Prismatic cohomology

Outline

(i) Sites and sheaves

(ii) The prismatic site

(iii) "Computational tools", Hodge–Tate comparison map.

Sites and sheaves [ignore set theory]

Definition A *site* $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ consists of a category $\mathcal{C}$ and a collection of *coverings* $\mathrm{Cov}(\mathcal{C}) \ni \{U_i \to U\}_{i \in I}$ (fixed target)

such that the following hold:

(i) $U_i \xrightarrow{\sim} U \Rightarrow \{U_i \xrightarrow{\sim} U\} \in \mathrm{Cov}(\mathcal{C})$.

(ii) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$, $\{U_{ij} \to U_i\}_{j \in J(i)} \in \mathrm{Cov}(\mathcal{C})$ for all $i \in I$, then $\{U_{ij} \to U_i \to U\}_{j \in J(i), i \in I} \in \mathrm{Cov}(\mathcal{C})$.

(iii) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$, then $\{\underbrace{V \times_U U_i}_{\text{existence part of axiom}} \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$.

Example Let X be a topological space $\rightsquigarrow$ category with objects open sets, morphisms inclusions.

With ordinary open covers, this gives X the structure of a site: The *Zariski site* X_{Zar}.

Example Let $\mathcal{C}$ be a category. This always admits a *chaotic* site structure:

$\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C}) \Leftrightarrow I = \{i\}$, $U_i \xrightarrow{\sim} U$.

Definition A presheaf $\mathcal{F}: \mathcal{C}^{op} \to \mathrm{Set}$ is a sheaf if the diagram

$$
\begin{tikzcd}
\mathcal{F}(U) \arrow[r] & \prod_{i \in I} \mathcal{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{j,h} \mathcal{F}(U_j \times_U U_h) \\
X \arrow[u, dashed, "\exists!"] \arrow[ur] & &
\end{tikzcd}
$$

is a limit diagram for each $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$.

Example For $\mathcal{C} = X_{zar}$, this recovers the usual definition of a sheaf.

Example If $\mathcal{C}$ is a category with the chaotic site structure, the sheaf condition is vacuous: Every presheaf is a sheaf.

The prismatic site

Let $(A, I) \in \mathrm{Prism}$, $R \in \mathrm{CRing}_{A/I}/$ [formally smooth for some purposes.]

Informally, $(R/A)_{\Delta}$ is defined as the following "pullback":

$$
\begin{tikzcd}
(R/A)_{\Delta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{CRing}_{R}/ \ni C \arrow[d] \arrow[d, bend left, start anchor=east] \\
\mathrm{Prism}_{(A,I)}/ \arrow[r] & \mathrm{CRing}_{A/I}/ \ni C \\
(B, IB) \arrow[r, mapsto] & B/IB
\end{tikzcd}
$$

[should consider the opposite category?]

Definition The prismatic site $(R/A)_{\Delta}$ is the following category with the chaotic site structure: The objects are prisms over (A,I) with an A/I-alg. map $R \to B/IB$.

We shall write

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A/I \arrow[r] & R \arrow[r] & B/IB
\end{tikzcd}
\quad = \quad (R \to B/IB \leftarrow B)
$$

for objects of $(R/A)_{\Delta}$.

Remark Bhatt-Scholze use something closer to the flat topology. We will use the chaotic topology in the affine case, and globalize via Hodge-Tate.

Example If $R = A/I$, then $(R/A)_{\Delta} = \mathrm{Prism}_{(A,I)/}$.

Example Let $R = A/I\langle x\rangle := A/I[x]^{\wedge}_p$.

$\delta(x)=0$, $\tilde{R} := \widehat{A\langle x\rangle}_{(p,I)} \rightsquigarrow (R \xrightarrow{\sim} \tilde{R}/I\tilde{R} \leftarrow \tilde{R})$.

If $R \in \mathrm{CRing}_{A/I}$ formally smooth, can choose $\tilde{R}$ with $\tilde{R}/I\tilde{R} \cong R$.

Prismatic cohomology

Definition Let $\mathcal{C}$ be a site with a terminal object X. For $\mathcal{F} : \mathcal{C}^{op} \to \mathrm{Ab}$ a sheaf on $\mathcal{C}$, set $\Gamma(\mathcal{C}, \mathcal{F}) := \mathcal{F}(X)$. This is left-exact

$\rightsquigarrow R\Gamma(\mathcal{C}, \mathcal{F}) := \Gamma(\mathcal{F}, I^{\bullet})$, $\mathcal{F} \xrightarrow{\sim} I^{\bullet}$ injective resolution.

Example If $\mathcal{C} = X_{zar}$, $R\Gamma(X_{zar}, \mathcal{F})$ is sheaf cohomology.

Remark Since $X \in \mathcal{C}$ is terminal, $\mathcal{F}(X) = \lim_{Y \in \mathcal{C}} \mathcal{F}(Y)$.

Definition If $\mathcal{C}$ is a category with the chaotic site structure, we <u>define</u> $\Gamma(\mathcal{C}, \mathcal{F}) := \lim_{X \in \mathcal{F}} \mathcal{F}(X)$.

$$\rightsquigarrow R\Gamma(\mathcal{C}, \mathcal{F}) = R\lim_{X \in \mathcal{F}} \mathcal{F}(X).$$

Remark This cohomology is always trivial if $\mathcal{C}$ has a terminal object.

Definition We define sheaves on $(R/A)_{\Delta}$ by

$$\mathcal{O}_{\Delta} : (R/A)_{\Delta} \to (\mathrm{CRing}_{\delta})^{\wedge(p,I)}_{A/}, \quad (R \to B/IB \leftarrow B) \mapsto B$$

$$\overline{\mathcal{O}}_{\Delta} : (R/A)_{\Delta} \to \mathrm{CRing}^{\wedge p}_{A/I/}, \quad (R \to B/IB \leftarrow B) \mapsto B/IB.$$

From this we obtain

- <u>prismatic cohomology</u> $\Delta_{R/A} := R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$
- <u>Hodge-Tate cohomology</u> $\overline{\Delta}_{R/A} := R\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta})$.

Example If $R = A/I$, $(R/A)^{op}_{\Delta} = \mathrm{Prism}^{op}_{(A,I)}$ has a terminal object $\Rightarrow \Delta_{R/A} \cong A$, $\overline{\Delta}_{R/A} \cong A/I$.

Towards Hodge–Tate

Let $E^* \in \mathrm{cdga}_{B/}$. We say E^* is <u>strictly commutative</u> if $x^2 = 0$ for $x \in E^{2i+1}$. Since $x^2 = -x^2$, this is only additional data in characteristic 2.

Let $B \to C$ be a map of rings. The <u>de Rham complex</u>

$$(\Omega^*_{C/B}, d_{dR}) = (C \xrightarrow{d} \Omega^1_{C/B} \xrightarrow{d} \Omega^2_{C/B} \xrightarrow{d} \cdots)$$

is initial among cdgas over B with a map $C \to E^0$.

A slightly sharper formulation is:

Lemma $(\Omega^*_{C/B}, d_{dR})$ is initial among $(E^*, d) \in \mathrm{cdga}_{B/}$

(i) $E^i = 0 \ \forall i < 0$;

(ii) $\exists\, \eta : C \to E^0$;

(iii) $d(\eta(c))^2 = 0$.

Proof sketch

$$\begin{tikzcd} C \arrow[r, "\eta"] \arrow[dr] & E^0 \arrow[r, "d"] & E^1 \\ & \Omega^1_{C/B} \arrow[ur, dashed] & \end{tikzcd} \in \mathrm{Der}_B(C, E^1) \rightsquigarrow \begin{tikzcd} \Omega^1_{C/B} \otimes_C \Omega^1_{C/B} \arrow[r] \arrow[d] & E^1 \otimes_C E^1 \arrow[d] \\ \Omega^2_{C/B} \arrow[r, dashed] & E^2 \end{tikzcd}. \quad \square$$

Since $\overline{\mathcal{O}}_{\Delta}$ takes values in $\mathrm{CRing}^{op}_{A/I/}$, $H^*(\overline{\Delta}_{R/A})$ will be a graded commutative ring. Moreover, there is a natural differential: If $I = (d)$, then the short exact sequence

$$0 \to \overline{\mathcal{O}}_{\Delta} \xrightarrow{\cdot d} \mathcal{O}_{\Delta}/d^2 \to \overline{\mathcal{O}}_{\Delta} \to 0$$

gives rise to a map $H^*(\bar{\Delta}_{R/A}) \xrightarrow{\beta_d^*} H^{*+1}(\bar{\Delta}_{R/A})$ on cohomology.

Goal Construct a map $(\Omega^*_{(A/I)/R}, d_{dR}) \to (H^*(\bar{\Delta}_{R/A}), \beta_d^*)$.
For this, it suffices to show that $\beta_d^0 (\eta(t))^2 = 0 \quad \forall t \in R$.

More on cohomology over the cha0tic site

Currently, we sit in-between two extremes:

(i) If $\mathcal{C}$ has a terminal object, all higher cohomology vanishes.

(ii) If $\mathcal{C}$ does not have a terminal object, we have little control over "$R\lim_{X \in \mathcal{C}} \mathcal{F}(X)$" in general.

The following is a useful compromise:

Definition An object $X \in \mathcal{C}$ is <u>weakly terminal</u> if for all $Y \in \mathcal{C}$ there exists a map $Y \to X$.

Example If $\mathcal{C} = \mathrm{Set}$, every nonempty set is weakly terminal. Terminal $\Leftrightarrow$ singleton.

Lemma If $X \in \mathcal{C}$ is weakly terminal, then $R\Gamma(\mathcal{C}, \mathcal{F})$ can be computed by the <u>Čech-Alexander complex</u>

$$\mathcal{F}(X) \rightrightarrows \mathcal{F}(\underbrace{X \times X}_{\text{additional assumption}}) \substack{\to \\ \to \\ \to} \mathcal{F}(X \times X \times X) \cdots$$

Subgoal Find a weakly terminal object in $(R/A)_{\Delta}^{op}$.

Lemma (Existence of prismatic envelopes) Let (A, I) be a prism and let $(A, I) \to (B, J)$ be a map of δ-pairs.

Then there is a universal map of prisms

$$\begin{tikzcd} (A, I) \arrow[rr] \arrow[dr] & & (C, IC) \\ & (B, J) \arrow[ur] & \end{tikzcd}$$

Sometimes one writes $C = B\{\frac{J}{I}\}^{\wedge}$.

Remark If $(p, d, x_1, \dots, x_r) = J$, then $B\{\frac{J}{I}\} = B\{\frac{x_i}{d}\}$ under some hypotheses: Details next talk!

Corollary $CR(A)_\Delta$ admits finite coproducts.

Proof sketch $(R \to B/IB \leftarrow B)$ and $(R \to C/IC \leftarrow C)$, may consider $D_0 := B \otimes_A C$.

There is no reason that

$$\begin{tikzcd} R \arrow[r] \arrow[d] & B/IB \arrow[d] \\ C/IC \arrow[r] & D_0/ID_0 \end{tikzcd}$$

should commute, so we fix the failure: $J := \ker(D_0 \to C/IC \otimes_R B/IB)$, and then we form $(A, I) \to (D_0, J) \to (D_0\{\frac{J}{I}\}, ID_0\{\frac{J}{I}\})$.
This is the coproduct. $\square$

Construction of Cech-Alexander complexes

goal: Construct a weakly initial object in $CR(A)_\Delta$.

Let $F_0 := A\{\tilde{R}\}$, where $\tilde{R}$ is a formally smooth lift of R.
This comes with a surjection $F_0 \twoheadrightarrow R$; let J be the kernel.

We obtain a diagram

send generators to lifts of images in B/IB

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] \arrow[rr, bend left] & F_0\{\frac{J}{I}\} =: F \arrow[r, dashed] \arrow[d] & B \arrow[d] \\ A/I \arrow[r] & F_0/J \arrow[r] & F/IF \arrow[r] & B/IB \\ & R \arrow[u, phantom, "\cong"] \arrow[urr, bend right] & & \end{tikzcd}$$

The claim is that $R \to F/IF \leftarrow F$ is weakly initial, which follows from the orange diagram.

Upshot $H^*(\Delta_{R/A})$ can be computed as the cohomology of the chain complex
$$F^0 \xrightarrow{\varepsilon_0 - \varepsilon_1} F^1 \xrightarrow{\varepsilon_0 - \varepsilon_1 + \varepsilon_2} F^2 \to \cdots,$$
where $F^q := \mathcal{O}_{\Delta_{R/A}}(F^{\otimes(1+q)})$.

! We write $\varepsilon_i : F^q \to F^{1+q}$ for the cosimplicial structure map.

Remark This is the cochain complex associated to a diagram of δ-rings. Consequently, each differential will commute with ϕ.

Lemma Consider $H^*(\bar{\Delta}_{R/A})$ as a cdga with differential β_d^* induced by $0 \to \bar{\mathcal{O}}_\Delta \to \mathcal{O}_\Delta/(d^2) \to \bar{\mathcal{O}}_\Delta \to 0$.

Then $\beta_d^0(\eta(t))^2 = 0 \quad \forall t \in R$.

Proof Let us first give a convenient description of $\beta_d^0(\eta(t))$.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
F^0 \arrow[r] \arrow[d, "\cdot d"] & F^1 \ni a \arrow[r] \arrow[d, "\cdot d"] & F^2 \arrow[r] \arrow[d, "\cdot d"] & \cdots \\
T \in F^0 \arrow[r] \arrow[d] & F^1 \ni \underset{\varepsilon^0(T)-\varepsilon^1(T)}{\overset{da}{\shortparallel}} \arrow[r] \arrow[d] & F^2 \arrow[r] \arrow[d] & \cdots \\
\eta(t) \in F^0/d \arrow[r] \arrow[d] & 0 \in F^1/d \arrow[r] \arrow[d] & F^2/d \arrow[r] \arrow[d] & \cdots \\
0 & 0 & 0
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d, "\beta_d^0(\eta(t))"] \\
F^0/d \arrow[r] \arrow[d, "\cdot d"] & F^1/d \arrow[r] \arrow[d, "\cdot d"] & \cdots \\
T \in F^0/d^2 \arrow[r] \arrow[d] & F^1/d^2 \ni \varepsilon^0(T)-\varepsilon^1(T) = da \arrow[r] \arrow[d] & \cdots \\
\eta(t) \in F^0/d \arrow[r] \arrow[d] & F^1/d \arrow[r] \arrow[d] & \cdots \\
0 & 0
\end{tikzcd}$$

Begin with $\eta(t) \in F^0/d$. Since $F^\bullet/d$ is R-linear (!), $\eta(t) \mapsto 0 \in F^1/d$. Pick a lift $T \in F^0$.

This maps to $\varepsilon^0(T) - \varepsilon^1(T) = da$, which comes from $a \in F^1$.

Playing the same game on the right-hand side, we find $\beta_d^0\, \eta(t) = [a]$.

So our goal is to show that $a \cup a = 0$ in $H^2(F^\cdot / dF^\cdot)$,

or equivalently $(\varepsilon^0(T) - \varepsilon^1(T)) \cup (\varepsilon^0(T) - \varepsilon^1(T)) = 0$

in $H^2(d^2F^\cdot / d^3F^\cdot)$.

The definition of this cup product is

$$\varepsilon^2(\varepsilon^0(T) - \varepsilon^1(T)) \cdot \varepsilon^0(\varepsilon^0(T) - \varepsilon^1(T))$$

$$= (\varepsilon^0\varepsilon^1(T) - \varepsilon^1\varepsilon^1(T)) \cdot (\varepsilon^0\varepsilon^0(T) - \varepsilon^0\varepsilon^1(T))$$

Claim $(\varepsilon^0\varepsilon^1(T) - \varepsilon^1\varepsilon^1(T)) \cdot (\varepsilon^0\varepsilon^0(T) - \varepsilon^0\varepsilon^1(T)) = \partial(d^2\delta(a))$.

We can assume $p = 2$. Hence

$$\underset{\text{def}}{\delta}(\varepsilon^0(T) - \varepsilon^1(T)) = \delta(\varepsilon^0(T)) + \delta(-\varepsilon^1(T)) + \frac{\varepsilon^0(T)^2 + \varepsilon^1(T)^2 - (\varepsilon^0(T) - \varepsilon^1(T))^2}{2}$$

$$= \delta(\varepsilon^0(T)) + ((-1)^2\delta(\varepsilon^1(T)) + \varepsilon^1(T)^2\delta(-1) + 2\delta(-1)\delta(\varepsilon^1(T)))$$

$$+ \varepsilon^0(T)\varepsilon^1(T)$$

$$= \delta(\varepsilon^0(T)) + \delta(\varepsilon^1(T)) - \varepsilon^1(T)^2 - 2\delta(\varepsilon^1(T)) + \varepsilon^0(T)\varepsilon^1(T)$$

$$= \underbrace{\delta(\varepsilon^0(T)) - \delta(\varepsilon^1(T))}_{\partial(\delta(T))} + \varepsilon^1(T)(\varepsilon^0(T) - \varepsilon^1(T)).$$

$$\Rightarrow \partial(\delta(\varepsilon^0(T) - \varepsilon^1(T))) = \partial(\varepsilon^1(T)(\varepsilon^0(T) - \varepsilon^1(T)))$$

$$= \varepsilon^0(\varepsilon^1(T))(\varepsilon^0(\varepsilon^0(T)) - \varepsilon^0\varepsilon^1(T))$$

$$- \varepsilon^1(\varepsilon^1(T))(\varepsilon^1(\varepsilon^0(T)) - \varepsilon^1(\varepsilon^1(T)))$$

$$+ \varepsilon^2(\varepsilon^1(T))(\varepsilon^2(\varepsilon^0(T)) - \varepsilon^2(\varepsilon^1(T)))$$

$$= \varepsilon^0(\varepsilon^1(T))(\varepsilon^0(\varepsilon^0(T)) - \varepsilon^0(\varepsilon^1(T)))$$

$$- \varepsilon^1(\varepsilon^1(T))(\varepsilon^0(\varepsilon^0(T)) - \varepsilon^1\varepsilon^1(T))$$

$$+ \varepsilon^1(\varepsilon^1(T)) (\varepsilon^0\varepsilon^1(T) - \varepsilon^1\varepsilon^1(T))$$

$$= (\varepsilon^0\varepsilon^1(T) - \varepsilon^1\varepsilon^1(T)) \cdot (\varepsilon^0\varepsilon^0(T) - \varepsilon^0\varepsilon^1(T))$$

On the other hand,

$$\delta(\varepsilon^0(T) - \varepsilon^1(T)) = \delta(da)$$

$$= d^2\delta(a) + a^2\delta(d) + 2\delta(a)\delta(d)$$

$$= d^2\delta(a) + \delta(d)(a^2 + 2\delta(a))$$

$$= d^2\delta(a) + \delta(d)\phi(a)$$

$$\Rightarrow \partial(\varepsilon^0(T) - \varepsilon^1(T)) = \partial(d^2\delta(a)),$$

since $\delta(d)\phi(a)$ is a cycle. ☒