PRISMATIC COHOMOLOGY SEMINAR

The theory of prismatic cohomology [BS22] grows out of an attempt to build a unified integral p-adic cohomology theory. The investigation started in the celebrated predecessors [BMS18] and [BMS19]. In [BMS18] Bhatt-Morrow-Scholze defined the $\mathbf{A}_{\mathrm{inf}}$-cohomology theory to be the hypercohomology of a certain complex $A\Omega$, which is built via the décalage functor, the pro-étale site and the theory of perfectoid rings. The authors establish various comparison results with de Rham-, p-adic étale-, and crystalline cohomology. In [BMS19], an alternative construction using methods of homotopy theory was given. Using the renewed theory of topological Hochschild homology (THH) in [NS18], the authors establish the deep connection of $A\Omega$ with the homotopy fixed points of THH under the circle action. The third construction of the $\mathbf{A}_{\mathrm{inf}}$-cohomology is given in [BS22]. In this work, Bhatt-Scholze provide a site-theoretic viewpoint of the $\mathbf{A}_{\mathrm{inf}}$-cohomology: it is the cohomology of the structure sheaf on the prismatic site. The comparison theorem between the prismatic cohomology and the $\mathbf{A}_{\mathrm{inf}}$-cohomology is given via the q-de Rham complex.

In this seminar, we will discuss the third construction given in [BS22], following [Bha18, Lectures II-VI, X–XI], see also the lecture notes [Ked22], [Eme20].

1. Overview (April 12)

[Bha18, Lecture 1, §3]. [Ked22, §1.3, §1.4]

Part 1. Commutative algebra around prisms (5 talks)

Fix a prime number p.

2. δ-rings (April 19)

The aim of this talk is to go through basic properties of δ-rings.

- To motivate yourself to this highly algebraic topic, see [Bor16]. But this does not need to appear in the lecture.

There are 3 equivalent ways to define a δ-structure on a p-torsion free ring A:

- [BS22, Definition 2.1]: Giving a set-theoretic map $\delta : A \to A$ satisfying two identities:
  - Addition rule: $\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p+y^p-(x+y)^p}{p}$.
  - Multiplication rule: $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$.
- [BS22, Remark 2.4]: Giving a (ring-theoretic) section $w : A \to W_2(A)$ of the restriction map $W_2(A) \to A$.
- [BS22, Remark 2.2]: Giving a (ring-theoretic) Frobenius lift $\phi : A \to A$. (A Frobenius lift is a ring map $\phi : A \to A$ such that $\overline{\phi} : A/p \to A/p$ is given by the p-power Frobenius map $\overline{x} \mapsto \overline{x}^p$.)

Define a morphism of δ-rings $f : (A, \delta_A) \to (B, \delta_B)$ to be a ring map $f : A \to B$ such that $f\delta_A = \delta_B f$.

- Remark that such a map f necessarily commutes with the Frobenius lift ϕ (see [Bha18, Lecture 2, Lemma 1.3]) and the section w.
- Quickly go through the examples in [Bha18, Lecture 2, Example 1.4 (1)-(5)]. Give explicitly what are the maps δ, ϕ, w.

The category of δ-rings has good categorical properties.

- [BS22, Example 2.6] The initial object is $\mathbb{Z}_{(p)}$ with
$$\delta : \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}, \quad x \mapsto \frac{x - x^p}{p}.$$
  Deduce that
  - p is never nilpotent in a non-zero δ-ring (cf [Bha18, Lecture 2, Lemma 1.5]); [1]
  - any prime $l \neq p$ is a unit in any δ-ring. (Hence when p is also invertible in A, A is necessarily a $\mathbb{Q}$-algebra.)
  - as the theory of p-typical Witt vectors, the theory of δ-rings is a p-typical theory: A δ-ring over $\mathbb{Q}$ is the same as a $\mathbb{Q}$-algebra together with a endomorphism (playing the role as the Frobenius lift).
- Free δ-rings: Identify the free object with one variable with the infinite polynomial ring via the universal property following [Bha18, Lecture 2, Lemma 2.5].
- All limits and colimits exist: [Bha18, Lecture 2, Lemma 2.3].

First properties of δ-rings:

- Zariski localization: [Bha18, Lecture 2, Lemma 2.7]

[1] The p-torsion elements in a δ-ring also has good description: $\mathrm{Ann}(p) \subset \mathrm{Ker}(\phi)$. The proof is a combination of clever observations, see [Bha18, Lecture 2, Lemma 3.3].

- Completion with respect to an ideal I containing p: [BS22, Lemma 2.17]
- Étale extensions: prove it under extra assumption as in [Bha18, Lecture 2, Lemma 2.9] suffices. (You can find the general proof in [BS22, Lemma 2.18].)
- Quotients: [Bha18, Lecture 2, Lemma 2.10]

3. Example of perfect δ-rings: the ring of Witt vectors (April 26)

The goal of this talk is the structure theorem of p-complete prefect δ-rings [Bha18, Lecture 2, Proposition 3.2].

- Construct the ring of Witt vectors following [Ked22, §3.1-§3.3]. Make sure that you define Verschiebung (Definition 3.2.3, which is additive but not multiplicative), Frobenius (Definition 3.2.4, which is both additive and multiplicative), the Teichmüller map (which is multiplicative but not additive) and the Teichmüller expansion ([Bha18, Lecture II, Construction 3.6]).
- Explain the universal lifting property: [Ked22, Lemma 3.3.5].

The structure theorem for p-complete perfect δ-rings: every p-complete perfect δ-ring is of the form $W(A)$ (with its natural Frobenius) for a perfect $\mathbf{F}_p$-algebra A.

- Define what is a perfect δ-ring.
- State and prove the categorical equivalence [Bha18, Lecture II, Proposition 3.2]. (Lemma 3.5 can be taken as a blackbox. For a short account of the cotangent complex, I recommend [Bha17, Construction 3.1.1 and Definition 3.1.2])

4. Derived completion (May 10)

This talk is independent of the other talks. The aim of this talk is to clear up the subtleties around the notion of completion and derived completion.

Go through [Bha18, Lecture III, §2]. See also the excellent treatment of this topic in the stacks project [Sta23, Tag 091N].

5. Distinguished elements and prisms (May 17)

Go through [Bha18, Lecture III, §1]. Omit the followings: Example 1.2 (3)(4), Remark 1.3, Remark 1.4.

Go through [Bha18, Lecture III, §3]. In particular, prove the rigidity lemma (Lemma 3.7).

6. Perfect prisms (May 24)

In this talk, we quickly go through [Bha18, Lecture IV]. Prove Theorem 2.3. State Proposition 2.10. Prove proposition 3.2. (Note that perfectoid rings are also called slices or lenses in [Ked22, Definition 7.3.1, Definition 8.1.1].)

Part 2. The prismatic cohomology and its computations (4 talks)

7. Prismatic cohomology (June 7)

The aim of this talk is to define the prismatic site and the prismatic complex. The emphasis is on the computational aspects. All the numberings refer to [Bha18, Lecture V], unless otherwise stated.

- Give a short introduction to what is a site. You can follow [Ked22, §11.1 and §11.3] or [Sta23, Tag 00VG].
- Define the prismatic site $(R/A)_{\Delta}$ following [Bha18, Definition 2.1]
- Define the prismatic complex $\Delta_{R/A}$ and the Hodge-Tate complex $\overline{\Delta}_{R/A}$ following [Bha18, Definition 2.10].
- Explain Example 2.11.
- State the existence of the prismatic envelop following Lemma 5.1. Skip the proof. State the explicit description of the prismatic envelope in Lecture VI, Corollary 2.3(2).
- Explain the details of the Cech-Alexander complexes following Construction 5.3. (In order to do this, one needs that the prismatic site admits finite coproducts. So if time permits, quickly sketch the proof of Corollary 5.2.)
- As an application of the Cech-Alexander complexes, prove carefully the details of Lemma 5.4.

8. The PD-envelope and the prismatic envelop (June 14)

The aim of this talk is to give an overview of the crystalline cohomology theory, and relate the PD-envelop with the prismatic envelope.

Give a recap of the crystalline cohomology materials following [Bha18, Lecture VI, §1]. Prove Lemma 2.1 and Corollary 2.3 in [Bha18, Lecture VI].

9. The crystalline comparison and the Hodge-Tate comparison (June 21)

The aim of this talk is to sketch the proof of the crystalline comparison and the Hodge-Tate comparison for prismatic cohomology.

Recall the construction of the Hodge-Tate comparison map following [Bha18, Lecture V, §3.2]. Prove [Bha18, Lecture VI, Theorem 0.1] following §4. State the globalization of prismatic cohomology [Bha18, Corollary 4.1].

Part 3. q-deformations and Scholze's co-ordinate independence conjecture (3 talks)

10. q-PD thickenings and q-PD-envelopes (June 28)

The aim of this talk is to set up the commutative algebra for the q-crystalline cohomology. The main reference is [Bha18, Lecture XI, §1].

- Define q-PD pair, q-PD-thickening according to Definition 1.1.
- Explain Example 1.3.
- Prove the existence of q-PD-envelopes following Lecture XI, Proposition 1.6. Compare it with the PD-envelop construction in Lecture VI, Construction 1.1.
- Explain carefully Example 1.7.

11. q-crystalline cohomology (July 5)

The aim of this talk is the comparison of q-crystalline cohomology and the prismatic cohomology. The main reference is [Bha18, Lecture XI, §2, p4-5].

- Define the q-crystalline site following Definition 2.1, and compare it with the classical crystalline site.
- Define the Cech-Alexander complex for q-crystalline cohomology, and compare it with Lecture VI, Construction 1.4 and Lecture V, Construction 5.3.
- Explain the q-crystalline vs de Rham comparison in Theorem 2.3. (The crystalline vs de Rham comparison is given in Lecture VI, Theorem 1.6. Taking a look in [BJ11, Theorem 2.12] might be of help.)
- Explain the q-crystalline vs prismatic comparison in Theorem 2.5. Compare it with the crystalline vs prismatic comparison in Lecture VI, Theorem 3.2
- Explain the Hodge-Tate comparison of the q-crystalline in Corollary 2.6.

12. q-de Rham and Scholze's co-ordinate independence conjecture (July 12)

The aim of this talk is to introduce the q-de Rham complex, relate it with the q-crystalline complex, and understand the proof of Bhatt-Scholze on Scholze's co-ordinate independence conjecture.

- State Scholze's co-ordinate independent conjecture: [Bha18, X, Conjecture 1.9]
- Define the q-de Rham complex as in [Bha18, Construction 1.2]
- (When time permits) Independency when R is a $\mathbb{Q}$-algebra: Explain [Ked22, Remark 26.3.2].
- (When time permits) Multiplicative structure [Bha18, X, Remark 1.4]
- (When time permits) Extend the construction of the q-de Rham complex to any formally smooth $\mathbb{Z}_p$-algbera equipped with étale co-ordinate: [Bha18, pp. X. 1.5–1.7]
- The q-de Rham complex is a q-deformation [Bha18, X, Remark 1.8] and it is nontrivial [Bha18, X, Remark 1.3]
- Hodge-Tate comparison via the q-de Rham: Lecture V, Example 3.9.
- (Omit from the talk: The conjecture holds after "perfection" base change: [Bha18, X, Remark 1.10])
- (Omit from the talk: The conjecture holds after "rationalization" base change: [Bha18, X, Remark 1.11])
- "Cartier isomorphism": when q is a root of unity: [Bha18, X, Remark 1.14]
- Frobenius is an isogeny: explain this according to [Bha18, Remark 2.10, paragraph 2] and [Ked22, §27.4].
- Explain the construction of the q-de Rham complex of a q-PD-envelop: [Bha18, Construction 2.7].
- Summarize the main idea of the proof of [BJ11, Theorem 2.12] and explain the proof of [Bha18, Theorem 2.9].

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