition. The outer triangle is commutative by Proposition 4.2.9 and the three arrows connecting the two triangles are bjections by Remark 4.2.7. Hence the inner triangle is commutative and this proves that $E^i(\mathcal{M})$ is a presheaf.

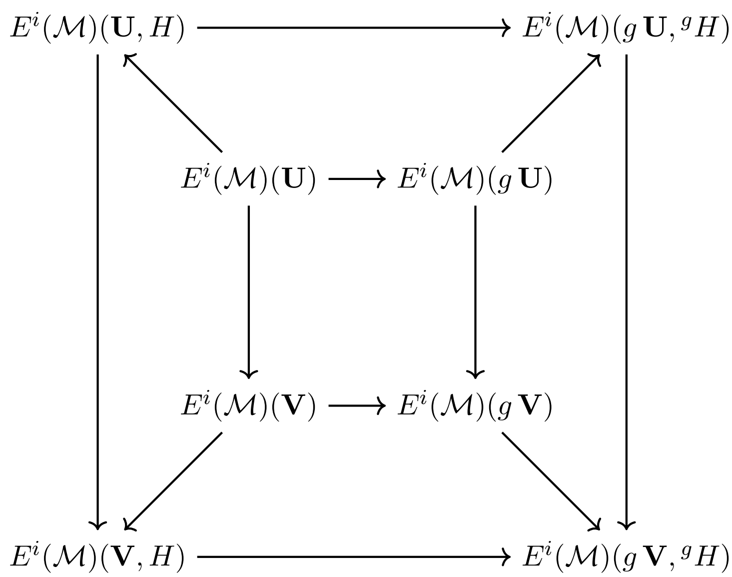
Next, fix $g \in G$ and $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$. We define

$$g^{E^i(\mathcal{M})}(\mathbf{U}): \; E^i(\mathcal{M})(\mathbf{U}) \longrightarrow E^i(\mathcal{M})(g\,\mathbf{U})$$

to be the inverse limit of the maps $g^{E^i(\mathcal{M})}_{\mathbf{U},H}$ in Proposition 4.2.14. Then

⋆ By (4.5) (Proposition 4.2.14), it is straightforward to see that $g^{E^i(\mathcal{M})}(m.a) = g^{E^i(\mathcal{M})}(m).g^{\mathcal{D}}(a)$ for any $a \in \mathcal{D}(\mathbf{U})$ and $m \in E^i(\mathcal{M})(\mathbf{U})$.

⋆ If $\mathbf{V} \subset \mathbf{U}$ are in $\mathbf{X}_w(\mathcal{T})$, then there exists a $\mathbf{U}$-small subgroup H of $G_{\mathbf{U}} \cap G_{\mathbf{V}}$ such that we have the following diagram:



Note that the outer square is commutative by Proposition 4.2.18, the four trapezia are commutative by definition and the arrows connecting the two squares are bijections. This proves that the inner square is commutative. Hence $g^{E^i(\mathcal{M})}: \; E^i(\mathcal{M}) \longrightarrow g^*(E^i(\mathcal{M}))$ is a morphism of presheaves on $\mathbf{X}_w(\mathcal{T})$.

⋆ Finally, if $g, h \in G$, we need to show that $(gh)^{E^i(\mathcal{M})} = g^{E^i(\mathcal{M})} \circ h^{E^i(\mathcal{M})}$. By taking a free resolution of $\mathcal{M}(\mathbf{U})$ by free $\widehat{\mathcal{D}}(\mathbf{U}, H)$-modules, it is enough to show that for any $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module P, the diagram

$$\begin{array}{ccc} Hom_{\widehat{\mathcal{D}}(\mathbf{U},H)}(P, \widehat{\mathcal{D}}(\mathbf{U}, H)) & \xrightarrow{\phi^h_{\mathbf{U},H}} & Hom_{\widehat{\mathcal{D}}(h\,\mathbf{U},hHh^{-1})}({}^hP, \widehat{\mathcal{D}}(h\,\mathbf{U}, hHh^{-1})) \\ & \searrow^{\phi^{gh}_{\mathbf{U},H}} & \downarrow{\scriptstyle \phi^g_{h\,\mathbf{U},{}^hH}} \\ & & Hom_{\widehat{\mathcal{D}}(gh\,\mathbf{U},ghHh^{-1}g^{-1})}({}^{gh}P, \widehat{\mathcal{D}}(gh\,\mathbf{U}, ghHh^{-1}g^{-1})) \end{array}$$

is commutative. Let $f \in Hom_{\widehat{\mathcal{D}}(\mathbf{U},H)}(P, \widehat{\mathcal{D}}(\mathbf{U}, H))$, then

$$\phi^g_{h\,\mathbf{U},{}^hH} \circ \phi^h_{\mathbf{U},H}(f) = \phi^g_{h\,\mathbf{U},{}^hH}(\widehat{h}_{\mathbf{U},H} \circ f) = \widehat{g}_{h\,\mathbf{U},{}^hH} \circ \widehat{h}_{\mathbf{U},H} \circ f$$

while $\phi^{gh}_{\mathbf{U},H} = \widehat{gh}_{\mathbf{U},H} \circ f$. Hence the commutativity of the diagram follows from the equality $\widehat{gh}_{\mathbf{U},H} = \widehat{g}_{g\,\mathbf{U},{}^hH} \circ \widehat{h}_{\mathbf{U},H}$, which is from [4, Lemma 3.4.3]

□

In the last part of this section, we intend to prove that for any $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$, the presheaf $E^i(\mathcal{M})$ on $\mathbf{X}_w(\mathcal{T})$ is in fact a sheaf and $E^i(\mathcal{M})$ can be therefore extended to a G-equivariant sheaf of right $\mathcal{D}_{\mathbf{X}}$-modules on $\mathbf{X}$. It then turns out that this sheaf in fact defines an object in $\mathcal{C}^r_{\mathbf{X}/G}$.

We first assume that $(\mathbf{X}, G)$ is small and let $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$ be a sheaf of coadmissible G-equivariant left $\mathcal{D}_{\mathbf{X}}$-modules.

Lemma 4.2.20. *Let $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and H be a $\mathbf{U}$-small subgroup of G. Then there is an isomorphism of right $\widehat{\mathcal{D}}(\mathbf{U}, H)$-modules*

$$\Phi^i_{U,H}: \; Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}), \widehat{\mathcal{D}}(\mathbf{X}, G))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\widehat{\mathcal{D}}(\mathbf{U}, H) \xrightarrow{\sim} Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, H)).$$

Proof. Recall that $\mathcal{M} \cong Loc_{\mathbf{X}}(\mathcal{M}(\mathbf{X}))$, so that

$$\mathcal{M}(\mathbf{U}) \simeq \widehat{\mathcal{D}}(\mathbf{U}, H)\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\mathcal{M}(\mathbf{X}).$$

By applying Proposition 4.2.1, we obtain an isomorphism of right $\widehat{\mathcal{D}}(\mathbf{X}, H)$-modules

$$\widehat{p}^i_{G,H}(X): Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}), \widehat{\mathcal{D}}(\mathbf{X}, G)) \xrightarrow{\sim} Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},H)}(\mathcal{M}(\mathbf{X}), \widehat{\mathcal{D}}(\mathbf{X}, H)). \tag{4.9}$$

Hence

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}), \widehat{\mathcal{D}}(\mathbf{X}, G))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\widehat{\mathcal{D}}(\mathbf{U}, H) \xrightarrow{\sim} Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},H)}(\mathcal{M}(\mathbf{X}), \widehat{\mathcal{D}}(\mathbf{X}, H))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\widehat{\mathcal{D}}(\mathbf{U}, H). \tag{4.10}$$

Finally, apply Lemma 4.2.8 gives:

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},H)}(\mathcal{M}(\mathbf{X}), \widehat{\mathcal{D}}(\mathbf{X}, H))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\widehat{\mathcal{D}}(\mathbf{U}, H) \xrightarrow{\sim} Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, H)). \tag{4.11}$$

□

Let us explain how the isomorphism in the above lemma looks like when $i = 0$ and H is open normal in G. Consider the morphism ($\Phi_{\mathbf{U},H} := \Phi^0_{\mathbf{U},H}$)

$$\Phi_{\mathbf{U},H}: \ Hom_{\widehat{\mathcal{D}}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}), \widehat{\mathcal{D}}(\mathbf{X},G))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\widehat{\mathcal{D}}(\mathbf{U},H) \xrightarrow{\sim} Hom_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U},H)).$$

Write $M := \mathcal{M}(\mathbf{X})$. Let us choose a G-stable Lie lattice $\mathcal{L}$ of $\mathcal{T}(\mathbf{X})$ such that $\mathbf{U}$ is $\mathcal{L}$-accessible and a good chain (J_n) for $\mathcal{L}$. Then we can take the sheaves $\mathcal{Q}_n$ into account and obtain that:

$$\widehat{\mathcal{D}}(\mathbf{X},G) = \varprojlim_n \mathcal{Q}_n(\mathbf{X},G),\ \widehat{\mathcal{D}}(\mathbf{X},H) = \varprojlim_n \mathcal{Q}_n(\mathbf{X},H) \text{ and } \widehat{\mathcal{D}}(\mathbf{U},H) = \varprojlim_n \mathcal{Q}_n(\mathbf{U},H).$$

The morphism $\Phi_{\mathbf{U},H}$ is defined as the inverse limit of an inverse system $(\Phi_{\mathbf{U},H,n})_n$ of morphisms

$$\Phi_{\mathbf{U},H,n}: Hom_{\mathcal{Q}_n(\mathbf{X},G)}(M_n, \mathcal{Q}_n(\mathbf{X},G))\otimes_{\mathcal{Q}_n(\mathbf{X},H)}\mathcal{Q}_n(\mathbf{U},H) \xrightarrow{\sim} Hom_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{M}_n(\mathbf{U}), \mathcal{Q}_n(\mathbf{U},H))$$

-with $\mathcal{M}_n(\mathbf{U}) = \mathcal{Q}_n(\mathbf{U},H)\otimes_{\mathcal{Q}_n(\mathbf{X},H)} M_n$, which is defined as follows. If $f_n: M_n \longrightarrow \mathcal{Q}_n(\mathbf{X},G)$ is a $\mathcal{Q}_n(\mathbf{X},G)$-linear morphism and $a \in \mathcal{Q}_n(\mathbf{U},H)$, then applying (4.9), we obtain the $\mathcal{Q}_n(\mathbf{X},H)$-linear morphism

$$p_{G,H,n}\circ f_n: \ M_n \longrightarrow \mathcal{Q}_n(\mathbf{X},H).$$

Next, $(p_{G,H,n}\circ f_n)\otimes a$ is the image of $f_n \otimes a$ via the isomorphism (4.10). Finally, by applying the isomorphism (4.11), we get the map

$$\begin{aligned} 1\bar{\otimes}((p_{G,H,n}\circ f_n).a): \ \mathcal{Q}_n(\mathbf{U},H)\otimes M_n &\longrightarrow Q_n(\mathbf{U},H) \\ b\otimes m &\longmapsto b.p_{G,H}(f_n(m)).a. \end{aligned}$$

Note that in the above formula, we identify $p_{G,H,n}(f_n(m)) \in \mathcal{Q}_n(\mathbf{X},H)$ with its image in $\mathcal{Q}_n(\mathbf{U},H)$ via the canonical morphism $\mathcal{Q}_n(\mathbf{X},H) \longrightarrow \mathcal{Q}_n(\mathbf{U},H)$. Therefore

$$\Phi_{U,H,n}(f_n) = id\bar{\otimes}((p_{G,H,n}\circ f_n).a) \in Hom_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{M}_n(\mathbf{U}), \mathcal{Q}_n(\mathbf{U},H)). \tag{4.12}$$

Recall that ${}^rLoc_{\mathbf{X}}(-)$ denotes the localisation functor on the category $\mathcal{C}^r_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ of coadmissible right $\widehat{\mathcal{D}}(\mathbf{X},G)$-modules.

Proposition 4.2.21. *Suppose that $(\mathbf{X},G)$ is small. There is an isomorphism of presheaves of right $\mathcal{D}_{\mathbf{X}}$-modules on $\mathbf{X}_w(\mathcal{T})$*

$$\Phi: \ {}^rLoc_{\mathbf{X}}(Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}), \widehat{\mathcal{D}}(\mathbf{X},G)) \xrightarrow{\sim} E^i(\mathcal{M}).$$

Proof. Write $M := \mathcal{M}(\mathbf{X})$ and fix an open affinoid subset $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$. By Lemma 4.2.20, for any $\mathbf{U}$-small subgroup of G, there is an isomorphism of right $\widehat{\mathcal{D}}(\mathbf{U},H)$-modules

$$\Phi^i_{U,H}: \ Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}), \widehat{\mathcal{D}}(\mathbf{X},G))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\widehat{\mathcal{D}}(\mathbf{U},H) \xrightarrow{\sim} Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U},H)).$$

If $H' \leq H$ is another $\mathbf{U}$-small subgroup of G, we need to show that

$$\begin{array}{ccc} Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X},G))\widehat{\otimes}_{\widehat{\mathcal{D}}(X,H')}\widehat{\mathcal{D}}(U,H') & \xrightarrow{\Phi^i_{U,H'}} & Ext^i_{\widehat{\mathcal{D}}(U,H')}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(U,H')) \\ \downarrow & & \downarrow \widehat{p}^i_{H',H} \\ Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X},G))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\widehat{\mathcal{D}}(\mathbf{U},H) & \xrightarrow{\Phi^i_{U,H}} & Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U},H)) \end{array} \tag{4.13}$$

is commutative. It suffices to assume that H' and H are normal in G. Then $\widehat{p}^i_{H'H}$ is the inverse of the map $\widehat{p}^i_{H,H'}$ (which is defined in Proposition 4.1.3), it is equivalent to show that the diagram

$$\begin{array}{ccc} Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M,\widehat{\mathcal{D}}(\mathbf{X},G))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H')}\widehat{\mathcal{D}}(\mathbf{U},H') & \longrightarrow & Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H')}(\mathcal{M}(\mathbf{U}),\widehat{\mathcal{D}}(\mathbf{U},H')) \\ \downarrow & & \uparrow \\ Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M,\widehat{\mathcal{D}}(\mathbf{X},G))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\widehat{\mathcal{D}}(\mathbf{U},H) & \longrightarrow & Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}),\widehat{\mathcal{D}}(\mathbf{U},H)) \end{array}$$

is commutative.
Fix a G-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $\mathcal{T}(\mathbf{X})$ for some G-stable affine formal model $\mathcal{A}$ of $\mathcal{O}(\mathbf{X})$ and a good chain (J_n) for $\mathcal{L}$. By rescaling $\mathcal{L}$ if necessary, we may suppose that $\mathbf{U}$ is $\mathcal{L}$-accessible. Recall the sheaves $\mathcal{Q}_n$ and $\mathcal{M}_n$ in 4.2, 4.6, and 4.7. Then

$$\widehat{\mathcal{D}}(\mathbf{X},G) = \varprojlim_n \mathcal{Q}_n(\mathbf{X},G),\ \widehat{\mathcal{D}}(\mathbf{U},H) = \varprojlim_n \mathcal{Q}_n(\mathbf{U},H) \text{ and } \widehat{\mathcal{D}}(\mathbf{U},H') = \varprojlim_n \mathcal{Q}_n(\mathbf{U},H').$$

Thus $M \cong \varprojlim_n M_n$, with $M_n := \mathcal{Q}_n(\mathbf{X},G) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} M$. Since the morphisms in the above square are linear between coadmissible modules, it is enough to prove that the diagram

$$\begin{array}{ccc} Ext^i_{\mathcal{Q}_n(\mathbf{X},G)}(M_n,\mathcal{Q}_n(\mathbf{X},G)) \otimes_{\mathcal{Q}_n(X,H')} \mathcal{Q}_n(U,H') & \longrightarrow & Ext^i_{\mathcal{Q}_n(\mathbf{U},H')}(\mathcal{M}_n(\mathbf{U}),\mathcal{Q}_n(\mathbf{U},H')) \\ \downarrow & & \uparrow \\ Ext^i_{\mathcal{Q}_n(\mathbf{X},G)}(M_n,\mathcal{Q}_n(\mathbf{X},G)) \otimes_{\mathcal{Q}_n(\mathbf{X},H)} \mathcal{Q}_n(\mathbf{U},H) & \longrightarrow & Ext^i_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{M}_n(\mathbf{U}),\mathcal{Q}_n(\mathbf{U},H)) \end{array}$$

is commutative.
Now, by taking a free resolution of M_n as a $\mathcal{Q}_n(\mathbf{X},G)$-module and by using the flatness of the morphisms $\mathcal{Q}_n(\mathbf{X},H') \longrightarrow \mathcal{Q}_n(\mathbf{U},H')$ and $\mathcal{Q}_n(\mathbf{X},H) \longrightarrow \mathcal{Q}_n(\mathbf{U},H)$ (Proposition 2.4.17), it remains to prove that, for any $\mathcal{Q}_n(\mathbf{X},G)$-module P, the diagram

$$\begin{array}{ccc} Hom_{\mathcal{Q}_n(\mathbf{X},G)}(P,\mathcal{Q}_n(\mathbf{X},G)) \otimes \mathcal{Q}_n(U,H') & \longrightarrow & Hom_{\mathcal{Q}_n(\mathbf{U},H')}(\mathcal{Q}_n(\mathbf{U},H') \otimes P,\mathcal{Q}_n(\mathbf{U},H')) \\ \downarrow & & \uparrow \\ Hom_{\mathcal{Q}_n(\mathbf{X},G)}(P,\mathcal{Q}_n(\mathbf{X},G)) \otimes \mathcal{Q}_n(\mathbf{U},H) & \longrightarrow & Hom_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{Q}_n(\mathbf{U},H) \otimes P,\mathcal{Q}_n(\mathbf{U},H)) \end{array}$$

is commutative.
Let $f \in Hom_{\mathcal{Q}_n(\mathbf{X},G)}(P,\mathcal{Q}_n(\mathbf{X},G))$ and $a \in \mathcal{Q}_n(\mathbf{U},H)$, then we need to show that:

$$p_{H,H',n} \circ (1\bar{\otimes}(p_{G,H,n} \circ f)i(a)) = 1\bar{\otimes}((p_{G,H',n} \circ f)a). \tag{4.14}$$

Where, $i : \mathcal{Q}_n(\mathbf{U},H') \longrightarrow \mathcal{Q}_n(\mathbf{U},H)$ is the natural inclusion. Let $b \in \mathcal{Q}_n(\mathbf{U},H')$ and $m \in P$, then we compute by using (4.12):

$$\begin{aligned} &p_{H,H',n} \circ (1\bar{\otimes}(p_{G,H,n} \circ f)i(a))(b \otimes m) = p_{H,H',n}(bp_{G,H,n}(f(m))i(a)) \\ &= bp_{H,H',n}(p_{G,H,n}(f(m)))a = bp_{H,H',n} \circ p_{G,H,n}(f(m))a = bp_{G,H',n}(f(m))a. \end{aligned}$$

Thus, the equality 4.14 is proved and so the commutativity of the diagram 4.13 follows. As a consequence of this, by taking the inverse limit of the maps $\Phi^i_{U,H}$, we obtain a right $\mathcal{D}(U)$-linear isomorphism

$$\Phi^i(\mathbf{U}) :\ {}^r Loc_{\mathbf{X}}(Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(\mathcal{M}(\mathbf{X}),\widehat{\mathcal{D}}(\mathbf{X},G))(\mathbf{U}) \xrightarrow{\sim} E^i(\mathcal{M})(\mathbf{U}).$$

Finally, Φ^i being a morphism of presheaves amounts to showing that if $\mathbf{V} \subset \mathbf{U}$ are open subsets in $\mathbf{X}_w(\mathcal{T})$ and H is an open normal subgroup of G which stabilizes $\mathbf{U}$ and $\mathbf{V}$, the following diagram is commutative:

$$\begin{array}{ccc} Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X},G))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\widehat{\mathcal{D}}(\mathbf{U},H) & \xrightarrow{\phi^i_{U,H}} & Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U},H)) \\ \downarrow & & \downarrow \\ Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X},G))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)}\widehat{\mathcal{D}}(V,H) & \xrightarrow{\phi^i_{\mathbf{V},H}} & Ext^i_{\widehat{\mathcal{D}}(\mathbf{V},H)}(\mathcal{M}(\mathbf{V}), \widehat{\mathcal{D}}(\mathbf{V},H)). \end{array}$$

This is indeed a consequence of Proposition 4.2.9, where it is proved that:

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{V},H)}(\mathcal{M}(\mathbf{V}), \widehat{\mathcal{D}}(\mathbf{V},H)) \cong Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U},H))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},H)}\widehat{\mathcal{D}}(\mathbf{V},H).$$

□

Corollary 4.2.22. *Let $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$ be a sheaf of coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-module on $\mathbf{X}$. The presheaf $E^i(\mathcal{M})$ is a sheaf on the basis $\mathbf{X}_w(\mathcal{T})$ of the Grothendieck topology on $\mathbf{X}$. In particular, this can be extended to a sheaf on $\mathbf{X}_{rig}$, which is still denoted by $E^i(\mathcal{M})$.*

Proof. Fix $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and let H be a $\mathbf{U}$-small open subgroup of G. Then following Proposition 4.2.21

$$E^i(\mathcal{M})|_{\mathbf{U}} \simeq {}^r Loc_{\mathbf{U}}(Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U},H)).$$

Since the right hand side is a sheaf on $\mathbf{U}_w(\mathcal{T}|_{\mathbf{U}})$, one has that $E^i(\mathcal{M}) \mid_{\mathbf{U}}$ is also a sheaf on $\mathbf{U}_w(\mathcal{T}|_{\mathbf{U}})$. It follows that the presheaf $E^i(\mathcal{M})$ is actually a sheaf on $\mathbf{X}_w(\mathcal{T})$ as claimed. □

Theorem 4.2.23. *Let $\mathcal{M}$ be a coadmissible G-equivariant left $\mathcal{D}_X$-module Then $E^i(\mathcal{M})$ is a coadmissible G-equivariant right $\mathcal{D}_{\mathbf{X}}$-module for every $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$ and all $i \geq 0$.*

Proof. First, let us show that $E^i(\mathcal{M})$ is a sheaf of G- equivariant locally Fréchet right $\mathcal{D}_{\mathbf{X}}$-modules. Let $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and H be a $\mathbf{U}$-small subgroup of G. Then the bijection

$$E^i(\mathcal{M})(\mathbf{U}) \simeq E^i(\mathcal{M})(\mathbf{U}, H) = Ext^i(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U},H))$$

from Remark 4.2.7 tells us that $E^i(\mathcal{M})(\mathbf{U})$ can be equipped with a canonical Fréchet topology transferred from the canonical topology on $Ext^i(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U},H))$. This topology does not depend on the choice of H, so that $E^i(\mathcal{M})(\mathbf{U})$ becomes a coadmissible (right) $\widehat{\mathcal{D}}(\mathbf{U},H)$-module. It remains to check that if $g \in G$ then each map $g^{E^i(\mathcal{M})}(\mathbf{U}) : E^i(\mathcal{M})(\mathbf{U}) \longrightarrow E^i(\mathcal{M})(g\,\mathbf{U})$ is continuous for any $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$. Indeed, note that the map $g^{E^i(\mathcal{M})}(\mathbf{U})$ is a linear isomorphism with respect to the K- algebras isomorphism $\widehat{g}_{\mathbf{U},H} : \widehat{\mathcal{D}}(\mathbf{U},H) \longrightarrow \widehat{\mathcal{D}}(g\,\mathbf{U}, gHg^{-1})$. We obtain that $g^{E^i(\mathcal{M})}(\mathbf{U})$ is continuous by [4, Lemma 3.6.5]. Thus $E^i(\mathcal{M})$ is in $Frech^r(G - \mathcal{D}_{\mathbf{X}})$.

Next, write $M := \mathcal{M}(\mathbf{X})$. In view of Theorem 4.2.19, Proposition 4.2.21 and Corollary 4.2.22, it remains to prove that when $(\mathbf{X}, G)$ is small, the morphism

$$\Phi^i : \ {}^r Loc_{\mathbf{X}}(Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X},G)) \longrightarrow E^i(\mathcal{M})$$

is indeed a G-equivariant morphism.
In the sequel, to simplify the notations, we write

$$\mathcal{N} := E^i(\mathcal{M}) \text{ and } \mathcal{N}' := {}^r Loc_{\mathbf{X}}(Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X},G)).$$

Let $U \in \mathbf{X}_w(\mathcal{T})$ and $g \in G$. Then by definition, $\Phi^i(\mathbf{U}) = \varprojlim_H \Phi^i_{\mathbf{U},H}$, it reduces to prove that for any $\mathbf{U}$-small subgroup H of G which is normal, the diagram

$$\begin{array}{ccc} Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X},G)) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)} \widehat{\mathcal{D}}(\mathbf{U},H) & \xrightarrow{g^{\mathcal{N}'}_{\mathbf{U},H}} & Ext^i_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X},G)) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},{}^gH)} \widehat{\mathcal{D}}(g\,\mathbf{U}, {}^gH) \\ \Phi^i_{\mathbf{U},H}\downarrow & & \downarrow \Phi^i_{g\,\mathbf{U},{}^gH} \\ Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\widehat{\mathcal{D}}(\mathbf{U},H) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)} M, \widehat{\mathcal{D}}(\mathbf{U},H)) & \xrightarrow{g^{\mathcal{N}}_{\mathbf{U},H}} & Ext^i_{\widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH)}(\widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},{}^gH)} M, \widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH)) \end{array}$$

is commutative. Here recall that $g^{\mathcal{N}}_{\mathbf{U},H}$ (resp. $g^{\mathcal{N}'}_{\mathbf{U},H}$) corresponds to the G-equivariant structure on the sheaf $\mathcal{N}$(resp. $\mathcal{N}'$).
Choose a Lie lattice $\mathcal{L}$ in $\mathcal{T}(\mathbf{X})$ and a good chain (J_n) for $\mathcal{L}$ such that

$$\widehat{\mathcal{D}}(\mathbf{X},G) = \varprojlim_n \mathcal{Q}_n(\mathbf{X},G).$$

By rescaling $\mathcal{L}$, we may suppose that $\mathbf{U}$ is $\mathcal{L}$-accessible. This implies

$$\widehat{\mathcal{D}}(\mathbf{U},H) = \varprojlim_n \mathcal{Q}_n(\mathbf{U},H).$$

Now, following Lemma 4.2.12, $g\,\mathbf{U}$ is also $\mathcal{L}'$-accessible with $\mathcal{L}' = g^{\mathcal{T}}(\mathcal{L})$. Thus $\mathcal{L}'$ together with the good chain (gJ_ng^{-1}) defines the Frechet-Stein structures

$$\widehat{\mathcal{D}}(\mathbf{X},{}^gH) = \varprojlim_n \mathcal{Q}_n(\mathbf{X},{}^gH) \text{ and } \widehat{\mathcal{D}}(g\,\mathbf{U},{}^gH) = \varprojlim_n \mathcal{Q}_n(g\,\mathbf{U},{}^gH).$$

Since each map of the above diagram is a linear map between coadmissible modules, they can be regarded as the inverse limits of systems of morphisms:

$$\Phi^i_{\mathbf{U},H} = \varprojlim_n \Phi^i_{\mathbf{U},H,n}, \quad \Phi^i_{g\,\mathbf{U},{}^gH} = \varprojlim_n \Phi^i_{g\,\mathbf{U},{}^gH,n}$$
$$g^{\mathcal{N}}_{\mathbf{U},H} = \varprojlim_n g^{\mathcal{N}}_{\mathbf{U},H,n} \; g^{\mathcal{N}'}_{\mathbf{U},H} = \varprojlim_n g^{\mathcal{N}'}_{\mathbf{U},H,n}.$$

As a consequence, it is enough to prove that the diagram

$$\begin{array}{ccc} Ext^i_{\mathcal{Q}_n(\mathbf{X},G)}(M_n, \mathcal{Q}_n(\mathbf{X},G)) \otimes \mathcal{Q}_n(\mathbf{U},H) & \xrightarrow{g^{\mathcal{N}'}_{\mathbf{U},H,n}} & Ext^i_{\mathcal{Q}_n(\mathbf{X},G)}(M_n, \mathcal{Q}_n(\mathbf{X},G)) \otimes \mathcal{Q}_n(g\,\mathbf{U},{}^gH) \\ \Phi^i_{\mathbf{U},H,n}\downarrow & & \downarrow \Phi^i_{g\,\mathbf{U},{}^gH,n} \\ Ext^i_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{Q}_n(\mathbf{U},H) \otimes M_n, \mathcal{Q}_n(\mathbf{U},H)) & \xrightarrow{g^{\mathcal{N}}_{\mathbf{U},H,n}} & Ext^i_{\mathcal{Q}_n(g\,\mathbf{U},{}^gH)}(\mathcal{Q}_n(g\,\mathbf{U},{}^gH) \otimes M_n, \mathcal{Q}_n(g\,\mathbf{U},{}^gH)) \end{array}$$

is commutative. Here we assume that $M = \varprojlim_n M_n$, with respect to the given Frechet-Stein structure on $\widehat{\mathcal{D}}(\mathbf{X},G)$. After taking a resolution of M_n by free $\mathcal{Q}_n(\mathbf{X},G)$-modules, it amounts to proving the commutativity of the above diagram for the case $i = 0$, which means that the following diagram is commutative:

$$\begin{array}{ccc} Hom_{\mathcal{Q}_n(\mathbf{X},G)}(M_n, \mathcal{Q}_n(\mathbf{X},G)) \otimes \mathcal{Q}_n(\mathbf{U},H) & \xrightarrow{g^{\mathcal{N}'}_{U,H,n}} & Hom_{\mathcal{Q}_n(\mathbf{X},G)}(M_n, \mathcal{Q}_n(\mathbf{X},G)) \otimes \mathcal{Q}_n(gU,{}^gH) \\ \Phi_{U,H,n}\big\downarrow & & \big\downarrow \Phi_{gU,{}^gH,n} \\ Hom_{\mathcal{Q}_n(\mathbf{U},H)}(\mathcal{Q}_n(\mathbf{U},H)\otimes M_n, \mathcal{Q}_n(\mathbf{U},H)) & \xrightarrow{g^{\mathcal{N}}_{U,H,n}} & Hom_{\mathcal{Q}_n(gU,{}^gH)}(\mathcal{Q}_n(gU,{}^gH)\otimes M_n, \mathcal{Q}_n(gU,{}^gH)) \end{array}$$

Let $f \in Hom_{\mathcal{Q}_n(\mathbf{X},G)}(M_n, \mathcal{Q}_n(\mathbf{X},G))$ and $a \in \mathcal{Q}_n(\mathbf{U},H)$. It is enough to show that:

$$\Phi_{gU,{}^gH}\left(g^{\mathcal{N}'}_{\mathbf{U},H,n}(f\otimes a)\right) = g^{\mathcal{N}}_{\mathbf{U},H,n}\left(\Phi_{\mathbf{U},H,n}(f\otimes a)\right)$$

Since $g^{\mathcal{N}'}_{\mathbf{U},H,n}(f\otimes a) = (f\gamma_n(g)).g^{\mathcal{Q}_n}_{\mathbf{U},H}(a)$ and $\Phi_{\mathbf{U},H,n}(f\otimes a) = 1\bar{\otimes}(p_{G,H,n}(f)).a$, it is equivalent to show that:

$$1\bar{\otimes}(p_{G,{}^gH,n}((f\gamma_n(g^{-1})).g^{\mathcal{Q}_n}_{\mathbf{U},H}(a)) = g^{\mathcal{Q}_n}_{\mathbf{U},H}\circ(1\bar{\otimes}(p_{G,H,n}(f)).a)\circ(g^{-1})^{\mathcal{M}_n}_{\mathbf{U},H}$$

where

$$\gamma_n : G \longrightarrow \mathcal{Q}_n(\mathbf{X},G)^\times = \left(\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G\right)^\times$$

is the canonical group homomorphism from Remark 2.2.17.
Let $m \in M_n, b \in \mathcal{Q}_n(g\,\mathbf{U},{}^gH)$, we compute

$$(1\bar{\otimes}(p_{G,{}^gH,n}\circ(f\gamma_n(g^{-1})).g^{\mathcal{Q}_n}_{\mathbf{U},H}(a)))(b\otimes m) = b.\,p_{G,{}^gH,n}(f(m)\gamma_n(g^{-1}))g^{\mathcal{Q}_n}_{\mathbf{U},H}(a)$$

and

$$(g^{\mathcal{Q}_n}_{\mathbf{U},H,}\circ(1\bar{\otimes}(p_{G,H,n}\circ f).a)\circ(g^{-1})^{\mathcal{M}_n}_{\mathbf{U},H})(b\otimes m) = g^{\mathcal{Q}_n}_{\mathbf{U},H}\circ(1\bar{\otimes}(p_{G,H,n}\circ f).a)(g^{-1}{}^{\mathcal{Q}_n}_{\mathbf{U},H}(b)\otimes\gamma_n(g^{-1})m)$$
$$= g^{\mathcal{Q}_n}_{\mathbf{U},H}(g^{-1}{}^{\mathcal{Q}_n}_{\mathbf{U},H}(b)).g^{\mathcal{Q}_n}_{\mathbf{U},H}(p_{G,H,n}(f(\gamma_n(g^{-1})m)a) = b.g^{\mathcal{Q}_n}_{\mathbf{U},H}(p_{G,H,n}(f(\gamma_n(g^{-1})m))g^{\mathcal{Q}_n}_{\mathbf{U},H}(a).$$

Here, we identify the element $p_{G,H,n}(f(\gamma_n(g^{-1})m) \in \mathcal{Q}_n(\mathbf{X},H)$ with its image in $\mathcal{Q}_n(\mathbf{U},H)$ via the natural restriction $\mathcal{Q}_n(\mathbf{X},H) \longrightarrow \mathcal{Q}_n(\mathbf{U},H)$ and the element $p_{G,{}^gH,n}(f(m)\gamma_n(g^{-1})) \in \mathcal{Q}_n(\mathbf{X},{}^gH)$ with its image in $\mathcal{Q}_n(g\,\mathbf{U},{}^gH)$ via $\mathcal{Q}_n(\mathbf{X},{}^gH) \longrightarrow \mathcal{Q}_n(g\,\mathbf{U},{}^gH)$. Thus, it remains to show that for any $m \in M_n$, one has

$$p_{G,{}^gH,n}(f(m)\gamma_n(g^{-1})) = g^{\mathcal{Q}_n}_{\mathbf{U},H}(p_{G,H,n}(\gamma_n(g^{-1})f(m)). \tag{4.15}$$

Consider the following diagram:

$$\begin{array}{ccc} \mathcal{Q}_n(\mathbf{X},G) & \xrightarrow{Ad_{\gamma_n(g)}} & \mathcal{Q}_n(\mathbf{X},G) \\ p_{G,H,n}\big\downarrow & & \big\downarrow p_{G,gHg^{-1},n} \\ \mathcal{Q}_n(\mathbf{X},H) & \xrightarrow{g^{\mathcal{Q}_n}_{X,H}} & \mathcal{Q}_n(\mathbf{X},gHg^{-1}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{Q}_n(\mathbf{U},H) & \xrightarrow{g^{\mathcal{Q}_n}_{\mathbf{U},H}} & \mathcal{Q}_n(g\,\mathbf{U},gHg^{-1}). \end{array} \tag{4.16}$$

By [4, Definition 3.4.9(c) and Propostion 3.4.10], we see that $Ad_{\gamma_n(g)} = g^{\mathcal{Q}_n}_{X,H}$ on $\mathcal{Q}_n(\mathbf{X},H) \subset \mathcal{Q}_n(\mathbf{X},G)$ and the commutativity of the lower diagram of the diagram 4.16 follows from loc.cit. On the other hand, it is proved in the proof of Proposition 4.2.18 that the upper diagram of 4.16 is commutative. Hence we may compute as follows:

$$\begin{aligned} p_{G,{}^gH,n}(f(m)\gamma_n(g^{-1})) &= p_{G,{}^gH,n}(\gamma_n(g)\gamma_n(g^{-1})f(m)\gamma_n(g^{-1})) \\ &= p_{G,{}^gH,n}(g^{\mathcal{Q}_n}_{X,H}(\gamma_n(g^{-1})f(m))) \\ &= g^{\mathcal{Q}_n}_{\mathbf{U},H}(p_{G,H}(\gamma_n(g^{-1})f(m)). \end{aligned}$$

Hence we obtain the commutativity of 4.16 and so the theorem follows.

□

Definition 4.2.24. *Let $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$, then we define for any non-negative integer $i \geq 0$:*

$$\mathcal{E}^i(\mathcal{M}) := \mathcal{H}om_{\mathcal{O}_{\mathbf{X}}}(\Omega_{\mathbf{X}}, E^i(\mathcal{M}))$$

Proposition 4.2.25. *For every $i \geq 0$, $\mathcal{E}^i$ is an endofunctor on the category $\mathcal{C}_{\mathbf{X},G}$ of coadmissible G-equivariant left $\mathcal{D}_{\mathbf{X}}$-modules.*

Proof. Following Theorem 2.4.26 and Theorem 4.2.23, the sheaf $\mathcal{E}^i(\mathcal{M})$ is a coadmissible G-equivariant left $\mathcal{D}_{\mathbf{X}}$-module. Now if $f : \mathcal{M} \longrightarrow \mathcal{M}'$ is a morphism of coadmissible G-equivariant left $\mathcal{D}_{\mathbf{X}}$-modules, then for any $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and any $\mathbf{U}$-small subgroup H of G, it follows that the $\widehat{\mathcal{D}}(\mathbf{U}, H)$-linear map $f(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \longrightarrow \mathcal{M}'(\mathbf{U})$ induces a morphism

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}'(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, H)) \longrightarrow Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\mathcal{M}(\mathbf{U}), \widehat{\mathcal{D}}(\mathbf{U}, H)),$$

which is right $\widehat{\mathcal{D}}(\mathbf{U}, H)$-linear. Hence by [4, Lemma 3.6.5], this is a continuous map with respect to the natural Fréchet topologies on both sides. In this way we obtain a morphism of G-equivariant locally Fréchet $\mathcal{D}_{\mathbf{X}}$modules

$$E^i(f) : E^i(\mathcal{M}') \longrightarrow E^i(\mathcal{M}),$$

whose local sections are continuous. Now, if $g : \mathcal{M}' \longrightarrow \mathcal{M}''$ is another morphism in $\mathcal{C}_{\mathbf{X}/G}$, then it is straightforward to show that $E^i(id) = id$ and $E^i(g \circ f) = E^i(f) \circ E^i(g)$, which ensures that E^i is a functor from $\mathcal{C}_{\mathbf{X}/G}$ into $\mathcal{C}^r_{\mathbf{X}/G}$. Finally $\mathcal{E}^i$ is a composition of two functors, so it is a functor from $\mathcal{C}_{\mathbf{X}/G}$ into itself, as claimed. □

4.3 Equivariant weakly holonomic $\mathcal{D}$-modules

4.3.1 Dimension

In this section, we fix a smooth rigid analytic K-variety $\mathbf{X}$ of dimension d and a p-adic Lie group G acting continuously on $\mathbf{X}$. We are now ready to introduce the notion of dimension for coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules.

First, recall that the set $\mathbf{X}_w(\mathcal{T})$ is a basis for the Grothendieck topology on the rigid analytic space $\mathbf{X}$.

Definition 4.3.1. *Let $\mathcal{U}$ be an admissible covering of $\mathbf{X}$ by affinoid subdomains in $\mathbf{X}_w(\mathcal{T})$ and $\mathcal{M}$ be a coadmissible G-equivariant left $\mathcal{D}_{\mathbf{X}}$-module on $\mathbf{X}$. Then the dimension of $\mathcal{M}$ with respect to $\mathcal{U}$ is defined as follows:*

$$d_{\mathcal{U}}(\mathcal{M}) := \sup\left\{d(\mathcal{M}(\mathbf{U}))|\, \mathbf{U} \in \mathcal{U}\right\},$$

where $d(\mathcal{M}(\mathbf{U}))$ is the dimension of the coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module $\mathcal{M}(\mathbf{U})$ for some $\mathbf{U}$-small subgroup H of G.

Proposition 4.3.2. *Suppose that $\mathcal{U}$ and $\mathcal{V}$ are two admissible coverings of $\mathbf{X}$ by elements in $\mathbf{X}_w(\mathcal{T})$. Then $d_{\mathcal{U}}(\mathcal{M}) = d_{\mathcal{V}}(\mathcal{M})$.*

Proof. We may assume that $\mathcal{V}$ is a refinement of $\mathcal{U}$ and every element of $\mathcal{U}$ has an admissible covering by elements of $\mathcal{V}$. Let $\mathbf{U}_1, ..., \mathbf{U}_k \in \mathcal{V}$ be a cover of $\mathbf{U}_0 \in \mathbf{U}$ (which is quasi-compact!). We fix an open compact subgroup H of G such that $(\mathbf{U}_0, H)$ is small and choose a H-stable affine formal model $\mathcal{A}$ in $\mathcal{O}(\mathbf{U}_0)$ and a H-stable smooth $\mathcal{A}$-Lie lattice $\mathcal{L}$ in $\mathcal{T}(\mathbf{U}_0)$.Then by [4, Lemma 4.4.1], we may assume that H stabilises $\mathcal{A}$, $\mathcal{L}$ and each member $\mathbf{U}_i$ in $\mathcal{V}$. By replacing $\mathcal{L}$ by a sufficiently large π-power multiple, we may also assume that each $\mathbf{U}_i$ is a $\mathcal{L}$-accessible affinoid subspace in $\mathbf{U}_0$ so that $\mathbf{U}_1, ..., \mathbf{U}_k \in (\mathbf{U}_0)_{ac}(\mathcal{L}, H)$ and they form an $(\mathbf{U}_0)_{ac}(\mathcal{L}, H)$-covering. Recall the sheaf of rings $\mathcal{Q}_n(-, H)$ and the sheaf of modules $\mathcal{M}_n$ induced by $\mathcal{M}$ from Section 4.2. These are sheaves on the Grothendieck topology $\mathbf{X}_{ac}(\mathcal{L}, H)$. Then

$$\widehat{\mathcal{D}}(\mathbf{U}_i, H) \simeq \varprojlim_n \mathcal{Q}_n(\mathbf{U}_i, H) \quad \text{and} \quad \mathcal{M}(\mathbf{U}_i) \simeq \varprojlim_n \mathcal{M}_n(\mathbf{U}_i) \quad \text{for all } i = 0, 1, .., k.$$

Each $\mathcal{M}(\mathbf{U}_i)$ is a coadmissible $\widehat{\mathcal{D}}(\mathbf{U}_i, H)$-module and by Definition 3.2.2, $d(\mathcal{M}(\mathbf{U}_i)) = 2d - j_H(\mathcal{M}(\mathbf{U}_i))$ for each i.
Now by [4, Theorem 4.3.14], one has that $\bigoplus_{i=1}^k \mathcal{Q}_n(\mathbf{U}_i, H)$ is a faithfully flat right $\mathcal{Q}_n(\mathbf{U}_0, H)$-module. Thus applying [6, Proposition 7.5(c)] gives that $\bigoplus_{i=1}^k \widehat{\mathcal{D}}(\mathbf{U}_i, H)$ is c-faithfully flat over $\widehat{\mathcal{D}}(\mathbf{U}_0, H)$. On the other hand, the completed tensor product commutes with finite direct sum, so that:

$$\begin{aligned}
&Ext^i_{\widehat{\mathcal{D}}(\mathbf{U}_0,H)}(\mathcal{M}(\mathbf{U}_0), \widehat{\mathcal{D}}(\mathbf{U}_0, H))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U}_0,H)} \oplus_{i=1}^k \widehat{\mathcal{D}}(\mathbf{U}_i, H)\\
&\simeq \oplus_{i=1}^k Ext^i_{\widehat{\mathcal{D}}(\mathbf{U}_0,H)}(\mathcal{M}(\mathbf{U}_0), \widehat{\mathcal{D}}(\mathbf{U}_0, H))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U}_0,H)}\widehat{\mathcal{D}}(\mathbf{U}_i, H)\\
&\simeq \oplus_{i=1}^k Ext^i_{\widehat{\mathcal{D}}(\mathbf{U}_i,H)}(\mathcal{M}(\mathbf{U}_i), \widehat{\mathcal{D}}(\mathbf{U}_i, H)).
\end{aligned}$$

By consequence, one has

$$j_H(\mathcal{M}(\mathbf{U}_0)) = \inf\{j_H(\mathcal{M}(\mathbf{U}_i)) : \mathcal{M}(\mathbf{U}_i) \neq 0, i = 1, 2, ..., k\},$$

so the proposition follows immediately. □

Remark: The above proposition tells us that the dimension of a coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-module $\mathcal{M}$ does not depend on the choice of an admissible covering $\mathcal{U}$ of $\mathbf{X}$. Hence we can now ignore the symbol $\mathcal{U}$ and denote it simply by $d(\mathcal{M})$. By definition, $0 \leq d(\mathcal{M}) \leq 2d$.

4.3.2 Dimension and the pushforward functor

First, we recall some material from [5] which will be used later. Let I be an ideal in a commutative R-algebra A and L be a (R, A)-Lie algebra. Then we say that a finite set $\{x_1, ..., x_d\}$ of elements in L is an *I-standard basis* if it satisfies the following conditions:

(i) $\{x_1, ..., x_d\}$ is a basis of L as an A-module (which implies that L is free over A),

(ii) There exists a set $F = \{f_1, ..., f_r\} \subset I$ with $r \leq d$ which generates I such that

$$x_i.f_j = \delta_{ij} \text{ for all } 1 \leq i \leq d \text{ and } 1 \leq j \leq r.$$

Let $i: \mathbf{Y} = Sp(A) \longrightarrow \mathbf{X} = Sp(A/I)$ be a closed embedding of smooth affinoid varieties and G be a compact p-adic Lie group which acts continuously on $\mathbf{X}$. We suppose that:

(a) $\mathcal{T}(\mathbf{X})$ admits a free $\mathcal{A}$-Lie lattice $\mathcal{L} = \mathcal{A}\partial_1 \oplus ... \oplus \mathcal{A}\partial_d$ for some affine formal model $\mathcal{A} \subset \mathcal{O}(\mathbf{X})$ and such that $[\mathcal{L}, \mathcal{L}] \subset \pi\mathcal{L}$, $\mathcal{L}.\mathcal{A} \subset \pi\mathcal{A}$,

(b) $\{\partial_1, ..., \partial_d\}$ is an I-standard basis with respect to a generating set $\{f_1, ..., f_r\} \subset I$.

(c) G preserves $\mathbf{Y} \subset \mathbf{X}$, $\mathcal{A} \subset \mathcal{O}(\mathbf{X})$ and $\mathcal{L} \subset \mathcal{T}(\mathbf{X})$.

Definition 4.3.3. *Let N be a coadmissible (right) $\widehat{\mathcal{D}}(\mathbf{Y}, G)$-module. We define the pushforward functor $i_+ : \mathcal{C}^r_{\widehat{\mathcal{D}}(\mathbf{Y},G)} \longrightarrow \mathcal{C}^r_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ by*

$$i_+N := N\widehat{\otimes}_{\widehat{\mathcal{D}}(Y,G)}\widehat{\mathcal{D}}(\mathbf{X}, G)/I\widehat{\mathcal{D}}(\mathbf{X}, G).$$

We can see explicitly that i_+N is a coadmissible (right) $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module as follows. Let $\mathcal{I} := I \cap \mathcal{A}$ and set

$$\mathcal{N}_{\mathcal{L}}(\mathcal{I}) := \{x \in \mathcal{L} : x(\mathcal{I}) \subset \mathcal{I}\}.$$

Then $\mathcal{N} := \mathcal{N}_{\mathcal{L}}(\mathcal{I})/\mathcal{I}\mathcal{L}$ is a G-stable $\mathcal{A}/\mathcal{I}$-Lie lattice in $\mathcal{T}(Y) = A/I\partial_1 \oplus ... \oplus A/I\partial_d$. Thus, for a good chain $(J_n)_n$ of G, we have (note that $G_{\mathcal{N}} \subset G_{\mathcal{L}}$ by [1, Lemma 4.3.2] so we can choose a good chain of G such that each J_n is contained in $G_{\mathcal{N}}$):

$$\widehat{\mathcal{D}}(\mathbf{X}, G) \cong \varprojlim_n \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G \text{ and } \widehat{\mathcal{D}}(\mathbf{Y}, G) \cong \varprojlim_n \widehat{U(\pi^n\mathcal{N})}_K \rtimes_{J_n} G.$$

Write $S_n := \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G$ and $T_n := \widehat{U(\pi^n\mathcal{N})}_K \rtimes_{J_n} G$, then

$$i_+N \cong \varprojlim_n N_n \otimes_{T_n} S_n/IS_n, \text{ with } N_n = N \otimes_{\widehat{\mathcal{D}}(Y,G)} T_n.$$

We recall the following result from [2, Proposition 6.1]

Proposition 4.3.4. *Let A, I, F, $\mathcal{L}$ be as above and denote by $\mathcal{C} := C_{\mathcal{L}}(F) = \{x \in \mathcal{L} : x.f = 0 \ \forall f \in F\}$ the centraliser of F in $\mathcal{L}$. Then*

$$j_{\widehat{U(\mathcal{L})}_K}(\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{C})}_K} M) = j_{\widehat{U(\mathcal{C})}_K/F\widehat{U(\mathcal{C})}_K}(M) + r.$$

for every finitely generated $\widehat{U(\mathcal{C})}_K/F\widehat{U(\mathcal{C})}_K$-module M

Proposition 4.3.5.

$$d_{\widehat{\mathcal{D}}(\mathbf{X},G)}(i_+N) = d_{\widehat{\mathcal{D}}(Y,G)}(N) + \dim A - \dim A/I$$

for every coadmissible right $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module $N \in \mathcal{C}^r_{\widehat{\mathcal{D}}(\mathbf{X},G)}$.

Proof. Since i_+N is a coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module, there exist n sufficiently large such that

$$j_{\widehat{\mathcal{D}}(\mathbf{X},G)}(i_+N) = j_{T_n}(i_+N \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} T_n) = j_{\widehat{U(\pi^n\mathcal{L})}_K}(i_+N \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} T_n).$$

Here, the last equality follows from Proposition 3.1.4 and Lemma 3.1.5. Note that:

$$i_+N \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} T_n = N_n \otimes_{S_n} T_n/IT_n,$$

where $N \cong \varprojlim_n N_n$ with $N_n \otimes_{\widehat{\mathcal{D}}(Y,G)} S_n$. Furthermore

$$\begin{aligned} N_n \otimes_{S_n} T_n/IT_n &= N_n \otimes_{\widehat{U(\pi^n\mathcal{N})}_K \rtimes_{J_n} G} \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G/I(\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G) \\ &\cong N_n \otimes_{\widehat{U(\pi^n\mathcal{L})}_K} \widehat{U(\pi^n\mathcal{L})}_K/I\widehat{U(\pi^n\mathcal{L})}_K. \end{aligned}$$

On the other hand, the $\mathcal{A}/\mathcal{I}$-Lie lattice $\pi^n\mathcal{N}$ of $\mathcal{T}(Y)$ is isomorphic to $N_{\pi^n\mathcal{L}}(\mathcal{I})/\mathcal{I}(\pi^n\mathcal{L})$. It follows that $\widehat{U(\pi^n\mathcal{L})}_K \cong \widehat{U(\mathcal{C}_n)}_K/I\widehat{U(\mathcal{C}_n)}_K$., so we have

$$N_n \otimes_{\widehat{U(\pi^n\mathcal{L})}_K} \widehat{U(\pi^n\mathcal{L})}_K/I\widehat{U(\pi^n\mathcal{L})}_K \cong N_n \otimes_{\widehat{U(\mathcal{C}_n)}_K} \widehat{U(\pi^n\mathcal{L})}_K.$$

Hence applying Proposition 4.3.4 gives

$$\begin{aligned} j_{\widehat{U(\pi^n\mathcal{L})}_K}(i_+N \otimes_{\widehat{\mathcal{D}}(\mathbf{X},G)} T_n) &= j_{\widehat{U(\pi^n\mathcal{L})}_K}(N_n \otimes_{\widehat{U(\mathcal{C}_n)}_K} \widehat{U(\pi^n\mathcal{L})}_K) \\ &= j_{\widehat{U(\mathcal{C})}_K/F\widehat{U(\mathcal{C})}_K}(N_n) + r = j_{\widehat{U(\pi^n\mathcal{N})}_K}(N_n) + r \\ &= j_{S_n}(N_n) + r. \end{aligned}$$

Here, the last equality follows from Proposition 3.1.4 and Lemma 3.1.5. Finally, for n sufficiently large, one has that:

$$\begin{aligned} d_{\widehat{\mathcal{D}}(\mathbf{X},G)}(i_+N) &= 2d - j_{\widehat{\mathcal{D}}(\mathbf{X},G)}(i_+N) \\ &= 2d - j_{T_n}(i_+N \otimes_{\widehat{\mathcal{D}}(\mathbf{Y},G)} T_n) \\ &= 2d - (r + j_{S_n}(N_n)) \\ &= r + (2d - 2r - j_{\widehat{\mathcal{D}}(\mathbf{Y},G)}(N)) \\ &= d_{\widehat{\mathcal{D}}(\mathbf{Y},G)}(N) + \dim A - \dim A/I. \end{aligned}$$

□

Now, let $i : \mathbf{Y} \longrightarrow \mathbf{X}$ be a closed embedding of smooth rigid varieties, G be a p-adic Lie group which acts continuously on $\mathbf{X}$ and which preserves $\mathbf{Y}$. The following result is necessary for introducing the pushforward functor in the general case:

Theorem 4.3.6. *([5, Theorem 6.2] Let $\mathbf{Y} \hookrightarrow \mathbf{X}$ be a closed embedding of smooth rigid varieties whose the ideal of definition is $\mathcal{I} \subset \mathcal{O}_{\mathbf{X}}$. Then the set $\mathcal{B}$ of connected affinoid subdomains $\mathbf{U}$ such that*

(i) there is a free $\mathcal{A}$-Lie lattice $\mathcal{L} = \partial_1\mathcal{A} \oplus ... \oplus \partial_d\mathcal{A}$ for some affine formal model $\mathcal{A} \subset \mathcal{O}(\mathbf{U})$ satisfying $[\mathcal{L},\mathcal{L}] \subset \pi.\mathcal{L}$ and $\mathcal{L}.\mathcal{A} \subset \pi\mathcal{A}$,

(ii) either $\mathcal{I}(\mathbf{U}) = \mathcal{I}(\mathbf{U})^2$, or $\mathcal{I}(\mathbf{U})$ admits a generating set $F = \{f_1, ..., f_r\}$ with $\partial_i(f_j) = \delta_{ij}$ for every $i = 1..., d$ and $j = 1, ..., r$.

Let $\mathbf{U} \in \mathcal{B}$. By definition, there is a free $\mathcal{A}$-Lie lattice $\mathcal{L} = \partial_1\mathcal{A} \oplus ... \oplus \partial_d\mathcal{A}$ for some affine formal model $\mathcal{A} \subset \mathcal{O}(\mathbf{U})$ satisfying the conditions (i) and (ii) in Theorem 4.3.6. Following [1, Lemma 4.4.2], there exists a compact open subgroup H of G which stabilies $\mathbf{U}$, $\mathcal{A}$ and $\mathcal{L}$. H is then called $\mathbf{U}$-*good*.

Let $\mathcal{N} \in \mathcal{C}^r_{Y/G}$ be a coadmissible G-equivariant $\mathcal{D}_Y$-module. Then the pushforward $i_+\mathcal{N}$ of $\mathcal{N}$ along the closed embedding i can be defined (locally) as follows:

$$i_+\mathcal{N}(\mathbf{U}) := \varprojlim_H M[\mathbf{U}, H]$$

for any $\mathbf{U} \in \mathcal{B}$, where $M[\mathbf{U}, H] := \mathcal{N}(\mathbf{U} \cap \mathbf{Y}) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U} \cap \mathbf{Y}, H)} \widehat{\mathcal{D}}(\mathbf{U}, H)/\mathcal{I}(\mathbf{U})\widehat{\mathcal{D}}(\mathbf{U}, H)$ and H runs over the set of all U-good subgroups of G.

Proposition 4.3.7. *Let $i : \mathbf{Y} \longrightarrow \mathbf{X}$ be a closed embedding of smooth rigid varieties, G be a p-adic Lie group which acts continuously on $\mathbf{X}$ and which preserves $\mathbf{Y}$. Then for every $\mathcal{N} \in \mathcal{C}^r_{\mathbf{Y}/G}$*

$$d_{\mathbf{X}}(i_+\mathcal{N}) = d_{\mathbf{Y}}(\mathcal{N}) + \dim \mathbf{X} - \dim \mathbf{Y}\,.$$

Proof. This is a consequence of Proposition 4.3.5 and Theorem 4.3.6. □

4.3.3 Bernstein's inequality for rigid flag varieties

The objective of the rest of this dissertation is to define the category of equivariant weakly holonomic modules. In order to do this, we have to gain the so-called *Bernstein's inequality.*

Definition 4.3.8. *Let $\mathbf{X}$ be a smooth rigid analytic variety and G be a p-adic Lie group acting continuously on $\mathbf{X}$. Then $(\mathbf{X}, G)$ is 'good' if Bernstein's inequality holds for the category $\mathcal{C}_{\mathbf{X}/G}$. More precisely, if $d(\mathcal{M}) \geq \dim \mathbf{X}$ for any non-zero module $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$.*

Even though, we don't know whether all smooth rigid-spaces (on which a p-adic Lie group G acts continuously) satisfy this condition, we know some special cases where Bernstein's inequality holds, as explaining in the following:

Lemma 4.3.9. *Let $\mathbf{X} = Sp(K\langle x_1, ..., x_d\rangle$ be the unit polydisc of dimension d and G be a compact p-adic Lie group acting continuously on $\mathbf{X}$ such that $(\mathbf{X}, G)$ is small. Then $(\mathbf{X}, G)$ is good.*

Proof. Let $\mathcal{M}$ be a non-zero module in $\mathcal{C}_{\mathbf{X}/G}$ and $M := \mathcal{M}(\mathbf{X}) \in \mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{X},G)}$ be its global section. Denote by $\partial_1, ..., \partial_n$ the partial derivations with respect to coordinates $x_1, ..., x_d$. Write $\mathcal{A} := \mathcal{R}\langle x_1, ..., x_d\rangle$ and $\mathcal{L} := Der_{\mathcal{R}}(\mathcal{A}) = \partial_1\mathcal{A} \oplus ... \oplus \partial_d\mathcal{A}$. Then $\mathcal{A}$ is an affine formal model of $\mathcal{O}(\mathbf{X}) = K\langle x_1, ..., x_d\rangle$ and $\mathcal{L}$ is a free $\mathcal{A}$-Lie lattice in $\mathcal{T}(\mathbf{X})$. Now, we can choose an open subgroup H of G which stabilises $\mathcal{A}$ and $\mathcal{L}$ ([3, Lemma 4.4.1]. Thus, $(\mathbf{X}, H)$ is small and

$$\widehat{\mathcal{D}}(\mathbf{X}, H) \cong \varprojlim_n \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} H$$

for any choice of a good chain $(J_n)_n$ for $\mathcal{L}$. Note that $d_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M) = d_{\widehat{\mathcal{D}}(\mathbf{X},H)}(M)$ (Remark 3.2.3 (ii)) and there exist n sufficiently large such that

$$j_{\widehat{\mathcal{D}}(\mathbf{X},H)}(M) = j_{\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} H}(M_n), \text{ with } M_n = (\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} H) \otimes_{\widehat{\mathcal{D}}(\mathbf{X},H)} M.$$

On the other hand, Proposition 3.1.4 and Lemma 3.1.5 tell us that

$$j_{\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} H}(M_n) = j_{\widehat{U(\pi^n\mathcal{L})}_K}(M_n).$$

Now applying [4, Corollary 7.4] gives $j_{\widehat{U(\pi^n\mathcal{L})}_K}(M_n) \leq d$ and so $d(\mathcal{M}) \geq d$ as claimed. □

Now let $\mathbb{G}$ be a connected, simply connected, split semisimple algebraic group scheme over K. Let $\mathbb{X}$ be the flag variety of $\mathbb{G}$, which is defined as the set of all Borel subgroups of $\mathbb{G}$ ([15, II.1.8]. Then the group $\mathbb{G}$ acts on $\mathbb{X}$ by conjugation, since any two Borel groups of G are conjugate. Let $\mathbf{X}$ be the rigid analytification of $\mathbb{X}$. Let $G := \mathbb{G}(K)$. The $\mathbb{G}$-action on $\mathbb{X}$ induces a G-action on $\mathbf{X}$. Moreover, G acts continuously on $\mathbf{X}$ by [4, Theorem 6.3.4]

Theorem 4.3.10. *The pair $(\mathbf{X}, G)$ of the rigid flag variety $\mathbf{X}$ and its induced G-action is good*

Proof. Since $\mathbb{G}$ is connected split semisimple, there exists a group scheme $\mathbb{G}_0$ over $\mathcal{R}$ such that $\mathbb{G} \simeq \mathbb{G} \times_{\mathcal{R}} K$. Let $\mathbb{B}_0$ be a closed, flat Borel $\mathcal{R}$-subgroup scheme of $\mathbb{G}_0$ and write $\mathbb{B} := \mathbb{B}_0 \times_{\mathcal{R}} K$ $\mathbb{X}_0 := \mathbb{G}_0/\mathbb{B}_0$. Then following [3, Proposition 6.4.3], the rigid analytification $\mathbf{X}$ of the flag variety $\mathbb{G}/\mathbb{B}$ is isomorphic to $(\widehat{\mathbb{X}_0})_{rig}$- the rigid analytic space associated to the (smooth) formal scheme $\widehat{\mathbb{X}_0}$. We then identify $(\widehat{\mathbb{X}_0})_{rig}$ with $\mathbf{X}$.
Let $d = \dim \mathbf{X}$ and W be the Weyl group. Then the Weyl translates $(U_w)_{w\in W}$ of the big cell in $\mathbb{X}_0$ form an affine covering of $\mathbb{X}_0$ ([15, II.1.10], the set $\{(\widehat{U_w})_{rig}\}_{w\in W}$ is then an admissible covering of $\mathbf{X}$. Now each $(U_w))_{w\in W}$ is isomorphic to the affine space $\mathcal{R}[x_1, ..., x_d]$ of dimension d ([15, II.1.7,], it follows that each $((\widehat{U_w})_{rig})_{w\in W}$ is isomorphic to the polydisc of dimension d. By choosing an open compact subgroup H_w of G such that $((\widehat{U_w})_{rig}), H_w)$ is small, we may apply Lemma 4.3.9 to the case of $((\widehat{U_w})_{rig}, H_w)$ and hence the result follows. □

Remark 4.3.11. 1. *The arguments used in Theorem* 4.3.10 *can be applied to any smooth rigid analytic space on which G-acts continuously and which admits an admissible affinoid covering by unit polydiscs of dimension* $\dim X$*. The rigid analytification $\mathbb{P}^{d,an}_K$ of the projective scheme $\mathbb{P}^d_K$ over K with the induced $G := GL_{d+1}(K)$-action is such an example. More generally, let $\mathbb{G}$ be an an affine algebraic group of finite type over K and V be a finite-dimensional $\mathbb{G}$-representation. Let $G := \mathbb{G}(K)$ and $\mathbb{P}(V)^{an}$ be the analytification of the algebraic projective space $\mathbb{P}(V)$. Then $(\mathbb{P}(V)^{an}, G)$ is good.*

2. *Let $\mathbf{X}$ be a smooth rigid analytic variety and G acts continuously on $\mathbf{X}$. If $(\mathbf{X}, G)$ is good, then for every Zariski closed subspace $\mathbf{Y}$ of $\mathbf{X}$ which is stable under the G-action, $(\mathbf{Y}, G)$ is also good. This is a direct consequence of Proposition 4.3.7 and Proposition 3.3.1.*

4.3.4 Equivariant weakly holonomic $\mathcal{D}$-modules

Let $\mathbf{X}$ be a smooth rigid analytic variety and G be a p-adic Lie group which acts continuously on $\mathbf{X}$. We assume from now on to the end of this section that $\mathbf{X}$ is good (i.e Bernstein's inequality holds for the category $\mathcal{C}_{\mathbf{X}/G}$).

Definition 4.3.12. *A G-equivariant coadmissible (left or right) $\mathcal{D}$-module $\mathcal{M}$ on $\mathbf{X}$ is called weakly holonomic if $d(\mathcal{M}) \leq \dim \mathbf{X}$.*

It follows from Bernstein's inequality that $d(\mathcal{M}) = \dim \mathbf{X}$ for every non-zero G-equivariant weakly holonomic module $\mathcal{M}$.

Proposition 4.3.13. *Let*

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{M}_2 \longrightarrow 0$$

is an exact sequence in $\mathcal{C}_{\mathbf{X}/G}$. Then $\mathcal{M}_0$ is G-equivariant weakly holonomic if and only if $\mathcal{M}_1, \mathcal{M}_2$ are G-equivariant weakly holonomic.

Proof. Let $\mathcal{U}$ be an admissible coverings of $\mathbf{X}$ by affinoid subdomains in $\mathbf{X}_w(\mathcal{T})$. For every $\mathbf{U} \in \mathcal{U}$, it follows from Proposition 3.2.4 that

$$d(\mathcal{M}_0(\mathbf{U})) = \max\{d(\mathcal{M}_1(\mathbf{U})), d(\mathcal{M}_2(\mathbf{U}))\}.$$

Then $d(\mathcal{M}_0(\mathbf{U})) \leq d$ if and only if both $d(\mathcal{M}_1(\mathbf{U}))$ and $d(\mathcal{M}_2(\mathbf{U}))$ are not greater than d, so the result follows. □

The following example is given by Proposition 3.2.6:

Example 4.3.14. *Let* $\mathbf{X}$ *be a smooth affinoid variety of dimension* 1 *and* G *be a compact* p*-adic Lie group which acts continuously on* $\mathbf{X}$ *and such that* $(\mathbf{X}, G)$ *is good. Let* $\mathcal{M} = Loc_{\mathbf{X}}(M)$ *is the coadmissible* G*-equivariant* $\mathcal{D}_{\mathbf{X}}$*-module associated to the coadmissible* $\widehat{\mathcal{D}}(\mathbf{X}, G)$*-module* $M := \widehat{\mathcal{D}}(\mathbf{X}, G)/\widehat{\mathcal{D}}(\mathbf{X}, G)P$*, where* $P \in \mathcal{D}(\mathbf{X})$ *is a regular differential operator. Then* $\mathcal{M}$ *is* G*-equivariant weakly holonomic.*

The category of G- equivariant weakly holonomic $\mathcal{D}_{\mathbf{X}}$-modules is denoted by $\mathcal{C}^{wh}_{\mathbf{X}/G}$. This is a full abelian subcategory of $\mathcal{C}_{\mathbf{X}/G}$ and is closed under extension (Proposition 4.3.13).

Theorem 4.3.15. *Suppose that* $\mathbf{X}$ *is of dimension* d*. The functor* $\mathcal{E}^d$ *defined in Definition 4.2.24 preserves* G*-equivariant weakly holonomic left* $\mathcal{D}_{\mathbf{X}}$*-modules.*

Proof. We may suppose that $\mathbf{X}$ is a smooth affinoid variety , i.e $\mathbf{X} = Sp(A)$ and G is compact such that $(\mathbf{X}, G)$ is small. Write $L = Der_K(A)$. Let $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$ be a non-zero G-equivariant weakly holonomic modules, then $\mathcal{M} \simeq Loc^{X}_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M)$, with $M = \mathcal{M}(\mathbf{X})$ is a non-zero coadmissible left $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module of dimension d. In particular, $j(M) = d$ implying that:

$$Ext^d_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G)) \neq 0.$$

By Proposition 4.13 and Theorem 2.4.26(ii), one has that:

$$\mathcal{E}^d\mathcal{M} \simeq Loc^{X}_{\widehat{\mathcal{D}}(\mathbf{X},G)}(Hom_A(\Omega_L, Ext^d_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G)))).$$

On the other hand, thanks to Auslander's condition, the (non-zero) coadmissible right $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module $Ext^d_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G))$ has grade $j(Ext^d_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G))) \geq d$, so that its dimension is less than d. Now apply Proposition 3.3.1, one has

$$d(Hom_A(\Omega_L, Ext^d_{\widehat{\mathcal{D}}(\mathbf{X},G)}(M, \widehat{\mathcal{D}}(\mathbf{X}, G)))) \leq d.$$

This proves that $\mathcal{E}^d(\mathcal{M})$ is a G-equivariant weakly holonomic $\mathcal{D}_{\mathbf{X}}$-module. □

Remark 4.3.16. *Let* $\mathcal{M} \in \mathcal{C}^{wh}_{\mathbf{X}/G}$*, then Auslander's condition together with Bernstein's inequality implies that* $\mathcal{E}^i\mathcal{M} = 0$ *for any* $i \neq d$*.*

Definition 4.3.17. *The duality functor* $\mathbb{D}$ *on* $\mathcal{C}^{wh}_{\mathbf{X}/G}$ *into itself is defined as follows:*

$$\mathbb{D}(\mathcal{M}) := \mathcal{E}^d = \mathcal{H}om_{\mathcal{O}}(\Omega_{\mathbf{X}}, E^d(\mathcal{M}))$$

for any $\mathcal{M} \in \mathcal{C}^{wh}_{\mathbf{X}/G}$.

Proposition 4.3.18. *Let* $\mathcal{M} \in \mathcal{C}^{wh}_{\mathbf{X}/G}$*. Then there is an isomorphism in* $\mathcal{C}^{wh}_{\mathbf{X}/G}$

$$\mathbb{D}^2(\mathcal{M}) \cong \mathcal{M}.$$

Proof. This can be proved along the lines of the proof of [2, Proposition 7.3] for weakly holonomic $\mathcal{D}_{\mathbf{X}}$-modules. Let $\mathcal{M} \in \mathcal{C}^{wh}_{X/G}$. Since $\mathbf{X}_w(\mathcal{T})$ is a basis for the G-topology on $\mathbf{X}$, it is enough to show that

$$\Gamma(\mathbf{U}, \mathbb{D}^2(\mathcal{M})) \simeq \Gamma(\mathbf{U}, \mathcal{M}),$$

for any $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$). Without loss of generality, we may suppose that $\mathbf{U} = \mathbf{X}$, which means that $\mathbf{X}$ is a smooth affinoid of dimension d and that $(\mathbf{X}, G)$ is small. Note that $\mathcal{T}(\mathbf{X})$ admits a G-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ for some affine formal model $\mathcal{A}$ of $\mathcal{O}(\mathbf{X})$. Choose a good chain J_n for $\mathcal{L}$. Then

$$\widehat{D} := \widehat{\mathcal{D}}(\mathbf{X}, G) \simeq \varprojlim_n D_n \text{ with } D_n := \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G.$$

Write $M := \Gamma(\mathbf{X}, \mathcal{M}) \cong \varprojlim_n M_n$ with $M_n := D_n \otimes_{\widehat{D}} M$. By Proposition 3.3.1, one has:

$$\begin{aligned}\Gamma(\mathbf{X}, \mathbb{D}^2(\mathcal{M})) &= Hom_A(\Omega, Ext^d_{\widehat{D}}(Hom_A(\Omega, Ext^d_{\widehat{D}}(M, \widehat{D})), \widehat{D}) \\ &\simeq Ext^d_{\widehat{D}}(\Omega \otimes_A Hom_A(\Omega, Ext^d_{\widehat{D}}(M, \widehat{D})), \widehat{D}) \\ &\simeq Ext^d_{\widehat{D}}(Ext^d_{\widehat{D}}(M, \widehat{D}), \widehat{D}).\end{aligned}$$

In other words, it remains to prove that:

$$Ext^d_{\widehat{D}}(Ext^d_{\widehat{D}}(M, \widehat{D}), \widehat{D}) \simeq M.$$

Recall that(Lemma 2.4.19) $Ext^d_{\widehat{D}}(M, \widehat{D}) \cong \varprojlim_n Ext^d_{D_n}(M_n, D_n)$ implying that:

$$\begin{aligned}Ext^d_{\widehat{D}}(Ext^d_{\widehat{D}}(M, \widehat{D}), \widehat{D})) &\cong \varprojlim_n Ext^d_{D_n}(Ext^d_{D_n}(M_n, D_n), D_n)) \\ &\cong \varprojlim_n M_n \cong M.\end{aligned}$$

Here the second isomorphism follows from [14, Theorem 4]. This proves that $\mathbb{D}^2(\mathcal{M}) \simeq \mathcal{M}$ as claimed. □

4.3.5 Extension

In this subsection, we give a way to construct equivariant weakly holonomic modules. As in [2, Section 7.2], we are going to define a kind of so-called extension functor. This functor is defined on the category of G-equivariant coherent $\mathcal{D}_{\mathbf{X}}$-module and takes values in the category $\mathcal{C}_{\mathbf{X}/G}$. We also want to prove that the extension functor preserves weakly holonomicity.

Let $\mathbf{X}$ be a smooth affinoid variety and G is a compact p-adic Lie group acting continuously on $\mathbf{X}$ such that $(\mathbf{X}, G)$ is good. We first begin by proving the following lemma:

Lemma 4.3.19. *Let H be an open subgroup of G. The natural map*

$$\widehat{\mathcal{D}}(\mathbf{X}, H) \otimes_{\mathcal{D}(\mathbf{X}) \rtimes H} (\mathcal{D}(\mathbf{X}) \rtimes G) \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, G)$$

is an isomorphism.

Proof. Following [4, Proposition 3.4.10] there is a bijection

$$\widehat{\mathcal{D}}(\mathbf{X}, H) \otimes_{K[H]} K[G] \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, G).$$

Furthermore this morphism factors into

$$\widehat{\mathcal{D}}(\mathbf{X}, H) \otimes_{K[H]} K[G] \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, H) \otimes_{\mathcal{D}(\mathbf{X}) \rtimes H} (\mathcal{D}(\mathbf{X}) \rtimes G) \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, G).$$

The first morphism is surjective, which implies that the second map is an bijection as claimed. □

Corollary 4.3.20. *If M is a $\mathcal{D}(\mathbf{X}) \rtimes G$-module and H is an open subgroup of G. Then the natural morphism*

$$\widehat{\mathcal{D}}(\mathbf{X}, H) \otimes_{\mathcal{D}(\mathbf{X})\rtimes H} M \xrightarrow{\sim} \widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})\rtimes G} M$$

is bijective.

Proof. Applying Lemma 4.3.19 one has that

$$\widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})\rtimes G} M \cong \widehat{\mathcal{D}}(\mathbf{X}, H) \otimes_{\mathcal{D}(\mathbf{X})\rtimes H} (\mathcal{D}(\mathbf{X}) \rtimes G) \otimes_{\mathcal{D}(\mathbf{X})\rtimes G} M \cong \widehat{\mathcal{D}}(\mathbf{X}, H) \otimes_{\mathcal{D}(\mathbf{X})\rtimes H} M.$$

□

Proposition 4.3.21. *Let M be a $\mathcal{D}(\mathbf{X}) \rtimes G$-module wich is coherent as a $\mathcal{D}(\mathbf{X})$-module. Then the tensor product*

$$\widehat{M} := \widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})\rtimes G} M$$

is a coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module.

Proof. Since G is compact, there exists an uniform pro-p subgroup N which is normal in G [4, Lemma 3.2.1]. So G is topologically finitely generated. As M is finitely presented as a $\mathcal{D}(\mathbf{X})$-module, it follows that the $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module $\widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})} M$ is coadmissible [25, Corollary 3.4(v)]. Now, let $g_1, g_2, ..., g_r$ be a set of topological generators for the compact p-adic Lie group G, $m_1, ..., m_s$ generate M as a $\mathcal{D}(\mathbf{X})$-module and let I be the $\widehat{\mathcal{D}}(\mathbf{X}, G)$-submodule generated by the finite set $\{g_i \otimes m_j - 1 \otimes g_i m_j\}$. Then I is a coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module by [25, Corollary 3.4(iv)]. There is a surjective map

$$f : \widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})} M \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})\rtimes G} M$$

We will show that I is exactly the kernel of this map. Let $x \in L = \mathcal{T}(\mathbf{X})$. Then $g_i x g_i^{-1} = g_i.x$ in $\mathcal{D}(\mathbf{X}) \rtimes G = U(L) \rtimes G$, so that we can compute as follows:

$$g_i \otimes x m_j - 1 \otimes g_i x m_j = (g_i x g_i^{-1}) g_i \otimes m_j - 1 \otimes (g_i x g_i^{-1}) g_i m_j = (g_i.x) g_i \otimes m_j - 1 \otimes (g_i.x) g_i m_j = (g_i.x)(g_i \otimes m_j - 1 \otimes g_i m_j)$$

Hence I contains all elements of the form $g_i \otimes m - 1 \otimes g_i m$ with $m \in M$. Now, let $g \in G$ and $(g_n) \in \langle g_1, ..., g_r \rangle$ such that $\lim g_n = g$. Note that the coadmissible module $\widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})} M$ has a natural Fréchet topology such that the map $\widehat{\mathcal{D}}(\mathbf{X}, G) \longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})} M$ is continuous. This implies that:

$$\lim(g_n \otimes m - 1 \otimes g_n m) = g \otimes m - 1 \otimes g.$$

Here we note that $G \subset \widehat{\mathcal{D}}(\mathbf{X}, G)$. Combining with the fact that I is a closed subspace of $\widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})} M$ [25, Lemma 3.6], we have that $g \otimes m - 1 \otimes gm \in I$ for any $g \in G$ and $m \in M$. Thus $I = \ker(f)$. By consequence the $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module $\widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})\rtimes G} M$ is coadmissible. □

Now let $\mathbf{X}$ be a smooth rigid analytic variety and G acts continuously on $\mathbf{X}$. Let $\mathcal{M}$ be a G-equivariant $\mathcal{D}_{\mathbf{X}}$-module which is coherent as a $\mathcal{D}_{\mathbf{X}}$-module. Then we define the presheaf $E_{\mathbf{X}/G}(\mathcal{M})$ on $\mathbf{X}_w(\mathcal{T})$ as follows. Let $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$. Then define

$$E_{\mathbf{X}/G}(\mathcal{M})(\mathbf{U}) := \varprojlim_{H} \widehat{\mathcal{D}}(\mathbf{U}, H) \otimes_{\mathcal{D}(\mathbf{U})\rtimes H} \mathcal{M}(\mathbf{U})$$

where the inverse limit is taken over the set of all $\mathbf{U}$-small subgroups H of G.

Theorem 4.3.22. *The presheaf $E_{\mathbf{X}/G}(\mathcal{M})$ is a sheaf on $\mathbf{X}_w(\mathcal{T})$, thus extends naturally to a coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-module on $\mathbf{X}$ and we still denote it by $E_{\mathbf{X}/G}(\mathcal{M})$.*

Proof. We suppose that $(\mathbf{X}, G)$ is small. Denote

$$M := \mathcal{M}(\mathbf{X}) \text{ and } \widehat{M} = \widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X}) \rtimes G} M.$$

Let $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$. Then

$$\widehat{\mathcal{D}}(\mathbf{U}, H) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)} \widehat{M} \cong \widehat{\mathcal{D}}(\mathbf{U}, H) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)} \widehat{\mathcal{D}}(\mathbf{X}, H) \otimes_{\mathcal{D}(\mathbf{X}) \rtimes H} M \cong \widehat{\mathcal{D}}(\mathbf{X}, H) \otimes_{\mathcal{D}(\mathbf{X}) \rtimes H} M.$$

Furthermore, since $\mathcal{M}$ is a coherent $\mathcal{D}_{\mathbf{X}}$-module, one has that

$$\mathcal{M}(\mathbf{U}) \cong \mathcal{D}(\mathbf{U}) \otimes_{\mathcal{D}(X)} M.$$

Consequently, $\mathcal{M}(\mathbf{U}) \cong \mathcal{D}(\mathbf{U}) \otimes_{\mathcal{D}(\mathbf{X})} M \cong \mathcal{D}(\mathbf{U}) \rtimes H \otimes_{\mathcal{D}(\mathbf{X}) \rtimes H} M$. This implies that

$$E_{\mathbf{X}/G}(\mathcal{M})(\mathbf{U}) \cong \widehat{\mathcal{D}}(\mathbf{U}, H) \otimes_{\mathcal{D}(\mathbf{U}) \rtimes H} \mathcal{M}(\mathbf{U}) \cong \widehat{\mathcal{D}}(\mathbf{U}, H) \otimes_{\mathcal{D}(\mathbf{X}) \rtimes H} M \cong \widehat{\mathcal{D}}(\mathbf{U}, H) \widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{X},H)} \widehat{M}.$$

This proves that $E_{\mathbf{X}/G}(\mathcal{M}) \cong Loc_{\mathbf{X}}(\widehat{M})$. So this is a G-equivariant coadmissible $\mathcal{D}_{\mathbf{X}}$-module. □

Remark 4.3.23. *Let $Coh(G - \mathcal{D}_{\mathbf{X}})$ be the category of G-equivariant coherent $\mathcal{D}_{\mathbf{X}}$-modules. Then it is straightforward to verify that the mapping $E_{\mathbf{X}/G} : Coh(G - \mathcal{D}_{\mathbf{X}}) \longrightarrow \mathcal{C}_{\mathbf{X}/G}$, which sends $\mathcal{M}$ to $E_{\mathbf{X}/G}(\mathcal{M})$ is a functor.*

Recall ([18]) that we can also define the dimension for a finitely generated $\mathcal{D}_{\mathbf{X}}$-module $\mathcal{M}$ on the smooth rigid analytic variety $\mathbf{X}$. The module $\mathcal{M}$ is said to be of minimal dimension if its dimension is not greater than $\dim \mathbf{X}$.

Theorem 4.3.24. *Let $\mathcal{M}$ is a G-equivariant $\mathcal{D}_{\mathbf{X}}$-module of minimal dimension. Then $E_{\mathbf{X}/G}(\mathcal{M})$ is a G-equivariant weakly holonomic module.*

Proof. Since the question is local, we may assume that $\mathbf{X}$ is smooth affinoid and G is compact such that $(\mathbf{X}, G)$ is small. Choose a G-stable free Lie lattice $\mathcal{L}$ and a good chain $(J_n)_n$ for $\mathcal{L}$ such that $\widehat{\mathcal{D}}(\mathbf{X}, G) \cong \widehat{U(\pi^n \mathcal{L})_K} \rtimes_{J_n} G$. Write $D := \mathcal{D}(\mathbf{X})$, $\widehat{D} := \widehat{\mathcal{D}}(\mathbf{X}, G)$, $D_n := \widehat{U(\pi^n \mathcal{L})_K} \rtimes_{J_n} G$, $M := \mathcal{M}(\mathbf{X})$ and let $d := \dim \mathbf{X}$. Recall that by definition $\widehat{M} = \widehat{D} \otimes_{D \rtimes G} M$.
As $\mathcal{M}$ is of minimal dimension, we obtain that $d(M) = d$. Now the $\widehat{D}$-module $\widehat{D} \otimes_D M$ is coadmissible, there is a n sufficiently large such that $j_{D_n}(D_n \otimes_D M) = j_{\widehat{D}}(\widehat{D} \otimes_D M)$. As we know that D_n is flat over D, it follows that

$$Ext^i_D(M, D) \otimes_D D_n \cong Ext^i_{D_n}(D_n \otimes_D M, D_n).$$

Thus $j_{D_n}(D_n \otimes_D M) \geq j_D(M)$, which implies

$$d(\widehat{D} \otimes_D M) = d(D_n \otimes_D M) \leq d(M) = d.$$

This also proves that $d(\widehat{M}) \leq d$, since we have the following exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \widehat{D} \otimes_D M \xrightarrow{f} \widehat{M} \longrightarrow 0.$$

So $E_{\mathbf{X}/G}(\mathcal{M})$ is weakly holonomic. □

Chapter 5

Examples

5.1 A class of equivariant weakly holonomic $\mathcal{D}$-modules

Let $\mathbf{X}$ be a smooth rigid analytic variety and G be a p-adic Lie group which acts continuously on $\mathbf{X}$. We assume throughout this section that $(\mathbf{X}, G)$ is good, i.e every non-zero module $\mathcal{M} \in \mathcal{C}_{\mathbf{X}/G}$ satisfies Bernstein's inequality.
In this section, we study a class of equivariant weakly holonomic modules whose each module is coherent as a $\mathcal{O}_{\mathbf{X}}$-module. In particular, we will see that the structure sheaf $\mathcal{O}_{\mathbf{X}}$ is G-equivariant weakly holonomic.

Recall that an integrable connection on $\mathbf{X}$ is a $\mathcal{D}_{\mathbf{X}}$-module which is locally free of finite rank as an $\mathcal{O}_{\mathbf{X}}$-module (so it is coherent as a $\mathcal{D}_{\mathbf{X}}$-module). In this subsection, all integrable connections on $\mathbf{X}$ will be G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules and we call them *G-equivariant integrable connections.*

Proposition 5.1.1. *Let $\mathcal{M}$ be a G-equivariant integrable connection on $\mathbf{X}$. Then $\mathcal{M} \in Frech(G-\mathcal{D})$.*

Proof. Let $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ be an affinoid subdomain. Then $\mathcal{M}|_{\mathbf{U}}$ is a coherent $\mathcal{O}_{\mathbf{U}}$-module, so that by Kiehl's theorem, $\mathcal{M}(\mathbf{U})$ is a coherent $\mathcal{O}(U)$-module. This implies that $\mathcal{M}(\mathbf{U})$ has a canonical Banach topology by [7, Chapter 3, Proposition 3.7.3.3]. For any $g \in G$, the map

$$g^{\mathcal{M}}(\mathbf{U}) : \mathcal{M}(\mathbf{U}) \longrightarrow \mathcal{M}(g\,\mathbf{U})$$

is a bijection which is linear with respect to the continuous morphism of K-Banach algebras $g^{\mathcal{O}}(\mathbf{U}) : \mathcal{O}(\mathbf{U}) \longrightarrow \mathcal{O}(g\,\mathbf{U})$. If we consider the $\mathcal{O}(g\,\mathbf{U})$-module $\mathcal{M}(g\,\mathbf{U})$ as a $\mathcal{O}(\mathbf{U})$-module, then $\mathcal{M}(g\,\mathbf{U})$ is coherent as a $\mathcal{O}(\mathbf{U})$-module such that the map $\mathcal{O}(\mathbf{U}) \times \mathcal{M}(g\,\mathbf{U}) \longrightarrow \mathcal{M}(g\,\mathbf{U})$ is continuous and $g^{\mathcal{M}}(\mathbf{U})$ is a $\mathcal{O}(U)$-linear map. By [7, Chapter 3, Proposition 3.7.3.2], $g^{\mathcal{M}}(\mathbf{U})$ is a continuous mapping between Banach spaces. Since every Banach space is in particular a Fréchet space, this proves that $\mathcal{M} \in Frech(G-\mathcal{D})$. □

Definition 5.1.2. *A G-equivariant integrable connection $\mathcal{M}$ is called strongly G-equivariant if $\mathcal{M}$, together with the topology explained in Proposition 5.1.1, lies in $\mathcal{C}_{\mathbf{X}/G}$.*

Proposition 5.1.3. *Suppose that $\mathbf{X} = Sp(A)$ is affinoid and G is compact. Let M be a $\mathcal{D}(\mathbf{X}) \rtimes G$-module which is coherent as an A-module. Let $\mathcal{L}$ be a G-stable $\mathcal{A}$-Lie lattice in $\mathcal{T}(\mathbf{X}) = Der_K(A)$ for some G-stable affine formal model $\mathcal{A}$ of A. Then there exists $m \geq 0$ such that there is a structure of $\widehat{U(\pi^n\mathcal{L})}_K \rtimes G$-module on M for all $n \geq m$ which extends the given $\mathcal{D}(\mathbf{X}) \rtimes G$-action.*

Proof. By assumption M is a coherent A-module. Let S be a generating set of M as an A-module. Then $\mathcal{M} := \mathcal{A}S$ is an $\mathcal{A}$-submodule of M which generates M over K. Furthermore, since $\mathcal{L}$ is a $\mathcal{A}$-Lie lattice by assumption, there exists $m \geq 0$ such that for all $n \geq m$, $\pi^n \mathcal{L}\mathcal{M} \subset \mathcal{M}$, forcing $\mathcal{M}$ to be a $U(\pi^n\mathcal{L})$-module. Now, since $\mathcal{A}$ is π-adically complete, $\mathcal{M}$ is also π-adically complete and

$$\mathcal{M} \cong \widehat{U(\pi^n\mathcal{L})} \otimes_{U(\pi^n\mathcal{L})} \mathcal{M},$$

so that $\mathcal{M}$ is also a $\widehat{U(\pi^n\mathcal{L})}$-module. Therefore, $M \cong K \otimes \mathcal{M}$ is a $\widehat{U(\pi^n\mathcal{L})}_K$-module. On the other hand, we see that the structure of $\widehat{U(\pi^n\mathcal{L})}_K$-module (which extends the given $\mathcal{D}(\mathbf{X})$-action) on M is compatible with the G-action. This proves that M is therefore a $\widehat{U(\pi^n\mathcal{L})}_K \rtimes G$-module. □

Proposition 5.1.4. *We suppose the conditions as in Proposition* 5.1.3*. Then the* $\mathcal{D}(\mathbf{X}) \rtimes G$*-action on* M *extends to a coadmissible* $\widehat{\mathcal{D}}(\mathbf{X}, G)$*-module structure if there exists a* G*-stable free* $\mathcal{A}$*-Lie lattice* $\mathcal{L}$ *such that for all n sufficiently large, the following equality holds*

$$g.m = \beta_{\pi^n\mathcal{L}}(g)m \text{ for all } m \in M \text{ and } g \in G_{\pi^n\mathcal{L}}. \tag{5.1}$$

Proof. Let $\mathcal{L}$ be a G-stable free $\mathcal{A}$-Lie lattice in $L = Der_K(\mathcal{O}(\mathbf{X}))$. Following proposition 5.1.3, for n sufficiently large, M is a $\widehat{U(\pi^n\mathcal{L})}_K \rtimes G$-module. Suppose that the condition (5.1) holds. Let (J_n) be a good chain for $\mathcal{L}$. Then the $\widehat{U(\pi^n\mathcal{L})_K} \rtimes G$-action on M factors through a $\widehat{U(\pi^n\mathcal{L})_K} \rtimes_{J_n} G$-action, so that M is a $\widehat{U(\pi^n\mathcal{L})_K} \rtimes_{J_n} G$-module.

Next, we note that the natural morphism $i : M \longrightarrow \widehat{U(\pi^n\mathcal{L})}_K \rtimes G \otimes_{U(L)\rtimes G} M$ is an isomorphism. Indeed, as we see that M is a $\widehat{U(\pi^n\mathcal{L})}_K \rtimes G$-module, there exists a $\widehat{U(\pi^n\mathcal{L})}_K \rtimes G$-linear map

$$\begin{aligned} j: \quad \widehat{U(\pi^n\mathcal{L})}_K \rtimes G \otimes_{U(L)\rtimes G} M &\longrightarrow M \\ a \otimes m &\longmapsto am \end{aligned}$$

such that $j \circ i = id_M$. Therefore i is injective. For the surjectivity of i, we note that the ring $\widehat{U(\pi^n\mathcal{L})}_K \rtimes G$ (resp. $U(L) \rtimes G$) consists of elements of the form $\sum a_i g_i$, where the sum is finite with $g_i \in G$ and $a_i \in \widehat{U(\pi^n\mathcal{L})}_K$ (resp. $a_i \in U(L)$). As a consequence, the map $\widehat{U(\pi^n\mathcal{L})}_K \otimes_{U(L)} M \longrightarrow \widehat{U(\pi^n\mathcal{L})}_K \rtimes G \otimes_{U(L)\rtimes G} M$ is surjective. On the other hand, the map i factors through

$$M \xrightarrow{\sim} \widehat{U(\pi^n\mathcal{L})}_K \otimes_{U(L)} M \longrightarrow \widehat{U(\pi^n\mathcal{L})}_K \rtimes G \otimes_{U(L)\rtimes G} M.$$

Where the first map is an isomorphism by [5, Lemma 7.2]. Therefore i is surjective, so it is an isomorphism as claimed.

As a consequence, this proves that the canonical morphism

$$M \longrightarrow \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G \otimes_{U(L)\rtimes G} M$$

is an isomorphism of $\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G$-modules, as M is also a $\widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G$-module.

Now we prove that M is a coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module as follows. Write $D_n := \widehat{U(\pi^n\mathcal{L})}_K \rtimes_{J_n} G$ for all n. Then $\widehat{\mathcal{D}}(\mathbf{X}, G) \cong \varprojlim_n D_n$. Consider the following commutative diagram:

$$\begin{array}{ccc} M & \longrightarrow & D_n \otimes_{\mathcal{D}(X)\rtimes G} M \\ \downarrow & & \downarrow \\ D_n \otimes_{D_{n+1}} M & \longrightarrow & D_n \otimes_{D_{n+1}} (D_{n+1} \otimes_{\mathcal{D}(\mathbf{X})\rtimes G} M) \end{array}$$

For each n, M has a structure of D_n-module and the natural morphism $M \longrightarrow D_n \otimes_{\mathcal{D}(\mathbf{X})\rtimes G} M$ is an isomorphism. Hence the horizontal arrows are isomorphisms. The right vertical arrow is also an isomorphism by the associativity of tensor product. It follows that the left vertical arrow is isomorphism, so that $M = \lim M$ is a coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module which extends the given $\mathcal{D}(\mathbf{X}) \rtimes G$-action. □

Remark 5.1.5. *The above proposition tells us that an integrable connection $\mathcal{M}$ is strongly G-equivariant if there exists a $\mathbf{X}_w(\mathcal{T})$-covering $\mathcal{U}$ of $\mathbf{X}$ such that for every $\mathbf{U} \in \mathcal{U}$, there is a $\mathbf{U}$-small subgroup of G and a H-stable free Lie lattice of $\mathcal{T}(\mathbf{U})$ such that the condition 5.1 holds for $\mathcal{M}(\mathbf{U})$.*

Corollary 5.1.6. *The structure sheaf $\mathcal{O}_\mathbf{X}$ is strongly G-equivariant. More generally, the sheaf $\mathcal{O}^n_\mathbf{X}$ is strongly G-equivariant for all integer $n \geq 1$.*

Proof. Without loss of generality, we may suppose that $\mathbf{X}$ is affinoid, G is compact such that $(\mathbf{X}, G)$ is small and consider the case where $n = 1$. As a consequence of Proposition 5.1.4, it is enough to show that the module $A := \mathcal{O}(\mathbf{X})$ satisfies the condition (5.1).
Let $\mathcal{A}$ be a G-stable affine formal model of A and suppose that $\mathcal{L}$ is a G-stable $\mathcal{A}$-Lie lattice in $Der_K(A)$. Recall that each $g \in G$ acts on $\mathcal{A}$ via the morphism of groups $\rho : G \longrightarrow Aut(\mathcal{A})$ and on $\mathcal{L}$ via

$$g.x := \rho(g) \circ x \circ \rho(g^{-1}) \text{ for all } x \in \mathcal{L}.$$

Now if $g \in G_\mathcal{L}$, we can write $\rho(g) = \exp(p^\epsilon x)$ with $x \in \mathcal{L}$. Then for $a \in \mathcal{A}$,

$$\beta_\mathcal{L}(g).a = \exp(p^\epsilon \iota(x)).a = \textstyle\sum_n \frac{p^{\epsilon n}}{n!}\iota(x)^n.a = \sum_n \frac{p^{\epsilon n}}{n!}x^n.a = \exp(p^\epsilon x)(a) = \rho(g)(a) = g.a.$$

This proves that $\beta_\mathcal{L}(g) - g$ acts trivially on $\mathcal{A}$, so that the condition (5.1) holds for A. □

Proposition 5.1.7. *Let $\mathcal{M}$ be a strongly G-equivariant connection. Then the natural morphism*

$$\mathcal{M} \longrightarrow E_{\mathbf{X}/G}(\mathcal{M})$$

is an isomorphism in $\mathcal{C}_{\mathbf{X}/G}$. In particular every strongly G-equivariant connection is weakly holonomic.

Proof. Let $\mathcal{M}$ be a strongly G-equivariant connection. We may suppose that $\mathbf{X}$ is affinoid, G is compact and $(\mathbf{X}, G)$ is small. Write $M := \mathcal{M}(\mathbf{X})$. Then it suffices to show that the natural morphism

$$\begin{aligned} i : M &\longrightarrow \widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})\rtimes G} M \\ m &\longmapsto 1 \otimes m. \end{aligned}$$

is an isomorphism. Note that M is a coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-module by assumption. Now it is straightforward that i is $\mathcal{D}(\mathbf{X}) \rtimes G$-linear. Combining with the fact that the ring $\mathcal{D}(\mathbf{X}) \rtimes G$ is (of image) dense in $\widehat{\mathcal{D}}(\mathbf{X}, G)$, we conclude that i is also $\widehat{\mathcal{D}}(\mathbf{X}, G)$-linear. As a consequence

$$1 \otimes am = a \otimes m \text{ for all } a \in \widehat{\mathcal{D}}(\mathbf{X}, G) \text{ and } m \in M.$$

Hence i is surjective.
To see that i is injective, we note that $j \circ i = id_M$, where j denotes the following morphism:

$$\begin{aligned} \widehat{\mathcal{D}}(\mathbf{X}, G) \otimes_{\mathcal{D}(\mathbf{X})\rtimes G} M &\longrightarrow M \\ a \otimes m &\longmapsto am. \end{aligned}$$

So i is an isomorphism of coadmissible $\widehat{\mathcal{D}}(\mathbf{X}, G)$-modules as claimed.
Finally, as integrable connections are of minimal dimension ([12, 2.3.7]), Theorem 4.3.15, together with the isomorphism $\mathcal{M} \cong E_{\mathbf{X}/G}(\mathcal{M})$, implies that $\mathcal{M}$ is weakly holonomic. □

5.2 Examples of weakly holonomic $\mathcal{D}$-modules on rigid analytic flag varieties

It is well-known in the classical theory (see [12]), where X is the complex flag variety $X \simeq G/B$ associated to a semisimple complex Lie group G whose Lie algebra is denoted by $\mathfrak{g}$, that the localisation functor $Loc_X^{U(\mathfrak{g})}(-)$ is an equivalence of categories between the category $Mod(U(\mathfrak{g})_0)$ of $U(\mathfrak{g})_0$-modules (resp. coherent $U(\mathfrak{g})_0$-modules) and the category $Mod(\mathcal{D}_X)$ of $\mathcal{D}_X$-modules (resp. coherent $\mathcal{D}_X$-modules). Here $U(\mathfrak{g})_0$ denotes the quotient ring $U(\mathfrak{g})/\mathfrak{m}_0 U(\mathfrak{g})$ with maximal ideal $\mathfrak{m}_0$ of the center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$. Moreover, the sheaf associated to a B-equivariant $U(\mathfrak{g})_0$-module is a B-equivariant holonomic $\mathcal{D}_X$-module. In this section we study a similar example of equivariant weakly holonomic module on a rigid analytic flag variety induced from the BGG category $\mathcal{O}^{\mathfrak{p}}$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, via the equivalence of categories in the rigid analytic setting ([4, Theorem 6.4.8])

5.2.1 Induction functor

Let $\mathbf{X}$ be a smooth rigid analytic space and G be a p-adic Lie group acting continuously on $\mathbf{X}$. Suppose that P is a closed subgroup of G such that G/P is compact. Note that under this condition, the set of double cosets $|H\backslash G/P|$ is finite for every open subgroup $H \leq G$ ([3, Lemma 3.2.1]).

We recall from [1, 2.2] the geometric induction functor

$$\mathrm{ind}_P^G : \mathcal{C}_{\mathbf{X}/P} \longrightarrow \mathcal{C}_{\mathbf{X}/G}$$

which is locally defined as follows. Let $\mathcal{N} \in \mathcal{C}_{\mathbf{X}/P}$. Let $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ be an affinoid open subset, H be a $\mathbf{U}$-small subgroup of G and $s \in G$. If $J \leq G$ is a subgroup, we write ${}^sJ = sJs^{-1}$, $J^s = s^{-1}Js$. Then we set

$$[s]\mathcal{N}(s^{-1}\mathbf{U}) := \{[s]m : m \in \mathcal{N}(s^{-1}\mathbf{U})\}.$$

Note that H is open in G, the subgroup $P \cap H^s$ is also open in P and the pair $(s^{-1}\mathbf{U}, P \cap H^s)$ is small. Hence $\mathcal{N}(s^{-1}\mathbf{U})\}$ is a $\widehat{\mathcal{D}}(s^{-1}\mathbf{U}, P \cap H^s)$-module. So $[s]\mathcal{N}(s^{-1}\mathbf{U})$ can be equipped with a structure of $\widehat{\mathcal{D}}(\mathbf{U}, {}^sP \cap H)$-module via the isomorphism of K-algebras

$$s^{-1} : \widehat{\mathcal{D}}(\mathbf{U}, {}^sP \cap H) \xrightarrow{\sim} \widehat{\mathcal{D}}(s^{-1}\mathbf{U}, P \cap H^s).$$

This is indeed a coadmissile $\widehat{\mathcal{D}}(\mathbf{U}, {}^sP \cap H)$-module [1, Lemma 2.2.3]. By consequence, we may form the following coadmissible $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module:

$$M(\mathbf{U}, H, s) = \widehat{\mathcal{D}}(\mathbf{U}, H)\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U}, {}^sP\cap H)}[s]\mathcal{N}(s^{-1}\mathbf{U})$$

The $\widehat{\mathcal{D}}(\mathbf{U}, H)$-module $M(\mathbf{U}, H, s)$ only depends on the double coset HsP which contains s ([1, Proposition 3.2.7]), which means that if $t \in HsP$ such that $s = h^{-1}th'$ with $h \in H, h' \in P$, then $M(\mathbf{U}, H, s) \cong M(\mathbf{U}, H, t)$ as $\widehat{\mathcal{D}}(\mathbf{U}, H)$-modules.
This allows us to define for each double coset $Z \in H \setminus G/P$:

$$M(\mathbf{U}, H, Z) := \lim_{s\in Z} M(\mathbf{U}, H, s).$$

Note that $M(\mathbf{U}, H, Z) \cong M(\mathbf{U}, H, s)$ in $\mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{U},H)}$ for all $s \in Z$. Since $| H \setminus G/P |$ is finite, we obtain that

$$M(\mathbf{U}, H) := \bigoplus_{Z\in H\backslash G/P} M(\mathbf{U}, H, Z)$$

is also a coadmissible $\widehat{\mathcal{D}}(\mathbf{U},H)$-module. If $J \leq H$ are $\mathbf{U}$-small subgroups of G then there is an isomorphism of $\widehat{\mathcal{D}}(\mathbf{U},J)$-modules ([1, Proposition 3.2.11]

$$M(\mathbf{U},J)\xrightarrow{\sim}M(\mathbf{U},H).$$

So we can define

$$\begin{aligned}\operatorname{ind}_P^G(\mathcal{N})(\mathbf{U}) &= \varprojlim_H \bigoplus_{Z\in H\backslash G/P} \lim_{s\in Z} \widehat{\mathcal{D}}(\mathbf{U},H)\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H)}[s]\mathcal{N}(s^{-1}\,\mathbf{U})\\ &= \varprojlim_H M(\mathbf{U},H).\end{aligned}$$

Where the inverse limit is taken over the set of $\mathbf{U}$-small subgroups H of G.

By construction, this is a coadmissible $\widehat{\mathcal{D}}(\mathbf{U},H)$-module and is isomorphic to $M(\mathbf{U},H)$ for every $\mathbf{U}$-small subgroup H of G. Also, we obtain a presheaf $\operatorname{ind}_P^G(\mathcal{N})$ of $\mathcal{D}_{\mathbf{X}}$-modules on $\mathbf{X}_w(\mathcal{T})$. Moreover, of course, we want to equip $\operatorname{ind}_P^G(\mathcal{N})$ with a G-action such that this becomes a coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-modules. This is a complicated work due to K. Ardakov ([1]). Since we will not be too interested in this G-equivariant structure, we accept that $\operatorname{ind}_P^G(\mathcal{N}) \in \mathcal{C}_{\mathbf{X}/G}$ for any $\mathcal{N} \in \mathcal{C}_{\mathbf{X}/P}$ as a known result without any concrete explanation.

5.2.2 The result

Let $\mathbf{X}$ be a smooth rigid analytic space and G be a p-adic Lie group acting continuously on $\mathbf{X}$. Suppose that P is a closed subgroup of G such that G/P is compact. We prove below that the induction functor ind_P^G preserves weakly holonomicity.

Proposition 5.2.1. *Let $\mathcal{N}$ be a P-equivariant weakly holonomic $\mathcal{D}_{\mathbf{X}}$-module. Then $ind_P^G(\mathcal{N})$ is a G-equivariant weakly holonomic $\mathcal{D}_{\mathbf{X}}$-module.*

Proof. Since the sum $M(\mathbf{U},H) := \bigoplus_{Z\in H\backslash G/P} M(\mathbf{U},H,Z)$ is finite, one has that

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(M(\mathbf{U},H),\widehat{\mathcal{D}}(\mathbf{U},H)) \cong \bigoplus_{Z\in H\backslash G/P} Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(M(\mathbf{U},H,Z),\widehat{\mathcal{D}}(\mathbf{U},H)).$$

In particular, $Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(M(\mathbf{U},H),\widehat{\mathcal{D}}(\mathbf{U},H)) = 0$ if and only if

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(M(\mathbf{U},H,Z),\widehat{\mathcal{D}}(\mathbf{U},H)) = 0 \text{ for all } Z \in H\setminus G/P.$$

This shows that

$$j(M(\mathbf{U},H)) = \inf\{j(M(\mathbf{U},H,Z)) : Z \in H\setminus G/P\}. \tag{5.2}$$

Now let $Z \in H\setminus G/P$. Since $M(\mathbf{U},H,Z) \cong M(\mathbf{U},H,s)$ in $\mathcal{C}_{\widehat{\mathcal{D}}(\mathbf{U},H)}$ for any choice of $s \in Z$ and the map $\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H) \longrightarrow \widehat{\mathcal{D}}(\mathbf{U},H)$ is faithfully $c-$flat [1, Lemma 3.5.3] (note that ${}^sP\cap H$ is closed in H), we obtain

$$\begin{aligned}&Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},H)}(\widehat{\mathcal{D}}(\mathbf{U},H)\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H)}[s]\mathcal{N}(s^{-1}\,\mathbf{U}),\widehat{\mathcal{D}}(\mathbf{U},H))\\ \cong\ &Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H)}([s]\mathcal{N}(s^{-1}\,\mathbf{U}),\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H))\widehat{\otimes}_{\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H)}\widehat{\mathcal{D}}(\mathbf{U},H).\end{aligned}$$

This implies that:

$$j_{\widehat{\mathcal{D}}(\mathbf{U},H)}(M(\mathbf{U},H,Z)) = j_{\widehat{\mathcal{D}}(\mathbf{U},H)}(M(\mathbf{U},H,s)) = j_{\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H)}([s]\mathcal{N}(s^{-1}\mathbf{U})). \tag{5.3}$$

Next, the isomorphism of K-algebras $\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H)\xrightarrow{\sim}\widehat{\mathcal{D}}(s^{-1}\mathbf{U},P\cap H^s)$ implies that

$$Ext^i_{\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H)}([s]\mathcal{N}(s^{-1}\mathbf{U}),\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H)) \cong Ext^i_{\widehat{\mathcal{D}}(s^{-1}\mathbf{U},P\cap H^s)}(\mathcal{N}(s^{-1}\mathbf{U}),\widehat{\mathcal{D}}(s^{-1}\mathbf{U},P\cap H^s)).$$

By consequence,

$$j_{\widehat{\mathcal{D}}(\mathbf{U},{}^sP\cap H)}([s]\mathcal{N}(s^{-1}\mathbf{U})) = j_{\widehat{\mathcal{D}}(s^{-1}\mathbf{U},P\cap H^s)}(\mathcal{N}(s^{-1}\mathbf{U})). \tag{5.4}$$

Finally, Since $\mathcal{N}$ is P-equivariant weakly holonomic, (5.2), (5.3), (5.4) imply that $d(\mathrm{ind}_P^G(\mathcal{N}))$ is at most $\dim\mathbf{X}$, so that $\mathrm{ind}_P^G(\mathcal{N})$ is also G-equivariant weakly holonomic. □

Now, let $L/\mathbb{Q}_p$ be a finite field extension, $\mathbb{G}_L$ be a connected semisimple algebraic group over L and $\mathbb{P}_L$ be a parabolic subgroup of $\mathbb{G}_L$ which contains a maximal torus $\mathbb{T}_L$ and a Levi subgroup $\mathbb{L}_L$. Let $P\subset G$ denote the corresponding groups of L-valued points. Let $L\subseteq K$ be a complete non-archimedien extension field such that K is a splitting field for $\mathbb{G}_L$. Let $\mathbb{G}=\mathbb{G}_L\times_L K$, $\mathbb{P}=\mathbb{P}_L\times_L K$, $\mathbb{L}=\mathbb{L}_L\times_L K$ and $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{l}$ be its Lie algebras respectively. Let $\mathbb{X}$ be the algebraic flag variety of the split algebraic K-group $\mathbb{G}$. Then there is a natural action of $\mathbb{G}$ on $\mathbb{X}$ given by conjugating the Borel subgroups of $\mathbb{G}$. We denote by $\mathbf{X}=\mathbb{X}^{an}$ the rigid analytification of $\mathbb{X}$, with its induced G-action. Recall that there is a morphism of locally ringed G-spaces

$$\rho:\mathbf{X}\longrightarrow\mathbb{X}$$

and the induced functor

$$\rho^*: Mod(\mathcal{O}_{\mathbb{X}})\longrightarrow Mod(\mathcal{O}_{\mathbf{X}})$$
$$\mathcal{M}\longmapsto\rho^*\mathcal{M}=\mathcal{O}_{\mathbf{X}}\otimes_{\rho^{-1}\mathcal{O}_{\mathbb{X}}}\rho^{-1}\mathcal{M}.$$

In particular, one has that $\rho^*\mathcal{D}_{\mathbb{X}}=\mathcal{D}_{\mathbf{X}}$ and so ρ^* induces a functor from (coherent) $\mathcal{D}_{\mathbb{X}}$-modules to (coherent) $\mathcal{D}_{\mathbf{X}}$-modules.

Our next result needs to make use of the following lemma:

Lemma 5.2.2. *Let $\mathbb{X}$ be a proper smooth K-scheme of dimension d. Suppose that $\mathbb{X}$ is $\mathcal{D}_{\mathbb{X}}$-affine. Let $\mathcal{M}$ be a coherent $\mathcal{D}_{\mathbb{X}}$-module which is holonomic over $\mathbb{X}$. Then $\rho^*\mathcal{M}$ is a $\mathcal{D}_{\mathbf{X}}$-module of minimal dimension on $\mathbf{X}$.*

Proof. Let $\mathcal{M}$ be a coherent $\mathcal{D}_{\mathbb{X}}$-module. As $\mathbb{X}$ is proper, [3, Proposition 2.2.1] implies that $\Gamma(\mathbf{X},\rho^*\mathcal{M})=\Gamma(\mathbb{X},\mathcal{M}):=M$. In particular $\Gamma(\mathbb{X},\mathcal{D}_{\mathbb{X}})=\Gamma(\mathbf{X},\mathcal{D}_{\mathbf{X}}):=D$, as $\rho^*\mathcal{D}_{\mathbb{X}}=\mathcal{D}_{\mathbf{X}}$ by [3, Proposition 2.2.2(*a*)]. Now, since $\mathbb{X}$ is $\mathcal{D}_{\mathbb{X}}$-affine, one has that $\mathcal{M}\cong Loc^{\mathcal{D}}_{\mathbb{X}}(M)=\mathcal{D}_{\mathbb{X}}\otimes_D M$. Combining with the fact that ρ^* is an exact and fully faithful functor (Proposition 2.1.11(i)), this implies that $\rho^*\mathcal{M}\cong Loc^{D}_{\mathbf{X}}(M)=\mathcal{D}_{\mathbf{X}}\otimes_D M$. Now let U be an open affine subdomain of $\mathbb{X}$ and $\mathcal{U}=\{\mathbf{U}_i, i\in I\}$ be an admissible covering of $\rho^{-1}U$ by affinoid subdomains of $\mathbf{X}$. As ρ is flat, we may even suppose that for every $i\in I$, the morphism $\mathcal{O}_{\mathbb{X}}(U)\longrightarrow\mathcal{O}_{\mathbf{X}}(\mathbf{U}_i)$ is flat. The above argument gives

$$\rho^*\mathcal{M}(\mathbf{U}_i)=\mathcal{D}_{\mathbf{X}}(\mathbf{U}_i)\otimes_D M\cong\mathcal{D}_{\mathbf{X}}(\mathbf{U}_i)\otimes_{\mathcal{D}_{\mathbb{X}}(U)}\mathcal{M}(U).$$

Now since $\mathcal{O}_{\mathbb{X}}(U)\longrightarrow\mathcal{O}_{\mathbf{X}}(\mathbf{U}_i)$ is flat, we deduce that $\mathcal{D}_{\mathbb{X}}(U)\longrightarrow\mathcal{D}_{\mathbf{X}}(\mathbf{U}_i)$ is also flat. Therefore

$$Ext^n_{\mathcal{D}_{\mathbf{X}}(\mathbf{U}_i)}(\rho^*\mathcal{M}(\mathbf{U}_i),\mathcal{D}_{\mathbf{X}}(\mathbf{U}_i))\cong Ext^n_{\mathcal{D}_{\mathbb{X}}(U)}(\mathcal{M}(U),\mathcal{D}_{\mathbb{X}}(U))\otimes_{\mathcal{D}_{\mathbb{X}}(U)}\mathcal{D}_{\mathbf{X}}(\mathbf{U}_i).$$

By consequence

$$j_{\mathcal{D}_{\mathbb{X}(U)}}(\mathcal{M}(U)) \leq j_{\mathcal{D}_{\mathbf{X}(\mathbf{U}_i)}}(\rho^*\mathcal{M}(\mathbf{U}_i)).$$

So if $\mathcal{M}$ is holonomic, then $j_{\mathcal{D}_{\mathbb{X}(U)}}(\mathcal{M}(\mathbf{U})) = d$, which implies that $j_{\mathcal{D}_{\mathbf{X}(\mathbf{U}_i)}}(\rho^*\mathcal{M}(\mathbf{U}_i)) \geq d$, so that $d_{\mathcal{D}_{\mathbf{X}(\mathbf{U}_i)}}(\rho^*\mathcal{M}(\mathbf{U}_i)) \leq d$ for every i and $\rho^*\mathcal{M}$ is of minimal dimension as claimed. □

The rest of this subsection is devoted to giving a class of G-equivariant weakly holonomic modules on the rigid analytic flag variety $\mathbf{X}$. First, let us recall that the BGG category $\mathcal{O}^{\mathfrak{p}}$ associated to the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be the Levi decomposition induced from $\mathbb{L}$. Then $\mathcal{O}^{\mathfrak{p}}$ consists of all $U(\mathfrak{g})$-modules M satisfying the following conditions:

(1) M is a finitely generated $U(\mathfrak{g})$-module.

(2) M is a direct sum of finite dimensional simple $U(\mathfrak{l})$-modules.

(3) M is locally $U(\mathfrak{u})$-finite.

We denote by $\mathcal{O}_0^{\mathfrak{p}}$ the subcategory of $\mathcal{O}^{\mathfrak{p}}$ consisting of all modules $M \in \mathcal{O}^{\mathfrak{p}}$ such that $\mathfrak{m}_0 M = 0$ for the maximal ideal $\mathfrak{m}_0 = Z(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{g}$ of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

Next, let us recall from [1, Section 6.2] that there is a Fréchet-Stein K-algebra $\widehat{U}(\mathfrak{g}, H)$ associated to each open compact subgroup H of G, which is defined as follows. For every H-stable Lie lattice $\mathcal{L}$ in $\mathfrak{g}$, there exists a normal subgroup $H_{\mathcal{L}}$ of $\mathfrak{g}$ and a morphism $H_{\mathcal{L}} \longrightarrow \widehat{U(\mathcal{L})_K}^{\times}$ which is a H-equivariant trivialisation. So we can form the ring $\widehat{U(\mathcal{L})_K} \rtimes_N H$ whenever N is an open normal subgroup of H contained in $H_{\mathcal{L}}$ and we define

$$\widehat{U}(\mathfrak{g}, H) = \varprojlim_{(\mathcal{L},N)} \widehat{U(\mathcal{L})_K} \rtimes_N H$$

with the limit is taken over all the pairs $(\mathcal{L}, N)$, where H-stable Lie lattice $\mathcal{L}$ of $\mathfrak{g}$ and N be an open normal subgroup of G contained in $H_{\mathcal{L}}$. So $\widehat{U}(\mathfrak{g}, H)$ can be viewed as a certain completion of the skew group ring $U(\mathfrak{g}) \rtimes H$.

In the sequel, we also make use of the following notions:

* $D(G, K), D(P, K)$ are the algebras of K-valued locally L-analytic distributions on G and P respectively.
* $D(\mathfrak{g}, P)$ is the K-subalgebra of $D(G, K)$ which is generated by $U(\mathfrak{g})$ and $D(P, K)$.
* $\widehat{U}(\mathfrak{g}, P)$ is an associative K-algebra which is equal to $\widehat{U}(\mathfrak{g}, H) \otimes_{K[H]} K[P]$ for some choice of open compact subgroup H of P. Note that this definition does not depend on the choice of H, meaning that if $N \leq H$ are open compact subgroups of G, then $\widehat{U}(\mathfrak{g}, N) \otimes_{K[N]} K[P] \cong \widehat{U}(\mathfrak{g}, H) \otimes_{K[H]} K[P]$. Similarly, we have the K-algebra $\widehat{U}(\mathfrak{g}, G)$.

It is worth pointing out that the K-algebras $\widehat{U}(\mathfrak{g}, P), \widehat{U}(\mathfrak{g}, G), D(P, K), D(G, K)$, are all Fréchet algebras. If G is compact (so is P), then they are Fréchet-Stein K-algebras.

By coadmissible $\widehat{U}(\mathfrak{g}, P)$-module, we mean that a $\widehat{U}(\mathfrak{g}, P)$-module M which is a coadmissible $\widehat{U}(\mathfrak{g}, H)$-module for every open compact subgroup H of P. Following [4], it is possible to localize a coadmissible $\widehat{U}(\mathfrak{g}, P)$-module M to obtain a coadmissible P-equivariant $\mathcal{D}_{\mathbf{X}}$-module $Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},P)}(M)$

on the rigid analytic flag variety $\mathbf{X}$. The construction of $Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},P)}(M)$ is exactly the same as for $Loc_{\mathbf{X}}^{\widehat{\mathcal{D}}(\mathbf{X},G)}(-)$, when $(\mathbf{X},G)$ small, where we interchange $\widehat{\mathcal{D}}(\mathbf{X},G)$ and $\widehat{U}(\mathfrak{g},P)$ in their construction. (see [4] for more details).

Now let $M \in \mathcal{O}_0^{\mathfrak{p}}$. Then it is proved ([3, Lemma 4.1.2] that M admits a P-action that lifts the given $\mathfrak{p}$-action on it. In the following, we let $\underline{M}$ denote the module $M \in \mathcal{O}_0^{\mathfrak{p}}$ together with this P-action in order to distinguish the initial module M (without P-action).
Each $\underline{M} \in \mathcal{O}_0^{\mathfrak{p}}$ has a structure of $D(\mathfrak{g},P)$-module ([19, 3.4]), so we can form the $\widehat{U}(\mathfrak{g},P)$-module

$$\widehat{\underline{M}} := \widehat{U}(\mathfrak{g},P) \otimes_{D(\mathfrak{g},P)} \underline{M}.$$

It is shown [3, Proposition 4.2.1] that $\widehat{\underline{M}}$ is a coadmissible $\widehat{U}(\mathfrak{g},P)$-module. Therefore, we may 'localize' $\widehat{\underline{M}}$ to obtain a P-equivariant coadmissible $\mathcal{D}_{\mathbf{X}}$-module $Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},P)}(\widehat{\underline{M}})$ on $\mathbf{X}$. On the other hand we may also 'localize' the coadmissible $\widehat{U}(\mathfrak{g})$-module $\widehat{M} := \widehat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$ to obtain a coadmissible $\widehat{\mathcal{D}}_{\mathbf{X}}$-module $Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g})}(\widehat{M})$.

Proposition 5.2.3. *Let $M \in \mathcal{O}_0^{\mathfrak{p}}$. Then $Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},P)}(\widehat{\underline{M}})$ is a P-equivariant weakly holonomic $\mathcal{D}_{\mathbf{X}}$-module.*

Proof. It is proved [3, Proposition 4.4.1] that

$$Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},P)}(\widehat{\underline{M}}) \cong Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g})}(\widehat{M}) \cong E_X \circ Loc_{\mathbf{X}}^{U(\mathfrak{g})}(M).$$

Here

$$\begin{aligned} E_X : Mod(\mathcal{D}_X) &\longrightarrow Mod(\widehat{\mathcal{D}}_{\mathbf{X}}) \\ \mathcal{M} &\longmapsto \widehat{\mathcal{D}}_{\mathbf{X}} \otimes_{\mathcal{D}_X} \mathcal{M} \end{aligned}$$

is the extension functor which sends coherent $\mathcal{D}_{\mathbf{X}}$-modules to coadmissible $\widehat{\mathcal{D}}_{\mathbf{X}}$-modules [2, 7.2] and $Loc_{\mathbf{X}}^{U(\mathfrak{g})}$ is the composition of the localisation functor

$$Loc_{\mathbb{X}}^{U(\mathfrak{g})} : coh(U(\mathfrak{g})) \longrightarrow coh(\mathcal{D}_{\mathbb{X}})$$

and the rigid analytification functor

$$\begin{aligned} \rho^* : Mod(\mathcal{O}_{\mathbb{X}}) &\longrightarrow Mod(\mathcal{O}_{\mathbf{X}}) \\ \mathcal{M} &\longrightarrow \mathcal{O}_{\mathbf{X}} \otimes_{\rho^{-1}\mathcal{O}_{\mathbb{X}}} \rho^{-1}\mathcal{M}. \end{aligned}$$

Note that $Loc_{\mathbb{X}}^{U(\mathfrak{g})}(M)$ is a P-equivariant coherent $\mathcal{D}_{\mathbb{X}}$-module. Hence by [12, Theorem 11.6.1(i)], this is a holonomic module. Using Lemma 5.2.2, we then obtain that $\rho^* Loc_{\mathbb{X}}^{U(\mathfrak{g})}(M) = Loc_{\mathbf{X}}^{U(\mathfrak{g})}(M)$ is a $\mathcal{D}_{\mathbf{X}}$-module of minimal dimension. Now, [2, Proposition 7.2] ensures that $Loc_{\mathbf{X}}^{\widehat{U(\mathfrak{g})}}(\widehat{M}) \cong E_{\mathbf{X}} \circ Loc_{\mathbf{X}}^{U(\mathfrak{g})}(M)$ is a weakly holonomic $\widehat{\mathcal{D}}_{\mathbf{X}}$-module. This can be used to prove the result. Indeed, it remains to prove that for any affinoid subdomain $\mathbf{U} \in \mathbf{X}_w(\mathcal{T})$ and any $\mathbf{U}$-small subgroup H of P, one has that

$$d_{\widehat{\mathcal{D}}(\mathbf{U})}(Loc_{\mathbf{X}}^{\widehat{U(\mathfrak{g})}}(\widehat{M})(\mathbf{U})) \geq d_{\widehat{\mathcal{D}}(\mathbf{U},H)}(Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},P)}(\widehat{\underline{M}})(\mathbf{U})).$$

Choose a H-stable free $\mathcal{A}$-Lie lattice $\mathcal{L}$ for some G-stable affine model $\mathcal{A}$ of $\mathcal{O}(\mathbf{U})$ and a good chain (J_n) for $\mathcal{L}$ such that

$$\widehat{\mathcal{D}}(\mathbf{U}, H) \cong \varprojlim_n U_n \rtimes_{J_n} H \quad \text{and} \quad \widehat{\mathcal{D}}(\mathbf{U}) \cong \varprojlim_n U_n,$$

where $U_n := \widehat{U(\pi^n \mathcal{L})}_K$ for all $n \geq 0$. Then

$$Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g})}(\widehat{M})(\mathbf{U}) = \widehat{\mathcal{D}}(\mathbf{U}) \widehat{\otimes}_{\widehat{U}(\mathfrak{g})} \widehat{M} \cong \varprojlim_n (U_n \otimes_{\widehat{U}(\mathfrak{g})} \widehat{M})$$

and

$$Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},P)}(\widehat{\underline{M}})(\mathbf{U}) \cong \widehat{\mathcal{D}}(\mathbf{U}, H) \widehat{\otimes}_{\widehat{U}(\mathfrak{g},H)} \widehat{\underline{M}} \cong \varprojlim_n (U_n \rtimes_{J_n} H) \otimes_{\widehat{U}(\mathfrak{g},H)} \widehat{\underline{M}}.$$

Write $N_n := U_n \otimes_{\widehat{U(\mathfrak{g})}} \widehat{M}$ and $N_n' := (U_n \rtimes_{J_n} H) \otimes_{\widehat{U(\mathfrak{g},H)}} \widehat{\underline{M}}$. As both $\widehat{\mathcal{D}}(\mathbf{U})$ and $\widehat{\mathcal{D}}(\mathbf{U}, H)$ are c-Auslander Gorenstein, we may assume that U_n and $U_n \rtimes_{J_n} H$ are all Auslander-Gorenstein of dimension at most $2d$. Furthermore there exist n sufficiently large such that

$$d_{\widehat{\mathcal{D}}(\mathbf{U})}(Loc_{\mathbf{X}}^{\widehat{U(\mathfrak{g})}}(\widehat{M})(\mathbf{U})) = d_{U_n}(N_n)$$

and

$$d_{\widehat{\mathcal{D}}(\mathbf{U},H)}(Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},P)}(\widehat{\underline{M}})(\mathbf{U})) = d_{U_n \rtimes_{J_n} H}(N_n') = d_{U_n}(N_n').$$

Here, the last equality follows from Proposition 3.1.4. It reduces to prove that $d_{U_n}(N_n') \leq d_{U_n}(N_n) \leq d$ ($Loc_{\mathbf{X}}^{\widehat{U(\mathfrak{g})}}(\widehat{M})$) is weakly holonomic). For this we note that the natural map

$$f_n : N_n \longrightarrow N_n'$$

is surjective [3, Proposition 4.4.1], which means that $N_n' = N_n/I_n$ for some finitely generated U_n-module I_n. We then obtain an exact sequence of U_n-modules

$$0 \longrightarrow I_n \longrightarrow N_n \longrightarrow N_n' \longrightarrow 0.$$

Now since each U_n is Auslander-Gorenstein, we can apply [16, Proposition 4.5] to obtain $d_{U_n}(N_n) \geq d_{U_n}(N_n')$, so that $d_{U_n}(N_n') \leq d$ and the result follows. □

Proposition 5.2.4. *Let $\underline{M} \in \mathcal{O}_0^{\mathfrak{g}}$. Then $ind_P^G(Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},P)}(\widehat{\underline{M}}))$ is a G-equivariant weakly holonomic $\mathcal{D}_{\mathbf{X}}$-module.*

Proof. This follows immediately from Proposition 5.2.1 and Propostion 5.2.3. □

Here is another point of view. There is also a localization functor $Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},G)}(-)$ which sends coadmissible $\widehat{U}(\mathfrak{g}, G)$-modules to $\mathcal{C}_{\mathbf{X}/G}$. Moreover, if we set

$$\widehat{U}(\mathfrak{g}, G)_0 = \widehat{U}(\mathfrak{g}, G)/\mathfrak{m}_0 \widehat{U}(\mathfrak{g}, G),$$

then this functor is an equivalence of categories between the category $\mathcal{C}_{\widehat{U}(\mathfrak{g},G)_0}$ of coadmissible $\widehat{U}(\mathfrak{g}, G)$-modules annihilated by $\mathfrak{m}_0$ and $\mathcal{C}_{\mathbf{X}/G}$. ([1, Theorem 6.4.8].
Note that $D(G, K) \cong \widehat{U}(\mathfrak{g}, G)$ ([1, Theorem 6.5.1]. We make use of the Orlik-Strauch functor ([19])

$$\begin{aligned} \mathcal{F}_P^G(-)' : \mathcal{O}_0^{\mathfrak{p}} &\longrightarrow \mathcal{C}_{D(G,K)_0} \\ \underline{M} &\longmapsto D(G, K) \otimes_{D(\mathfrak{g},P)} \underline{M}. \end{aligned}$$

(Recall that $\underline{M}$ denotes the module $M \in \mathcal{O}_0^{\mathfrak{p}}$ together with the induced P-action). Then for each $M \in \mathcal{O}_0^{\mathfrak{p}}$, $\mathcal{F}_P^G(\underline{M})'$ is a coadmissible $\widehat{U}(\mathfrak{g}, G)_0$-module. Hence we may form the coadmissible G-equivariant $\mathcal{D}_{\mathbf{X}}$-module $Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},G)}(\mathcal{F}_P^G(\underline{M})')$. Now, in [3, Theorem 4.4.2], the authors have proved that the diagram of functors

$$\begin{array}{ccc} \mathcal{O}_0^P & \xrightarrow{\mathcal{F}_P^G(-)'} & \mathcal{C}_{\widehat{U}(\mathfrak{g},G)_0} \\ {\scriptstyle E_X \circ Loc_{\mathbf{X}}^{U(\mathfrak{g})}} \downarrow & & \downarrow {\scriptstyle Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},G)}} \\ \mathcal{C}_{\mathbf{X}/P} & \xrightarrow{\mathrm{ind}_P^G} & \mathcal{C}_{\mathbf{X}/G} \end{array}$$

is commutative. Then the above proposition is equivalent to:

Proposition 5.2.5. $Loc_{\mathbf{X}}^{\widehat{U}(\mathfrak{g},G)}(\mathcal{F}_P^G(\underline{M})')$ *is a G-equivariant weakly holonomic module for any $U(\mathfrak{g})$-module $M \in \mathcal{O}_0^{\mathfrak{p}}$.*

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Titre : "Holonomicité faible pour les $\mathcal{D}$-modules équivariants sur les espaces rigides analytiques."

Mot clés : Espaces rigides analytiques, Équivariant $\mathcal{D}$-modules, Holonomicité , Représentations p-adiques.

Résumé : L'objectif de la thèse est d'établir une notion de la holonomicité faible pour les $\mathcal{D}$-modules équivariants coadmissibles sur quelques classes d'espaces rigides analytiques, dont les variétés (rigides analytiques) de drapeaux. Plus précisément, on develope une théorie de dimension pour la catégorie des $\mathcal{D}$-modules équivariant coadmissibles, puis on montre l'inégalité de Bernstein dans des cas plus concrètes, comme le cas des variétés (rigides analytiques) de drapeaux. Ce résulta nous permet donc à définir la notion de holonomicité faible. La dernière partie de cette thèse est consacrée à donner quelques examples typiques qui sont généralisés de la théorie classique.

Title: "Weak holonomicity for equivariant $\mathcal{D}$-modules on rigid analytic spaces."

Keywords: Rigid analytic spaces, Equivariant $\mathcal{D}$-modules, Holonomicity, p-adic representations.

Abstract: The aim of the dissertation is to define the notion of weak holonomicity for coadmissible equivariant $\mathcal{D}$-modules on some classes of rigid analytic spaces including rigid analytic flag varieties. More precisely, we develope a dimension theory for the category of coadmissible equivariant $\mathcal{D}$-modules, then we prove the so-called Bernstein's inequality for some specific varieties, as rigid analytic flag varieties. This result allow us to define weak holonomicity. The last section is devoted to give some typical examples generalized from the classical theory.