PRELIMINARY PROGRAM: $\mathbb{E}_\infty$-ALGEBRAS (WITH AN EMPHASIS ON DERIVED ∞-CATEGORIES)

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1. Introduction: $\mathbb{E}_\infty$-algebras and Wortschatz

Clarify a list of notions and have a first glance of the role they play in the big picture. Distribution of the talks.

Part 1. Basic notions

2. Simplicial sets and infinity categories

Recall the basics about simplicial sets.

- Define basic notions: ∞-categories (as quasicategories), Kan complexes (=spaces), homotopy categories. Equivalences of ∞-categories
- Please do at least one proof concerning "filling the horn". You can freely choose the one you like best.
- Discuss (in $\leq$ 3 minutes) the topological model for the ∞-category theory, i.e. the relation between quasicategories and topological categories following [Lur09, Chapter 1], motivating the notation $\mathrm{Map}_{\mathcal{C}}(X, Y)$. (If time permits.)

- Define simplicial abelian groups / R-modules / etc. Sketch the proof of Dold-Kan correspondence (mainly state the functors in both directions).
- Dold-Kan and multiplicative structure: does not work before passing to homotopy categories [CC04]
- Define the nerve of a category. Recall how to realize a commutative diagram as a category. (Hence we can apply the nerve functor to a commutative diagram.)

3. Symmetric monoidal categories

Please freestyle. Make sure to include the category of chain complexes of R-modules and the category of simplicial R-modules as examples. (R is any (commutative unital) ring.) Maybe [TV08, p19-20] is a possible reference. (Honestly I forgot from where I read these materials myself.)

4. Model categories

- Introduce the basic notions of a model category following [Hov99, Chapter 1] or [Lur09, A.2].
- Discuss the projective model structure of the category of chain complexes of R-modules, and the category of simplicial R-modules as examples. Discuss the notion of a homotopy category, left / right Quillen functors and Quillen equivalences. You can find all these materials in [Hov99].
- Explain whether the Dold-Kan correspondence is a Quillen equivalence.
- Simplicial model category, homotopy coherent nerve
- The notion of a *underlying ∞-category* of a model category;
- For a simplicial model category, the equivalence of homotopy coherent nerve of fibrant-cofibrant objects and the nerve of cofibrant objects after inverting weak equivalences: this is Dwyer-Kan theorem, [Lur17, 1.3.4.20]

5. Derived categories (as triangulated categories)

Recall the basics of derived categories in the triangulated category setting. You can follow [Ill, Chapter 1, §1-§4].

6. Derived functors (as triangulated functors)

[Ill, Chapter 1, §5-§7]

Part 2. Trip towards $\mathbb{E}_\infty$-algebras

7. Nonabelian derived category theory: Animation (of a 1-category)

Main references: [CS24], [BL, Appendix A].

The major aim of this talk is to understand the main theorem for nonabelian derived category, [Lur09, 5.5.8.15(1)], explain LF is the left Kan extension of $F|_{\{\text{compact projectives}\}}$. (Follow [Lur24, Tag 02Y1] for an account of Kan extensions.) In particular, explain why animated rings are the same as simplicial rings.

8. Excursion: classical cotangent complex

Recall the classical cotangent complex following [Bha17, 3.1.2]. Construct it explicitly, and discuss [Bha17, Example 3.1.3]. (Optional: discuss the relation of the cotangent complex to a concrete deformation problem if you can!)

9. Example of a nonabelian derived functor: the cotangent complex

Define cotangent complex for an animated ring following [Lur18, §25.2]. See also [TV08, §1.2.1].

10. Stable ∞-categories, e.g. $\mathcal{D}(\mathcal{A})$

In this talk we enter the homotopical algebra with more "abelian taste". [Lur17, Chapter 1] [Lur04, §2.1].

- Define stable ∞-categories.
- Define the (unbounded) derived ∞-category of a Grothendieck abelian category.
- Examples of Grothendieck abelian categories: abelian groups, R-modules, sheaf of $\mathcal{O}_X$-modules, quasi-coherent sheaves on a scheme.
- Nonexample: coherent sheaves on a scheme.
- Sketch the main theorem: the homotopy category of a stable ∞-category is a triangulated category. Follow [Lur17] Ch. 1 or [Lur04, 2.1.14]. (Optional: Explain this point if you can do it in ≤ 3 minutes: any stable ∞-category is naturally enriched over spectra.)
- Recall the definition of a t-structure in a triangulated category, and define the t-stricture of an stable ∞-category.
- Specify all these notions with $D(\mathcal{A})$ for an Grothendieck abelian category $\mathcal{A}$. What is its heart with the standard t-structure, e.g. when $\mathcal{A}$ is the 1-category of R-modules?

11. Symmetric monoidal ∞-categories via correspondences

We introduce two definitions of symmetric monoidal ∞-categories: one is with the span (or correspondence) category following Cranch's thesis, the other one is as a section of a certain map of ∞-operads.

In this section we follow Cranch's thesis [Cra10] (see also Bachmann-Hoyois's Astérisque [BH21, Appendix C]) to introduce symmetric monoidal ∞-categories.

12. Symmetric monoidal ∞-categories via ∞-operads

(Heavy talk. Might need to split into two.)

- Define ∞-operads following [Lur17, 2.1.1.10]. Explain its relation with the 1-categorical notions: operads and colored operads.
- Define the underlying ∞-category of an ∞-operad. Define maps of ∞-operads.
- Examples.
  (1) (Commutative ∞-operad) The (nerve of the 1-category) $\mathbf{Fin}_*$ together with $\mathrm{id} : \mathbf{Fin}_* \to \mathbf{Fin}_*$ is an ∞-operad. This ∞-operad is denoted by $\mathbf{Comm}^{\otimes}$. Its underlying ∞-category is Δ^0. [Lur17, 2.1.1.18]
  (2) (Associative ∞-operad) Define 1-category $\mathbf{Assoc}^{\otimes}$ as follows. The objects are the same as the objects in $\mathbf{Fin}_*$. For morphisms, define

$$\mathrm{Hom}_{\mathbf{Assoc}^{\otimes}}(\langle m\rangle, \langle n\rangle) := \left\{ (\alpha, (\preceq_1, \dots, \preceq_n)) \,\middle|\, \begin{array}{c} \alpha \in \mathrm{Hom}_{\mathbf{Fin}_*}(\langle m\rangle, \langle n\rangle), \\ \preceq_i \text{ is a total order of } \alpha^{-1}(i) \end{array} \right\}.$$

  The composition rule is given by composition of the α's and the lexicographical ordering. ([Lur17, 4.1.1.3]) The nerve of this 1-category (together with the forgetful functor to $\mathbf{Fin}_*$) is an ∞-operad. This ∞-operad is also denoted by $\mathbb{E}_1^{\otimes}$ [Lur17, 5.1.0.7]. Its underlying ∞-category is Δ^0.

(3) Define the 1-category $\mathbf{LM}^{\otimes}$ as follows. The objects are pairs

$$(\langle n\rangle, S)$$

where $\langle n\rangle \in \mathbf{Fin}_*$, $S \subset \langle n\rangle^{\mathrm{o}}$ is a subset. For morphisms, define

$$\mathrm{Hom}_{\mathbf{LM}^{\otimes}}((\langle n\rangle, S), (\langle n'\rangle, S')) := \left\{ (\alpha, \preceq_i) \in \mathrm{Hom}_{\mathbf{Assoc}^*}(\langle m\rangle, \langle n\rangle) \,\middle|\, \begin{array}{c} \alpha(S \cup \{*\}) \subset S' \cup \{*\} \\ \text{"each fiber of points in } S' \text{ contains} \\ \text{exactly one maximal element"} \end{array} \right\}$$

(See [Lur17, 4.2.1.6] for the precise statements.) The nerve of this 1-category (together with the forgetful functor to $\mathbf{Fin}_*$) is an ∞-operad. The underlying ∞-category is $\Delta^0 \coprod \Delta^0$.
There are two natural functors linking the ∞-operad $\mathbf{Assoc}^{\otimes}$ and $\mathbf{LM}^{\otimes}$: the forgetful functor

$$\mathbf{LM}^{\otimes} \to \mathbf{Assoc}^{\otimes},$$

and the natural inclusion

$$\mathbf{Assoc}^{\otimes} \hookrightarrow \mathbf{LM}^{\otimes}.$$

(4) $\mathbb{E}_0^{\otimes}$: [Lur17, 2.1.1.19]

- Define symmetric monoidal ∞-categories as sections of maps of ∞-operads [Lur17, 2.1.2.19] (See also [Lur17, 2.1.2.18] and [Lur17, 2.0.0.7].)
- For a symmetric monoidal 1-category $\mathcal{C}$, $\mathcal{C}$ naturally gives rise to a colored operad $\mathcal{C}^{\otimes}$. The nerve $N(\mathcal{C}^{\otimes})$ of this colored operad is a symmetric monoidal ∞-category, whose underlying ∞-category is $N(\mathcal{C})$. [Lur17, 2.1.2.21]
- The nerve of (the construction of [Lur17, 2.1.1.7]) applied to) a colored operad in the 1-categorical sense is an ∞-operad [Lur17, 2.1.1.21]. The following important examples are all of this kind: $\mathrm{CAlg}(\mathcal{C}), \mathrm{Alg}(\mathcal{C}), \mathrm{LMod}(\mathcal{C})$.
- Objects in $\mathrm{CAlg}(\mathcal{C})$ are called *$\mathbb{E}_\infty$-algebras in $\mathcal{C}$*.
- Example: An $\mathbb{E}_\infty$-algebra in $\mathbf{Cat}_\infty$ is a symmetric monoidal ∞-category.
- Example: a *monoidal ∞-category* is an object in $\mathrm{Alg}(\mathrm{Cat}_\infty)$.
- For $A \in \mathrm{Alg}(\mathcal{C})$, define

$$\mathrm{LMod}_A(\mathcal{C}) := \mathrm{LMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \{A\}.$$

- Intuition: State [Lur17, 5.1.1.5] and 5.1.1.7 (for n-categories)
- State Barr–Beck–Lurie

13. $\mathcal{D}(R) = \mathbf{Mod}_R$ and hence is symmetric monoidal

Do this talk together with the intuition introduction [HA §3.4].

14. A glance in different models for ∞-categories and homotopical algebraic geometry

The aim of this talk is to locate better where we are in the big picture.

- Compare quasicategories, topological categories and simplicial model categories following [Lur09, Chapter 1].
- Compare simplicial rings (= animated rings in [CS24] = derived rings in [Lur04]) and $\mathbb{E}_\infty$-rings following [Lur18, Chapter 25].

Grossary

References

[BH21] Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. *Astérisque*, (425):ix+207, 2021.

[Bha17] Bhargav Bhatt. The hodge-tate decomposition via perfectoid spaces arizona winter school 2017. https://swc-math.github.io/aws/2017/2017BhattNotes.pdf, 2017.

[CC04] José Luis Castiglioni and Guillermo Cortinas. Cosimplicial versus DG-rings: a version of the Dold-Kan correspondence. *J. Pure Appl. Algebra*, 191(1-2):119–142, 2004.

[Cra10] James Cranch. Algebraic theories and (infinity,1)-categories, 2010.

[CS24] Kestutis Cesnavicius and Peter Scholze. Purity for flat cohomology. *Ann. of Math. (2)*, 199(1):51–180, 2024.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[Ill] Luc Illusie. Topics in algebraic geometry. Lecture notes, available at http://staff.ustc.edu.cn/~yiouyang/Illusie.pdf.

[Lur04] Jacob Lurie. Derived algebraic geometry. PhD thesis, available at https://dspace.mit.edu/handle/1721.1/30144#files-area, 2004.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur17] Jacob Lurie. Higher algebra. https://people.math.harvard.edu/~lurie/papers/HA.pdf, 2017.

[Lur18] Jacob Lurie. Spectral algebraic geometry. https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf, 2018.

[Lur24] Jacob Lurie. Kerodon. https://kerodon.net, 2024.

[TV08] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.