Derived completions

Erik Barinaga

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In this text, we closely follow [1, Lecture III, §2]. For theory on derived categories, we use [2, Chapter 05QI].

Throughout this text, A denotes a commutative ring and we work in the derived category $D(A) := D(\mathrm{Mod}_A)$.

1 Derived Completeness

An element $M \in D(A)$ is a complex $M = (\cdots \to M^{-1} \to M^0 \to M^1 \to \cdots)$ of A-modules.

For $M \in D(A)$ and $f \in A$, define

$$T(M, f) := R\lim(\cdots \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} M) \in D(A).$$

Definition 1. Let $I \subset A$ be a finitely generated ideal. We call $M \in D(A)$ derived I-complete if for all $f \in I$ we have $T(M, f) = 0$. This is equivalent to requiring that the map

$$\varphi_f \colon \prod_{n \geq 0} M \to \prod_{n \geq 0} M,$$

where $(k_n) \mapsto (k_n - f k_{n+1})$, is an isomorphism in $D(A)$.

Proposition 2. *If $T(M, f) = 0$ for all f in a generating set of I, then M is derived I-complete.*

An A-module M is *derived I-complete* if it is so when regarded as a complex $(\cdots \to 0 \to M \to 0 \to \cdots) \in D(A)$ sitting in degree 0. Such a module is *classically I-complete* if $M \cong \lim_n M/I^n M$.

If $I = (f_1, \ldots, f_r)$ then for $M \in D(A)$ we can define

$$\hat{M} := R\lim_n (M \otimes^L_{\mathbb{Z}[x_1,\ldots,x_r]} \mathbb{Z}[x_1, \ldots, x_r]/(x_1^n, \ldots, x_r^n))$$

where x_i acts by f_i on M. The complex $\hat{M}$ is called the *derived I-completion* of M. More on this in Section 4.

2 Derived completion of a module with respect to a principal ideal

Let $I = (f) \subset A$ be a principal ideal, and let $M \in \mathrm{Mod}_A$ be viewed as a complex in $D(A)$. We use the standard resolution

$$(\mathbb{Z}[x] \xrightarrow{x^n} \mathbb{Z}[x]) \longrightarrow \mathbb{Z}[x]/(x^n),$$

to compute

$$M \otimes^L_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x^n) \cong \left(\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{f^n} M \longrightarrow 0 \longrightarrow \cdots\right)$$

sitting in degrees -1 and 0. The inverse system (K_n, f_n) that we can use is now

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M & \xrightarrow{f} & M & \longrightarrow & \cdots \\
 & & {\scriptstyle f^{n+1}}\downarrow & & \downarrow{\scriptstyle f^n} & & \\
\cdots & \longrightarrow & M & \xrightarrow{\mathrm{id}} & M & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

and the derived limit $K := \hat{M} = R\lim K_n$ is part of a distinguished triangle

$$K \longrightarrow \prod K_n \xrightarrow{\varphi} \prod K_n \longrightarrow K[1]$$

which induces a long exact sequence

$$\cdots \longrightarrow H^{i-1}(\textstyle\prod K_n) \longrightarrow H^i(K) \longrightarrow H^i(\prod K_n)$$

$$\longrightarrow H^i(\textstyle\prod K_n) \longrightarrow H^{i+1}(K) \longrightarrow \cdots$$

As $\prod K_n$ sits in degrees -1 and 0, we have that $H^i(K) = 0$ for $i \geq 2$ and $i \leq -2$. Hence, we can focus our attention to

$$\cdots \longrightarrow 0 \longrightarrow H^{-1}(K) \longrightarrow H^{-1}(\textstyle\prod K_n)$$

$$\longrightarrow H^{-1}(\textstyle\prod K_n) \longrightarrow H^0(K)$$

$$\longrightarrow H^0(\textstyle\prod K_n) \longrightarrow H^0(\prod K_n) \longrightarrow H^1(K) \longrightarrow 0 \longrightarrow \cdots$$

Note that $H^0(\prod K_n)$ is the cokernel of the non-trivial transition map in $\prod K_n$. Hence, we have a commutative diagram

$$\begin{array}{ccc}
\prod K_n & \xrightarrow{\varphi} & \prod K_n \\
\downarrow & & \downarrow \\
\prod K_n & \xrightarrow{\text{id}} & \prod K_n \\
\twoheaddownarrow & & \twoheaddownarrow \\
H^0(\prod K_n) & \longrightarrow & H^0(\prod K_n)
\end{array}$$

showing that the bottom horizontal map is surjective, so $H^1(K) = 0$.

Lemma 3. *The long exact cohomology sequence associated to the distinguished triangle breaks up into short exact sequences*

$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(K) \to \lim H^p(K_n) \to 0$$

Proof. Omitted. See [2, Lemma 0CQE]. □

Hence, we have

$$0 \to R^1 \lim M[f^n] \to H^0(K) \to \lim M/f^n M \to 0$$

and

$$H^{-1}(K) \cong \lim H^{-1}(K_n) = \lim M[f^n].$$

Lemma 4. *If for $C \in D(A)$ we have $H^i(C) = 0$ for all $i \neq i_0$, then we have that $H^{i_0}(C)[-i_0] \cong C$.*

Lemma 5. *Let $f \in A$. If $M \in Mod_A$ has bounded f^∞-torsion (there exists some $n > 0$ such that $M[f^\infty] = M[f^n]$), then the dervied f-completion of M as a complex is concentrated at degree 0 and coincides with the classical completion.*

Proof. The projective system $\{M[f^n]\}$ is essentially zero, i.e. any large enough composition of transition maps is zero. Hence, $\lim M[f^n] = 0 = R^1 \lim M[f^n]$. To see the second equality, let n be such that $M[f^n] = M[f^\infty]$ and note that

$$R^1 \lim_k M[f^k] = R^1 \lim_k M[f^{kn}] = \text{coker}((m_{kn}) \mapsto (m_{kn} - f^n m_{(k+1)n}) = 0.$$

By Lemma 4, we have $\hat{M} \cong H^0(\hat{M}) \cong \lim M/f^n M$. □

2.1 Example

Let $A = \mathbb{Z}$, $I = (p)$ and $M = \mathbb{Q}/\mathbb{Z} \in D(A)$. Explicitly, $M = (\cdots \to 0 \to \mathbb{Q}/\mathbb{Z} \to 0 \to \cdots)$. We want to compute $\hat{M}$.

In order to do this, we compute the cohomology groups of $\hat{M}$. With what was discussed above, we see that

$$H^{-1}(\hat{M}) = \lim_n M[p^n] = \lim_n (p^{-n}\mathbb{Z}/\mathbb{Z}) = \lim_n \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p.$$

On the other hand, we have that all the other cohomology groups are zero because $\lim M/p^n M = 0$ and $R^1 \lim M[p^n] = 0$. Indeed, to see the first equality note that any fraction can be written as the multiple of p^n for any n. To see the second equality, note that $R^1 \lim M[p^n]$ is the cokernel of the map $\prod \mathbb{Z}/p^n\mathbb{Z} \to \prod \mathbb{Z}/p^n\mathbb{Z}$ where $(x_n) \mapsto (x_n - \overline{x_{n+1}})$. This map is surjective, as for every $(y_n)_n \in \prod_n \mathbb{Z}/p^n\mathbb{Z}$, we can construct the following preimage. We set $x_1 = 0$, and we choose $x_2 \in \mathbb{Z}/p^2\mathbb{Z}$ such that $y_1 = x_1 - \overline{x_2} = \overline{x_2}$, where $\overline{x_2} \in \mathbb{Z}/p\mathbb{Z}$. Now we choose $x_3 \in \mathbb{Z}/p^3\mathbb{Z}$ such that $y_2 = x_2 - \overline{x_3}$, and we can continue like this for every $n \in \mathbb{N}$.

Hence, Lemma 4 gives us that $\hat{M} \cong \mathbb{Z}_p[1]$. This is interesting, because we have shown above that the classical (p)-adic completion of M is equal to zero.

3 Derived complete vs classically complete

Let $M \in \mathrm{Mod}_A$ be viewed as a complex in $D(A)$. By abuse of notation, we denote this complex by $M \in D(A)$. Let $f \in A$.

Proposition 6. *If M is classically (f)-complete, then M is derived (f)-complete. The converse is true when M is (f)-adically separated.*

Proof. Suppose that M is classically (f)-complete. Requiring the map φ_f to be an isomorphism is equivalent to requiring $\ker(\varphi_f) = \mathrm{coker}(\varphi_f) = 0$. We examine these two conditions.

For the kernel, we have the following. For $(m_n)_n \in \ker(\varphi)$, we have $m_n = fm_{n+1} = f^r m_{n+r}$ for all $r \geq 0$. Hence, $\bigcap_r f^r M = 0$ implies $\ker \varphi = 0$. For the cokernel we note that φ is surjective if and only if for all $(m_n)_n \in \prod M$ there exists a $(x_n)_n \in \prod M$ such that $m_n = x_n - fx_{n+1}$. For this we can take $x_n = \sum_{r=0}^{\infty} m_{n+r} f^r$, which exists because M is classically f-adically completed, i.e. $M = \lim_{\leftarrow r} M/f^r M$.

For the converse, note that I-adically separated implies that $\{M[f^n]\}$ is essentially zero, so $R^1 \lim M[f^n] = 0$. This finishes the proof by what was discussed in Section 2. □

Proposition 7. *Suppose that M is an A-module that is classically I-complete, so $\hat{M} = M$. Then M is also derived I-complete. The converse is true if M is I-adically separated.*

In fact, we have the following.

Proposition 8. *If A is Noetherian and M is finitely generated, then the notions of derived I-complete and classically I-complete are equivalent.*

Proof. By Proposition 7, we only need to show that if a A-module M is derived I-complete, then it is classically I-complete. This uses *Krull's intersection Theorem*:

Theorem 9. *Let $N = \bigcap I^n M$. Then there exists an $a \in A$ such that $a \equiv 1 \bmod I$ and $aN = 0$.*

Let $f := a - 1 \in I \subset A$ and $m \in \bigcap_n I^n M$. Consider

$$a\left(\sum_{j=0}^{N}(-f)^j\right)m = m - (-f)^{N+1}m.$$

On the one hand, it is equal to zero, because $am = 0$ but on the other it approaches m as $N \to \infty$ (here we use that φ_f is an isomorphism). Hence, $m = 0$ in M, so M is I-adically separated. □

4 Adjunctions

The derived I-complete A-complexes form a full triangulated subcategory

$$D_I(A) \subset D(A)$$

closed under inverse limits. This inclusion has a left adjoint $M \mapsto \hat{M}$.

In particular, this adjunction says that

$$\mathrm{Hom}_{D(A)}(\hat{M}, E) \cong \mathrm{Hom}_{D(A)}(M, E)$$

where $E \in D_I(A)$ and the bijection is given by composition with the natural map $M \to \hat{M}$.

The derived I-complete A-modules form an abelian subcategory

$$\mathrm{Mod}_A^I \subset \mathrm{Mod}_A$$

closed under kernels, cokernels and images. This inclusion has a left-adjoint $M \mapsto H^0(\hat{M})$.

In particular, this adjunction says that

$$\mathrm{Hom}_{\mathrm{Mod}_A}(H^0(\hat{M}), N) \cong \mathrm{Hom}_{\mathrm{Mod}_A}(M, N)$$

where $N \in \mathrm{Mod}_A^I$ and the bijection is given by composition with the natural map $M \to H^0(\hat{M})$.

If M is an A-module, we get two notions of *the* derived I-completion of M. For the first one, we regard M as a complex in $D(A)$ and we complete it: $\hat{M} \in D(A)$. For the second one, we need to take the cohomology group $H^0(\hat{M})$. Note the these may be different.

5 Properties

Proposition 10. *A derived I-complete A-complex $M \in D(A)$ is zero if and only if $M \otimes_A^L A/I \cong 0$.*

Proof. We follow [2, Lemma 0G1U] and [2, Lemma 0G1T].

Let $M \in D_I(A)$ and suppose $M \otimes_A^L A/I = 0$ where $I = (f_1, \dots, f_r)$. We have that $M = \hat{M} = R\lim(M \otimes^L_{\mathbb{Z}[\underline{x}]} \mathbb{Z}[\underline{x}]/(\underline{x}^n)) = R\lim(M \otimes_A^L K_n)$ where K_n is the Koszul complex on $f_1^n \dots f_r^n$ over A. We want to show that $M \otimes_A^L K_n = 0$. We will first show that $M \otimes_A^L N = 0$ where N is an A-module with $N = \cup N[I^n]$. Denote $A' := A/I^n \oplus N$. It suffices to show that $M' := M \otimes_A^L A' = 0$. Indeed, $M \otimes_A^L A' = (M \otimes_A^L A/I^n) \oplus (M \otimes_A^L N)$.

The map $A' \to A/I$ where $(x, y) \mapsto x$ has nilpotent kernel J. We have

$$0 = M \otimes_A^L A/I = (M \otimes_A^L A') \otimes_{A'}^L A/I = M' \otimes_{A'}^L A/I$$

because the derived tensor product is associative. Note that this also means that $M' \otimes_{A'}^L A'/J = 0$ as $A'/J \cong A/I$.

Claim: M' is bounded from above and if $H^b(M')$ is the right-most non-vanishing cohomology module, then it is a finite A'-module.

Proof. This is discussed in [2, Lemma 07LU]. □

As $M' \otimes_{A'}^L A'/J = 0$, we have $H^b(M')/JH^b(M') = H^b(M') \otimes_{A'}^L A'/J = 0$. To see the first equality, tensor the exact sequence $0 \to J \to A' \to A'/J \to 0$ with $H^b(M')$. Hence, $H^b(M') = JH^b(M')$ and we have that J is nilpotent, so $H^b(M') = 0$ by Nakayama and so $M' = 0$.

Now we can show by induction that $M \otimes_A^L K_n = 0$ by noting that the cohomology modules of K_n are I-power torsion (annihilated by I^{nr}). We omit the details. □

Proposition 11. *Let M be a derived complete A-module. If $M/IM = 0$, then $M = 0$.*

Proof. Suppose $I = (f_1, \dots, f_r)$ and that $M \neq 0$. Let $0 \leq i < r$ be maximal such that $N := M/(f_1, \dots, f_i)M \neq 0$. The map $f_{i+1} \colon N \to N$ is a surjective map of derived I-complete modules. Hence $\lim(\cdots \xrightarrow{f_{i+1}} N \xrightarrow{f_{i+1}} N) \neq 0$ as the limit must map surjectively to N. Contradiction, because N is derived I-complete. □

Proposition 12. *Let $f \colon : M \to N$ be a map in $D_I(A)$. If $\overline{f} : M/IM \to N/IN$ is surjecive, then f is surjective.*

Proof. Take W to be the cokernel of $\overline{f}$ and apply Proposition 11. □

6 Why is this important for prisms?

Proposition 13. *Let A be a δ-ring, and let $I \subset A$ be a finitely generated ideal containing p (A is dependent on a prime p). Then the derived I-completion $H^0(\hat{A})$ (as a module) admits a unique δ-structure compatible with the one on A.*

Proof. Let $w : A \to W_2(A)$ be the section classifying the δ-structure on A. Then we have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{w} & W_2(A) & \longrightarrow & W_2(H^0(\hat{A})) \\ \downarrow & & & \nearrow_{!} & \\ H^0(\hat{A}) & & & & \end{array}$$

where the dotted arrow is uniquely determined by the universal property of $A \to H^0(\hat{A})$. Explicitly, we have $W_2(H^0(\hat{A})) \in \mathrm{Mod}_A^I$ (because $W_2(H^0(\hat{A})) = H^0(\hat{A}) \times H^0(\hat{A})$ as a set), we have that

$$\mathrm{Hom}_{\mathrm{Mod}_A}(H^0(\hat{A}), W_2(H^0(\hat{A}))) \cong \mathrm{Hom}_{\mathrm{Mod}_A}(A, W_2(H^0(\hat{A})))$$

where a map is sent to the composition with $A \to H^0(\hat{A})$ (see [2, Lemma 091V]). The dotted arrow is now determined, and the diagram commutes. □

References

[1] Bhargav Bhatt. Geometric aspects of prismatic cohomology. https://www.math.ias.edu/ bhatt/teaching/prismatic-columbia/.

[2] The Stacks project authors. The stacks project, 2023.