Introduction to δ-Rings

Definition of δ-rings

Fix a prime $p \in \mathbb{Z}$.

Definition 1. A δ-ring is a pair (A, δ), where A is a commutative ring and $\delta : A \to A$ is a map of sets, which fulfills:

- $\delta(0) = \delta(1) = 0$
- $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$
- $\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p+y^p-(x+y)^p}{p} = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i}x^iy^{p-i}$

The map δ is sometimes called a p-derivation.

If the δ-structure on A is clear by the given context, we call A a δ-ring.
For a ring A that has no p-torsion, there are several equivalent ways to define a δ-structure on that ring. One of them is to give a lift of the Frobenius map modulo p.

Lemma 2. *Let A be a commutative ring.*

(1) If there is a δ-structure on A, the map

$$\begin{aligned} \phi : A &\to A \\ x &\mapsto x^p + p \cdot \delta(x) \end{aligned}$$

is a lift of the Frobenius $A/p \to A/p,\ x \mapsto x^p$.

(2) If A is p-torsionfree, there is a 1:1 correspondence between lifts of the Frobenius and δ-structures on A.

Proof. (1) It is clear from the definition of ϕ, that it lifts the Frobenius. Thus one has to check, that it is also a ring homomorphism. This follows by the properties of δ for sums and products. In particular, we have:

$$\begin{aligned} \phi(x + y) = (x + y)^p + p\delta(x + y) &= (x + y)^p + p\left(\delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p}\right) \\ &= x^p + y^p + p\delta(x) + p\delta(y) = \phi(x) + \phi(y) \end{aligned}$$

as well as

$$\begin{aligned} \phi(xy) = (xy)^p + p\delta(xy) &= (xy)^p + p(x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)) \\ &= (x^p + p\delta(x))(y^p + p\delta(y)) = \phi(x)\phi(y) \end{aligned}$$

and also $\phi(1) = 1^p + p\delta(1) = 1$.

(2) If ϕ is a lift of the Frobenius there is an equation of the form

$$\phi(x) = x^p + p\delta$$

for every $x \in A$. As A is p-torsionfree, $\delta = \delta(x) = \frac{\phi(x)-x^p}{p}$ is uniquely determined by the element x. What is left to show is, that the map δ defines a δ-structure on A. But this follows from the fact, that ϕ is a ring homomorphism. At first, we have $\phi(1) = 1 = 1^p$ and $\phi(0) = 0 = 0^p$, which ensures $\delta(0) = \delta(1) = 0$. Additionally it holds

$$\begin{aligned}\delta(x+y) = \frac{\phi(x+y)-(x+y)^p}{p} &= \frac{\phi(x)-x^p+\phi(y)-y^p+x^p+y^p-(x+y)^p}{p} \\ &= \delta(x)+\delta(y)+\frac{x^p+y^p-(x+y)^p}{p}\end{aligned}$$

and

$$\begin{aligned}&x^p\delta(y)+y^p\delta(x)+p\delta(x)\delta(y) = x^p\frac{\phi(y)-y^p}{p}+y^p\frac{\phi(x)-x^p}{p}+p\frac{\phi(x)-x^p}{p}\frac{\phi(y)-y^p}{p} \\ &= \frac{x^p\phi(y)-(xy)^p+y^p\phi(x)-(xy)^p+\phi(x)\phi(y)-x^p\phi(y)-y^p\phi(x)+(xy)^p}{p} \\ &= \frac{\phi(xy)-(xy)^p}{p} = \delta(xy).\end{aligned}$$

□

In the following, if we have a δ-ring (A, δ), we always denote the associated Frobenius lift (given by lemma 2(1)) by ϕ.

Definition 3. Let A be a ring. The ring of truncated Witt-vectors $W_2(A)$ is defined as follows:
$W_2(A) := A \times A$ as a set, with the addition and multiplication

$$\begin{aligned}(x,y)+(z,w) &:= \left(x+z, y+w+\frac{x^p+z^p-(x+z)^p}{p}\right) \\ (x,y)\cdot(z,w) &:= (xz, x^pw+z^py+pyw).\end{aligned}$$

This obviosly gives a ring homomorphism $\epsilon : W_2(A) \to A$, $(x,y) \mapsto x$.
This construction is also functorial, using $W_2(f)(x,y) := (f(x), f(y))$ for a given map of rings $f : A \to A$.

Lemma 4. *A δ-structure on a ring A is the same as a ring homomorphism $w : A \to W_2(A)$, such that $\epsilon \circ w = id$.*

Proof. Let $w : A \to W_2(A)$ be a ring homomorphism, such that $\epsilon \circ w = id$. Then we can write $w(x) = (w_1(x), w_2(x))$, and because of the condition on w it is clear, that $w_1(x) = x$ holds. We now define $\delta(x) := w_2(x)$ and have to show, that the properties of a δ-structure are fulfilled. We have $1_{W_2(A)} = (1,0)$, $0_{W_2(A)} = (0,0)$ and thus $\delta(0) = \delta(1) = 0$. Additionally it holds

$$\begin{aligned}w(x+y) &= (x+y, \delta(x+y)) \\ w(x)+w(y) &= (x,\delta(x))+(y,\delta(y)) = \left(x+y, \delta(x)+\delta(y)+\frac{x^p+y^p-(x+y)^p}{p}\right)\end{aligned}$$

and using the additivity of w shows the additive identity for δ. Using the multiplicativity of w also shows that the multiplicative identity of δ holds:

$$\begin{aligned}
&w(xy) = (xy, \delta(xy)) \\
&w(x)w(y) = (x, \delta(x)), (y, \delta(y)) = (xy, x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)).
\end{aligned}$$

If we otherwise have a δ-structure on A, the map $w : A \to W_2(A)$, $x \mapsto (x, \delta(x))$ defines a ring homomorphism with $\epsilon \circ w = id$, because:

$$\begin{aligned}
w(x+y) = (x+y, \delta(x+y)) &= \left(x+y, \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}\right) \\
&= (x, \delta(x)) + (y, \delta(y)) = w(x) + w(y)
\end{aligned}$$

and

$$w(xy) = (xy, \delta(xy)))(xy, x^p, \delta(y) + y^p\delta(x) + p\delta(x)\delta(y)) = (x, \delta(x))(y, \delta(y)) = w(x)w(y).$$

□

Thus for a p-torsionfree ring A there are three different equivalent ways to define a δ-ring. Each of the definitions can be useful in different situations. Especially for giving examples of δ-rings, it may be difficult to directly construct a δ-structure, whereas a lift of the Frobenius modulo p is easier to declare.

The δ-rings also form a category $\mathbf{Ring}_\delta$ with the following obvious definition of a morphism of δ-rings.

Definition 5. A map of δ-rings $f : (A, \delta_A) \to (B, \delta_B)$ is a ring homomorphism $f : A \to B$ which fulfills $f \circ \delta_A = \delta_B \circ f$.

Remark 6. Let $f : (A, \delta_A) \to (B, \delta_B)$ be a map of δ-rings. Then f also commutes with the associated Frobenius lifts and with the sections w_A and w_B. Using the definition of a δ-map shows that

$$f \circ \phi_A(x) = f(x^p + p\delta_A(x)) = f(x)^p + pf(\delta_A(x)) = f(x)^p + p\delta_B(f(x)) = \phi_B \circ f(x)$$

and

$$W_2(f) \circ w_A(x) = W_2(f)(x, \delta_A(x)) = (f(x), f(\delta_A(x))) = (f(x), \delta_B(f(x))) = w_B \circ f(x).$$

Lemma 7. *The associated Frobenius lift $\phi : A \to A$ of a δ-ring A is a morphism of δ-rings.*

Proof. If A is p-torsionfree, it holds that $\delta(x) = \frac{\phi(x) - x^p}{p}$. Using that, the commutativity directly follows out of the fact that ϕ is a ring homomorphism:

$$\delta(\phi(x)) = \frac{\phi(\phi(x)) - \phi(x)^p}{p} = \frac{\phi(\phi(x) - x^p)}{p} = \phi(\delta(x)).$$

In the general case we can use lemma 13 and remark 14 to find a surjection of δ-rings $f : B \twoheadrightarrow A$ with a p-torsionfree δ-ring B. For $x \in A$ we then find a $b \in B$ with $f(b) = x$. Using that f is a map of δ-rings, Lemma 6 and the p-torsionfree case shows the commutativity in the general case:

$$\phi(\delta(x)) = \phi(\delta(f(b))) = f(\phi(\delta(b))) = f(\delta(\phi(b))) = \delta(\phi(f(b))) = \delta(\phi(x)).$$

□

Example 8. Using that for a p-torsionfree ring a δ-structure is already determined by giving a lift of the Frobenius one can generate a few simple examples of δ-rings:

(1) The ring of integers $\mathbb{Z}$ with $\phi = id$ is a δ-ring. In this case we have $\delta(n) = \frac{n-n^p}{p}$ and $w(n) = (n, \frac{n-n^p}{p})$.
This δ-ring is the initial object in the category of δ-rings. For an arbitrary δ-ring (A, δ_A) there is exactly one ring homomorphism $\mathbb{Z} \to A$, because $\mathbb{Z}$ is the initial object in rings (it maps 1 to 1). This map is even a map of δ-rings, because by induction it holds

$$\begin{aligned}\delta_A(n) &= \delta_A((n-1)+1)) = \delta_A(n-1) + \delta_A(1) + \frac{(n-1)^p + 1^p - n^p}{p} \\ &= \delta_{\mathbb{Z}}(n-1) + \delta_{\mathbb{Z}}(1) + \frac{(n-1)^p + 1^p - n^p}{p} = \delta_{\mathbb{Z}}(n).\end{aligned}$$

Thus in all δ-rings A, even if it is not p-torsionfree, it is true that $\delta(n) = \frac{n-n^p}{p}$ for integers n.

(2) The polynomial ring $\mathbb{Z}[x]$ with the lift of the Frobenius determined by $\phi(x) = x^p + pg(x)$ for some $g(x) \in \mathbb{Z}[x]$. In this case we have $\delta(x) = g(x)$ and $w(x) = (x, g(x))$ and the value for an arbitrary $f \in \mathbb{Z}[x]$ is determined by the additivite and multiplicativite properties of δ.

(3) The ring of Witt vectors $W(k)$ for a perfect field of characteristic p, which will be introduced next week.

(4) If A is a $\mathbb{Z}[\frac{1}{p}]$-Algebra, which means that p has to be invertible in A, then every endomorphism of A is a lift of the Frobenius modulo p (because $pA = A$ and so ${}^{A}\!/_{pA} = 0$). Thus a δ-structure on A is given by an endomorphism ϕ of A. In this case we have $\delta(x) = \frac{\phi(x)-x^p}{p}$ and $w(x) = (x, \delta(x))$.

(5) An example for a δ-ring that contains p-torsion is the ring $A := {}^{\mathbb{Z}[x]}\!/_{(x^p, px)}$ with the δ-structure determined by $\delta(x) = 0$. This is a well definied structure. At first we notice per induction that for all $n \geq 1$

$$\delta(x^n) = \delta(x \cdot x^{n-1}) = x^p\delta(x^{n-1}) + x^{pn}\delta(x) + p\delta(x)\delta(x^{n-1}) = 0,$$

because $x^p = 0$ in A. Also knowing this it follows that for $m \in \mathbb{Z}$ and $n \geq 1$

$$\delta(mx^n) = m^p\delta(x^n) + x^{pn}\delta(m) + p\delta(m)\delta(x^n) = 0.$$

Now for some arbitrary $r := f \cdot x^p + g \cdot px \in (x^p, px)$ it holds

$$\begin{aligned}\delta(r) &= \delta(f \cdot x^p) + \delta(g \cdot px) + \frac{(f \cdot x^p)^p + (g \cdot px)^p - (f \cdot x^p + g \cdot px)^p}{p} \\ &= f^p\delta(x^p) + (x^p)^p\delta(f) + p\delta(f)\delta(x^p) + g^p\delta(px) + (px)^p\delta(g) + p\delta(g)\delta(px) = 0\end{aligned}$$

and thus for some arbitrary $h \in \mathbb{Z}[x]$

$$\delta(h+r) = \delta(h) + \delta(r) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} h^i r^{p-i} = \delta(h).$$

So this gives us a well defined δ-structure on A.

The Frobenius lift is defined by $\phi(x) = x^p$ and so for some $f = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ we have $\phi(f) = \sum_{i=0}^{n} a_i x^{pi} = a_0$ in A and $w(f) = (f, \delta(f))$.

Remark 9. (1) It is possible that a ring can not be equipped with a δ-structure, and so there is no possibility to lift the Frobenius. For example regard the ring $\mathbb{Z}[x, \frac{x^p}{p}]$ and assume there is a δ-structure δ with associated Frobenius lift ϕ. Then it holds that

$$\begin{aligned}\frac{1}{p}\left(\frac{x^p}{p}\right)^p &= \frac{1}{p}\left(\phi\left(\frac{x^p}{p}\right) - p\delta\left(\frac{x^p}{p}\right)\right)\\ &= \frac{\phi(x)^p}{p^2} - \delta\left(\frac{x^p}{p}\right)\\ &= \frac{(x^p + p\delta(x))^p}{p^2} - \delta\left(\frac{x^p}{p}\right)\\ &= p^{p-2}\left(\frac{x^p}{p} + \delta(x)\right)^p - \delta\left(\frac{x^p}{p}\right) \in \mathbb{Z}\left[x, \frac{x}{p}\right]\end{aligned}$$

in contradiction to $\frac{1}{p}(\frac{x^p}{p})^p \notin \mathbb{Z}[x, \frac{x^p}{p}]$.

(2) The theory of δ-rings is mostly applied for $\mathbb{Z}_{(p)}$-algebras, which is why it is often only defined for these. This changes nothing in the definition apart from A being a $\mathbb{Z}_{(p)}$-algebra.
In this case $\mathbb{Z}_{(p)}$ with $\delta(n) = \frac{n-n^p}{p}$ is the initial object and thus every prime $l \neq p$ is invertible in a δ-ring (A, δ_A).
Example 8 then changes to: If p is also invertible in a δ-ring A, then A is a $\mathbb{Q}$-algebra and a δ-structure on A is given by an endomorphism of A.

A few properties of δ-rings

Lemma 10. *In a nonzero δ-ring $A \neq 0$ the prime p is never nilpotent.*

Proof. The proof is done by contradiction. Assume that p nilpotent, so there exists an integer $n \geq 1$ such that $p^n = 0$. Then A is a $\mathbb{Z}_{(p)}$-algebra, because every prime $l \neq p$ has an inverse $q \in \mathbb{Z}$ modulo p. Thus there is some $a \in A$ such that $lq = 1 + pa$. Because 1 is a unit and $pn \in \mathrm{rad}(A)$, lq is invertible in A and so also l. By the universal property of localization the morphism $\mathbb{Z} \to A$ factorizes over $\mathbb{Z}_{(p)}$.
We now show that the following holds:

($\star$) $\forall u \in \mathbb{Z}_{(p)}^{\times}$, $m \geq 1$: $\delta(p^m u) = p^{m-1} v$ for some $v \in \mathbb{Z}_{(p)}^{\times}$.

Using ($\star$) the contradiction follows by induction, because $\delta^n(p^n)$ is a unit, but also $\delta^n(p^n) = \delta^n(0) = 0$.
To show the statement ($\star$) we first show it in the special case that $u = 1$. There we have

$$\delta(p^m) = \frac{\phi(p^m) - p^{mp}}{p} = \frac{p^m - p^{mp}}{p} = p^{m-1}(1 - p^{mp-m})$$

because ϕ is the identity on integers. This has the required form because $p^{mp-m} \in \mathrm{rad}(\mathbb{Z}_{(p)})$.

For the general case one uses the multiplicative identity of δ:

$$\delta(p^m u) = \phi(p^m)\delta(u) + u^p\delta(p^m) = p^m\delta(u) + u^p p^{m-1}(1 - p^{mp-m}) = p^{m-1}(p\delta(u) + u^p w)$$

with $w = 1 - p^{mp-m}$ a unit. This again has the required form because $u^p w$ is a unit and $p\delta(u) \in \mathrm{rad}(\mathbb{Z}_{(p)})$. ☐

Lemma 11. *The category of δ-rings has all limits and colimits and they commute with the forgetful functor from δ-rings to rings.*

Proof. For the existence of limits let $F : J \to \mathbf{Ring}_\delta$ be a functor from a small category J. Composed with the forgetful functor from δ-rings to rings this gives a functor $F : J \to \mathbf{Ring}$ and thus we already know, that there exists a limit (L, π) as rings. One can define a δ-structure on L using the projections: The element $\delta_L(x)$ for some $x \in L$ is uniquely defined by $\pi_X(\delta_L(x)) = \delta_X(\pi_X(x))$ for all X in J, because it holds $F(f) \circ \delta_X \circ \pi_X = \delta_Y \circ F(f) \circ \pi_X = \delta_Y \circ \pi_Y$. This is the only δ-structure such that the π_X are δ maps and thus the only possible δ-structure the limit can have to exist in the category of δ-rings. Also the properties of the δ-structure directly follow from the fact, that the properties are fulfilled in all $F(X)$.
Now for the universal property let (A, δ_A) be a δ-ring with δ-maps $\varphi_X : A \to F(X)$. Then there is a unique map of rings $\varphi : A \to L$ and it is left to check that this map is also a δ-map. For this we see that for every X in J it holds

$$\pi_X \circ \varphi \circ \delta_A = \varphi_X \circ \delta_A = \delta_X \circ \varphi_X = \delta_X \circ \pi_X \circ \varphi = \pi_X \circ \delta_L \circ \varphi$$

and thus also $\varphi \circ \delta_A = \delta_L \circ \varphi$ is valid.
For the colimit we use the description of a δ-structure by constructing a section from the ring of truncated Witt vectors. Let $F : J \to \mathbf{Ring}_\delta$ be a functor from a small category to the category of δ-rings, thus for every X in J there is a δ-structure δ_X on $F(X)$. Composed with the forgetful functor we receive a functor from J to the category of commutative rings, where the colimit of the given diagram exists. Thus we receive the colimit (C, ψ) for rings. Since every $F(X)$ for some object X in J is a δ-ring, there is a homomorphism $F(X) \to W_2(F(X))$, which is the identity in the first component. Using the functoriality of colimits there is a homomorphism $g : C \to D$, where (D, ψ') denotes the colimit of the diagram $W_2 \circ F : J \to \mathbf{Ring}$. Thus it holds $g \circ \psi_X = \psi'_X \circ w_X$ for all X, where w_X denotes the section $F(X) \to W_2(F(X))$, which we receive out of the δ-structure on $F(X)$.
Additionally for every X in J there is the natural morphism $\psi_X : F(X) \to C$. Using functoriality of $W_2(-)$ we receive a morphism $W_2(\phi_X) : W_2(F(X)) \to W_2(C)$. Since D is a colimit, there is a morphism $h : D \to W_2(C)$, which fulfills $h \circ \psi'_X = W_2(\psi_X)$ for all X. Altogether we obtain the map $w := h \circ g : C \to W_2(C)$. Composed with the projection on the first component $p_1 : W_2(C) \to C$ this is the identity on C, because for every X in J we have

$$p_1 \circ w \circ \psi_X = p_1 \circ h \circ g \circ \psi_X = p_1 \circ h \circ \psi'_X \circ w_X = p_1 \circ W_2(\psi_X) \circ w_X = \psi_X,$$

because w_X is the identity on the first component. So we obtain a δ-structure δ_C on the colimit C.
The ψ_X are also δ-maps, because they fulfill $h \circ g \circ \psi_X = W_2(\psi_X) \circ w_X$ and thus also commute with the δ-structures.

Now the pair (C, ψ) also fulfills the universal property of a colimit in $\mathbf{Ring}_\delta$: Let (A, δ_A) be a δ-ring and $\varphi_X : F(X) \to A$ δ-maps, such that for every $f : X \to Y$ in J it holds that $\varphi_Y \circ F(f) = \varphi_X$. Since (C, ψ) is a colimit in **Ring** there is a unique homomorphism of rings $\varphi : C \to A$, such that $\varphi \circ \psi_X = \varphi_X$ for all X in J. This is even a δ-map, because for every X in J it holds

$$\begin{aligned} w_A \circ \varphi \circ \psi_X = w_A \circ \varphi_X \circ w_A = W_2(\varphi_X) \circ w_X = W_2(\varphi \circ \psi_X) \circ w_X \\ = W_2(\varphi) \circ W_2(\psi_X) \circ w_X = W_2(\varphi) \circ w \circ \psi_X \end{aligned}$$

and so $w_A \circ \varphi = W_2(\varphi) \circ w$. Thus φ commutes with the sections w and w_A and also has to commute with the δ-structures. □

Remark 12. By lemma 11 and the Freyd adjoint functor theorem it follows that the forgetful functor from δ-rings to commutative rings has both a left and a right adjoint functor. We will in the following investigate the left adjoint functor, which gives us the free δ-ring $\mathbb{Z}\{S\}$ for a ring $\mathbb{Z}[S]$ with some set S.

Lemma 13. *The free δ-ring for a singleton is of the form $\mathbb{Z}\{x\} = \mathbb{Z}[x_0, x_1, x_2, ...]$ with $x = x_0$ and $\delta(x_i) = x_{i+1}$.*

Proof. With $A := \mathbb{Z}[x_0, x_1, x_2, ...]$ we have an endomorphism $\phi : A \to A$ determined by $\phi(x_i) = x_i^p + px_{i+1}$. This is a lift of the Frobenius, and because A is p-torsionfree this gives us a unique associated δ-structure on A with $\delta(x_i) = x_{i+1}$. That shows that the given object is well defined.
For the adjunction we check the universal property. Let (S, δ_S) be a δ-ring and $\varphi : \mathbb{Z}[x] \to S$ be a ring homomorphism determined by $x \mapsto f$ for some $f \in S$. We need to show that there is a unique δ-map $A \to S$ that maps x_0 to f. By induction it holds that $x_i \mapsto \delta^i(f)$, because

$$\varphi(x_{i+1}) = \varphi(\delta(x_i)) = \delta_S(\varphi(x_i)) = \delta_S(\delta_S^i(f)) = \delta_S^{i+1}(x_0).$$

So the map of δ-rings is uniquely defined by the image of x_0. Also by construction this map is a δ-map, because it commutes with the δ-structures on the generators of A. Thus we see that there is a unique δ-map that maps x_0 to f and the statement follows. □

Remark 14. For a set S the free δ-ring $\mathbb{Z}\{S\}$ is given by $\mathbb{Z}[S_0, S_1, S_2, ...]$ with each S_i being a copy of S. The δ-structure here is given as follows: For some $s \in S$ corresponding to $s_i \in S_i$ we have $\delta(s_i) = s_{i+1}$. Thus δ-ring $\mathbb{Z}\{S\}$ is p-torsionfree.
To evaluate the left adjoint on an arbitrary ring A, one can always find a surjection of a free δ-ring onto A, for example $\mathbb{Z}\{A\} \twoheadrightarrow A$ and then write A as a quotient of $\mathbb{Z}\{A\}$ (using that there is a unique δ-structure on the quotient, lemma 15).
By doing this, we also see, that we can write every δ-ring as a quotient of a free δ-ring hence a p-torsionfree δ-ring. That allows us to reduce many statements to the p-torsionfree case.

Lemma 15. *Let A be a δ-ring and $I \subset A$ an ideal with $\delta(I) \subset I$. Then there is a unique δ-structure on A/I that is compatible with the one on A.*

Proof. For $x \in A$ and $\epsilon \in I$ we have

$$\delta(x + \epsilon) = \delta(x) + \delta(\epsilon) - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^i \epsilon^{p-i} \equiv \delta(x) \mod I$$

and so $\delta(\overline{x}) := \overline{\delta(x)}$ for $\overline{x} \in {}^{A}\!/\!_{I}$ is well defined and the unique δ-structure that is compatible with the δ-structure on A. □

Lemma 16. *Let A be a δ-ring and $S \subset A$ a multiplicative subset with $\phi(S) \subset S$. Then there is a unique δ-structure on the localization $S^{-1}A$ that is compatible with the one on A.*

Proof. If A is p-torsionfree then $S^{-1}A$ is also p-torsionfree. Because $\phi(S) \subset S$ and the universal property of the localization there exists a unique extension $\phi : S^{-1}A \to S^{-1}A$ of ϕ, and it is also a lift of the Frobenius modulo p. Thus we receive a unique δ-structure on the localization that is compatible with the one on A.
If A is not p-torsionfree we can use lemma 13 to find a surjection $\pi : F \twoheadrightarrow A$ with a p-torsionfree δ-ring F. The preimage $T := \pi^{-1}(S) \subset F$ is multiplicative and fulfills $\phi(T) \subset T$. Thus using the previous p-torsionfree case we receive a unique δ-structure on the localization $T^{-1}F$ that is compatible with the one on F. Regarding the base change along $F \to A$ and using $T^{-1}F \otimes_F A \cong T^{-1}A \cong S^{-1}A$ as we regard A as a F-modul via π, we get a unique δ-structure on $S^{-1}A$, because a push-out is just a colimit. □

Lemma 17. *Let A be a δ-ring and $I \subset A$ a finitely generated ideal with $p \in I$. Then the map $\delta : A \to A$ is I-adically continuous. That means that for every $n \in \mathbb{N}$ there exists some $m \in \mathbb{N}$ such that for all $x \in A$: $\delta(x + I^m) \subset \delta(x) + I^n$.*
Furthermore there is a unique δ-structure on the I-adic completion of A that is compatible with the one on A.

Proof. For the continuity let $y \in I^m$. Then

$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$$

where the sum is an element of I^m, because $y \in I^m$ and $p - 1 \geq 1$. Therefore $\delta(x + I^m) - \delta(x) \subset \delta(I^m) + I^m$. Hence it suffices to show that for every $n \in \mathbb{N}$ there is some $m \geq n$ such that $\delta(I^m) \subset I^n$. To see this we first observe that for $J_1, J_2 \subset A$ ideals and $x \in J_1$, $y \in J_2$ we have

$$\begin{aligned} \delta(xy) &= x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y) \in J_1 + J_2 + p\delta(J_1)\delta(J_2) \\ &\Rightarrow \delta(J_1 J_2) \subset J_1 + J_2 + p\delta(J_1)\delta(J_2). \end{aligned}$$

Taking $J_1 = J_2 = I$ and using $p \in I$ we obtain $\delta(I^2) \subset I$. Per Induction and using $J_1 = J_2 = I^{2^n}$ it follows $\delta\left(I^{2^{n+1}}\right) \subset I^{2^n}$, because

$$\delta\left(I^{2^{n+1}}\right) \subset \delta\left(\left(I^{2n}\right)^2\right) \subset I^{2^n} + p\delta\left(I^{2^n}\right)^2 \subset I^{2^n} + p\left(I^{2^{n-1}}\right)^2 \subset I^{2^n}.$$

For some $k \in \mathbb{N}$ we can now find an $n \in \mathbb{N}$ such that $2^n \geq k$. Then $\delta\left(I^{2^{n+1}}\right) \subset I^{2^n} \subset I^k$. This shows the continuity of δ.
Now δ extends uniquely to a continuous map $\hat{\delta} : \hat{A} \to \hat{A}$, where $\hat{A}$ denotes the I-adic completion of A. Because of the continuity of $\hat{\delta}$ this is also a δ-structure. Every other δ-structure on $\hat{A}$ also has to be $I\hat{A}$-adically continuous by the previos result applied on $\hat{A}$. Thus $\hat{\delta}$ is the unique δ-structure on $\hat{A}$ that is compatible with the one on A. □

Lemma 18. *Let A, B be p-adically complete, p-torsionfree rings and $A \to B$ a ring homomorphism. Assume that A is equipped with a δ-structure and $A \to B$ is étale modulo p. Then B has a unique δ-structure that is compatible with the one on A.*

The proof uses the following result, which can be found in The Stacks Project, Tag 0BTY.

Theorem 19. *Let X and Y be schemes over a scheme S. Let $S' \to S$ be a universal homeomorphism. Denote X' and Y' the base change of X and Y via $S' \to S$. If X is étale over S, then the map*

$$Hom_S(Y, X) \to Hom_{S'}(Y', X')$$

is bijective.

Proof. (of the lemma) Since A and B are p-torsionfree, it is sufficient to show, that there is a unique Frobenius lift on B compatible with the one on A. As A and B are p-adically complete it suffices to construct it modulo p^n for all n.
For $n = 1$ the only lifts of the Frobeniusmorphisms on A and B are the Frobenius themselfs. These also commute with the morphism $A \to B$, because it is a ring homomorphism. For $n > 1$, the morphism $\operatorname{Spec}\left(A/p^nA\right) \to \operatorname{Spec}\left(A/p^{n+1}A\right)$ is a universal homeomorphism for all $n \geq 1$. Also $\operatorname{Spec}\left(B/p^nB\right) \to \operatorname{Spec}\left(A/p^nA\right)$ is étale. Using theorem 19 there is a bijection

$$\begin{aligned} &\operatorname{Hom}_{\operatorname{Spec}\left(A/p^{n+1}A\right)}\left(\operatorname{Spec}\left(B/p^{n+1}B\right), \operatorname{Spec}\left(B/p^{n+1}B\right)\right) \\ &\to \operatorname{Hom}_{\operatorname{Spec}\left(A/p^nA\right)}\left(\operatorname{Spec}\left(B/p^nB\right), \operatorname{Spec}\left(B/p^nB\right)\right) \end{aligned}$$

where we regard the right $\operatorname{Spec}\left(B/p^nB\right)$ and $\operatorname{Spec}\left(B/p^{n+1}B\right)$ as $\operatorname{Spec}\left(A/p^nA\right)$ and $\operatorname{Spec}\left(A/p^{n+1}A\right)$ schemes via the given map $A \to B$ and the left ones via the Frobenius lifts on A/p^nA and $A/p^{n+1}A$. Thus we have a bijection of ring homomorphisms

$$\operatorname{Hom}_{A/p^{n+1}A}\left(B/p^{n+1}B, B/p^{n+1}B\right) \to \operatorname{Hom}_{A/p^nA}\left(B/p^nB, B/p^nB\right)$$

where the left B/p^nB and $B/p^{n+1}B$ are A/p^nA- and $A/p^{n+1}A$-algebras via the given map $A \to B$ and the right via the given map $A \to B$ composed with the Frobenius lift on A. If we then have a Frobenius lift on B/p^nB, which is compatible with the one on A/p^nA, it lifts uniquely to a Frobenius lift in $B/p^{n+1}B$, that is compatible with the one on $A/p^{n+1}A$. □