Group rings, etc.

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Definition 0.1. *Let G be a finite group and let R be a commutative ring. We set the* group algebra $R[G]$ *to be the ring generated over R by x_g, $g \in G$, with the multiplication rule $x_g * x_h = gh$ for $g, h \in G$. If R is a topological ring, we equip $R[G]$ with the topology from R.*

The utility of this definition is the following: if R is a field, then the category of finite dimensional R representations of G is equivalent to the category of finite left modules over $R[G]$.

In general, there is a natural ring homomorphism:

$$\varepsilon\colon R[G] \to R,$$

the augmentation homomorphism, that maps $x_g \mapsto 1$ for all $g \in G$. The kernel of ε, I, is often called the augmentation ideal and is generated by elements of the form $x_g - 1$.

Definition 0.2. *Let G be a profinite group. The* completed group algebra, $\mathbb{Z}_\ell[\![G]\!]$ *is set to be:*

$$\mathbb{Z}_\ell[\![G]\!] := \mathsf{lim}_\lambda \mathbb{Z}_\ell[G/N_\lambda]$$

where N_λ is the (filtered) set of open normal subgroups, each $\mathbb{Z}_\ell[G/N_\lambda]$, being a finite free $\mathbb{Z}_\ell$-module, is given the ℓ-adic topology, and $\mathbb{Z}_\ell[\![G]\!]$ is given the inverse limit topology.

The ring $\mathbb{Z}_\ell[\![G]\!]$ is compact. There is a natural embedding $G \hookrightarrow \mathbb{Z}_\ell[\![G]\!]^\times$. It is easy to see that the augmentation homomorphism extends to a continuous augmentation map:

$$\varepsilon\colon \mathbb{Z}_\ell[\![G]\!] \to \mathbb{Z}_\ell$$

with augmentation ideal $\mathscr{I}$. The augmentation ideal is *not necessarily topologically nilpotent* if G is not pro-ℓ; however, if we assume G is pro-ℓ, it is topologically nilpotent. Even in this case, the $\mathscr{I}$-adic topology is clearly *coarser* than the profinite topology, as one may see from the following example (for instance).

When G is a free pro-ℓ group on k generators $g_1, \ldots, g_k$, then

$$\mathbb{Z}_\ell[\![G]\!] \to \mathbb{Z}_\ell \ll u_1, \ldots, u_k \gg,$$

where the latter is the non-commuative power series ring over $\mathbb{Z}_\ell$, given by $x_{g_i} - 1 \mapsto u_i$, is an isomorphism.

Definition 0.3. *Let G be a pro-ℓ group. The* $\mathbb{Q}_\ell$-unipotent group ring *is the following inverse limit.*

$$\mathbb{Q}_\ell[\![G]\!] := \mathsf{lim}_n \left(\mathbb{Z}_\ell[\![G]\!]/\mathscr{I}^n \otimes \mathbb{Q}_\ell \right)$$

Here is the basic property of the unipotent group ring: the category of modules over $\mathbb{Q}_\ell[\![G]\!]$ that are finite dimensional as $\mathbb{Q}_\ell$-vector spaces is equivalent to the category of finite dimensional unipotent $\mathbb{Q}_\ell$ representations of G.[1]

Finally, we have the following basic result.

[1] A representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ is unipotent if for all $g \in G$, $\rho(g)$ has characteristic polynomial $(x-1)^n$. Equivalently, ρ is unipotent if it is conjugate to a representation valued in the subgroup $U \subset \mathrm{GL}_n(\mathbb{Q}_\ell)$ of upper-triangular matrices with 1s on the diagonal.

Proposition 0.4. *Let G be a topologically finitely generated pro-ℓ group. Let R be $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$. Then the map $g \mapsto g - 1$ induces an isomorphism:*

$$G^{ab} \otimes_{\mathbb{Z}_\ell} R \to \mathscr{I}/\mathscr{I}^2.$$

This applies in the following situation. Let X/k be a smooth variety over a perfect field of characteristic prime-to-ℓ. Let $G := \pi_1(X)_{\bar{k}}^{(\ell)}$ be the pro-ℓ completion of the geometric fundamental group. Then $G^{\mathrm{ab}} \otimes_{\mathbb{Z}_\ell} R \cong H_1(X_{\bar{k}}, R)$, i.e., the left hand side of the above equation is the first ℓ-adic homology group of $X_{\bar{k}}$. This isomorphism is compatible under Gal_k.