DEFORMATIONS OF REPRESENTATIONS FOR PROFINITE GROUPS

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ABSTRACT. In this talk, we show the existence of a universal deformation ring.

CONTENTS

1. PROFINITE GROUPS AND REPRESENTATIONS

We first recall some basic definitions from [Ser97]. Another reference is [RZ00].

Definition 1.1. *A group G is called a p-group if $|G| = p^r$ for some $r \in \mathbb{N}$.*

Definition 1.2. *Let G be a topological group.*

(i) We call G a profinite group if

$$G = \varprojlim_i G_i$$

as topological groups, where G_i are finite groups with discrete topology.

(ii) Let p be a prime number. If we have

$$G = \varprojlim_i G_i$$

as topological groups, where G_i are finite groups with cardinality p^{r_i}, $r_i \in \mathbb{N}$ for each i, and discrete topology, then we call G a pro-p group.

Remark 1.3. (i) Closed subgroups $H \subseteq G$ are profinite and G/H is compact and totally disconnected. (ii) Let L/K be an extension of commutative fields, then $Gal(L/K)$ is a profinite group.

Definition 1.4. *Let p be a prime number.*

(i) Let G be a profinite group. If a subgroup $H \subseteq G$ is a pro-p-group and the index $(G : H)$ is prime to p, then we call H a Sylow p-subgroup of G.

(ii) Let G be a profinite group, G' a subgroup. Then a pro-p quotient of G' is a group of the form G'/H, which is a pro-p group. We call G'/H a maximal pro-p quotient if G'/H is the largest possible quotient that is a pro-p group.

We would like to further more define the profinite completion and pro-p completion of a group.

Definition 1.5. *Let G be a group.*

(i) The profinite completion of G is defined as

$$\hat{G} = \varprojlim_{\substack{N \lhd G \\ G/N \text{ finite}}} G/N$$

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(ii) The pro-p completion of G is defined as

$$G_{\hat{p}} = \varprojlim_{\substack{N \lhd G \\ G/N \text{ is a } p\text{-group}}} G/N$$

Example 1.6. (i) A profinite completion of $\mathbb{Z}$ is

$$\hat{\mathbb{Z}} = \varprojlim_{n \in \mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$$

(ii) A pro-p completion of $\mathbb{Z}$ is the p-adic integers

$$\mathbb{Z}_p = \varprojlim_{n} \mathbb{Z}/p^n\mathbb{Z}$$

(iii) Now consider the multiplicative subgroup $\mathbb{Z}_p^* \subseteq \mathbb{Z}_p$. We have the identification

$$\mathbb{Z}_p^* = \varprojlim_{n} (\mathbb{Z}/p^n\mathbb{Z})^\times$$

This is a profinite group but not a pro-p group. In fact, for $p \neq 2$, we have $\mathbb{Z}^* = \mu_{p-1} \times \mathbb{Z}_p$, and $\mathbb{Z}_2^* = \{\pm 1\} \times \mathbb{Z}_2$. Thus it has a pro-$p$ subgroup.

(iv) $GL_n(\mathbb{Z}_p)$ is a profinite group but not a pro-p group. However, the congruence subgroup

$$\Gamma_1 := \{A \in GL_n(\mathbb{Z}_p) | A \equiv \mathrm{Id} \mod p\}$$

is a pro-p group.

Remark 1.7. Let G be a profinite group. When we say a representation $\rho : G \to \mathrm{GL}(V)$ is continuous, we mean that it is continuous with respect to the profinite topology on G and the discrete topology on $GL(V)$. This implies that ρ being continuous means $stab(v) = \{g \in G | \rho(g)v = v\}$ is open in G for all $v \in V$.

2. Finiteness condition Φ_p for profinite groups

Fix p a prime number.

Definition 2.1. *A profinite group G satisfies the finiteness condition Φ_p if the following equivalent conditions hold for all $G_0 \subseteq G$ open subgroups of finite index.*

(i) The maximal pro-p quotient of G_0 is topologically finitely generated.
(ii) The set of continuous group homomorphisms $Hom(G_0, \mathbb{F}_p)$ is a finite dimensional $\mathbb{F}_p$-vector space.

Remark 2.2. A topological group G is topologically finitely generated means that G has a dense subgroup that is finitely generated. The equivalence of the conditions follows from Burnside basis theorem. Also note that a finite dimensional $\mathbb{F}_p$-vector space has finite cardinality.

Example 2.3. (i) Let $K/\mathbb{Q}_p$ finite, then $\mathrm{Gal}(\bar{K}/K)$ satisfies Φ_q for all primes q.

(ii) Let $F/\mathbb{Q}$ finite, so F is a number field. Let S be a finite set of places in F. Let F_S be the maximal extension of F that is unramified ouside of S in an algebraic closure $\bar{F}$ of F. Then $\mathrm{Gal}(F_S/F)$ satisfies Φ_p for all p.

Proof. (i) Let $K/\mathbb{Q}_p$ be a finite extension of degree d and its residue field k has cardinality q a power of p. Since $\mathrm{Hom}(G_0, \mathbb{F}_p) \cong \mathrm{Hom}(G_0^{\mathrm{ab}}, \mathbb{F}_p)$, we may check this on K^{ab}/K, as the maximal abelian subgroup of $\mathrm{Gal}(\bar{K}/K)$ is isomorphic to $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. By local class field theory, we have

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\sim} \mathcal{O}_K^\times \times \widehat{\mathbb{Z}} \xrightarrow{\sim} \mu_{q-1} \times \mu_{p^a} \times \mathbb{Z}_p^d \times \prod_{\ell \text{ prime}} \mathbb{Z}_\ell \xrightarrow{\sim} \mu_{q-1} \times \mu_{p^a} \times \mathbb{Z}_p^{d+1} \times \prod_{\ell \neq p} \mathbb{Z}_\ell$$

where μ_{p^a} is the subgroup of all roots of unity in K. Now take a finite index open subgroup $G_0 \subseteq \mathrm{Gal}(K^{\mathrm{ab}}/K)$. Since G_0 is open, it contains a subgroup H_0 of the form

$$H_0 = p^{r_1}\mathbb{Z}_p \times \cdots \times p^{r_{d+1}}\mathbb{Z}_p \times \prod_{l \neq p} \ell^{s_\ell}\mathbb{Z}_\ell$$

Now let $f : G_0 \to \mathbb{F}_p$ be any continuous group homomorphism. Since $p^{r_i}\mathbb{Z}_p$ and $\ell^{s_\ell}\mathbb{Z}_\ell$ are neighborhood of zero in $\mathbb{Z}_p$ and $\mathbb{Z}_\ell$ respectively, we must have $f(H_0) = 0$. Hence every continuous group homomorphism $f \in \mathrm{Hom}(G_0, \mathbb{F}_p)$ factorise through G_0/H_0. Since for $\ell \neq p$, there is no nontrivial group homomorphism between $\mathbb{Z}/\ell^{s_\ell}\mathbb{Z}$ and $\mathbb{F}_p$, we know that $f \in \mathrm{Hom}(G_0, \mathbb{F}_p)$ are zero on the subgroup

$$H = p^{r_1}\mathbb{Z}_p \times \cdots \times p^{r_{d+1}}\mathbb{Z}_p \times \prod_{l \neq p} \mathbb{Z}_\ell$$

We also know that G_0 has finite index in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, so H has finite index in G_0. Therefore every continuous group homomorphism $f \in \mathrm{Hom}(G_0, \mathbb{F}_p)$ factorise a finite group G_0/H, and hence $\mathrm{Hom}(G_0, \mathbb{F}_p)$ is a finite dimensional $\mathbb{F}_q$-vector space.

(ii) The number field case uses global class field theory

$$\widehat{\mathbb{I}_K/K^\times} \xrightarrow{\sim} \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

We leave this as an exercise.

□

3. Category of deformations of Galois representations

The references for the following sections are [Böc13], [Lee], [Maz89], and [Kis]. Fix a prime p and a finite field $\mathbb{F}$ with characteristic p. For example, $\mathbb{F}_q$ with $q = p^r$ for some positive integer r. Then the ring of Witt vectors $W(\mathbb{F})$ is the ring of integers $\mathcal{O}_K$ for a finite extension $K/\mathbb{Q}_p$ with degree r (K is in fact unique).

Let G be a profinite group, $V_\mathbb{F}$ a finite dimensional $\mathbb{F}$-vector space with a continuous G-action. Recall that this means that the stabliser $stab(v)$ for each $v \in V_\mathbb{F}$ is an open subgroup in G.

Let $\mathfrak{LR}_{W(\mathbb{F})}$ be the category of finite local Artinian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$.

Definition 3.1. *Let $A \in \mathfrak{LR}_{W(\mathbb{F})}$ and G be a profinite group. Let $V_\mathbb{F}$ be a finite dimensional $\mathbb{F}$-vector space with a continuous G-action.*

(i) A defomation of $V_\mathbb{F}$ to A is the datum (V_A, ι), where
- *V_A is a free A-module with a continuous G-action,*
- *$\iota : V_A \otimes_A \mathbb{F} \xrightarrow{\sim} V_\mathbb{F}$ is a G-equivariant isomorphism of $\mathbb{F}$-vector spaces.*

(ii) Fix a $\mathbb{F}$-basis β of $V_\mathbb{F}$, a framed deformation of $(V_\mathbb{F}, \beta)$ to A is the datum (V_A, ι, β_A), such that
- *V_A is a free A-module with a continuous G-action,*
- *$\iota : V_A \otimes_A \mathbb{F} \xrightarrow{\sim} V_\mathbb{F}$ is a G-equivariant isomorphism of $\mathbb{F}$-vector spaces.*
- *β_A is a A-basis of V_A such that $\iota(\beta_A) = \beta$.*

Naturally we may define a functor from $\mathfrak{LR}_{W(\mathbb{F})}$ to the category of isomorphism classes of deformations (resp. framed deformations) of $V_\mathbb{F}$ (resp. $(V_\mathbb{F}, \beta)$).

Definition 3.2. *(i) Given $V_\mathbb{F}$, we have*

$$D_{V_\mathbb{F}}(A) := \{\text{isomorphism classes of deformations of } V_\mathbb{F} \text{ to } A\}$$

(ii) Given $(V_\mathbb{F}, \beta)$, we have

$$D^\square_{V_\mathbb{F}}(A) := \{\text{isomorphism classes of framed deformations of } (V_\mathbb{F}, \beta) \text{ to } A\}$$

Remark 3.3. Let $\overline{\rho} : G \to GL_n(\mathbb{F})$ be a continuous representation of a profinite group G, then a framed deformation of $\overline{\rho}$ is of the form $\rho : G \to GL_n(A)$. We also have that

$$D_{\overline{\rho}}(A) = D^\square_{\overline{\rho}}(A)/\ker(GL_n(A) \to GL_n(\mathbb{F}))$$

as sets, where $\ker(GL_n(A) \to GL_n(\mathbb{F}))$ acts on $D_{\overline{\rho}}(A)$ by conjugation.

Proposition 3.4. *Let p be a prime number and let G be a profinite group that satisfies the finiteness condition Φ_p. Let $V_\mathbb{F}$ be a finite dimensional $\mathbb{F}$-vector space with a continuous G-action.*

(i) The functor $D^\square_{V_\mathbb{F}}$ is pro-representable by a complete local Noetherian $W(\mathbb{F})$-algebra $R^\square_{V_\mathbb{F}}$. Namely, for all $A \in \mathfrak{LR}_{W(\mathbb{F})}$, there exists an isomorphism

$$D^\square_{V_\mathbb{F}}(A) \xrightarrow{\sim} \mathrm{Hom}_{W(\mathbb{F})}\left(R^\square_{V_\mathbb{F}}, A\right)$$

functorial in A.

(ii) If $End_{\mathbb{F}[G]}(V_\mathbb{F}) = \mathbb{F}$, then $D_{V_\mathbb{F}}$ is pro-representable by a complete local Noetherian $W(\mathbb{F})$-algebra $R_{V_\mathbb{F}}$.

Definition 3.5. *We call such $R^{\square}_{V_{\mathbb{F}}}$ the universal framed deformation ring and $R_{V_{\mathbb{F}}}$ the universal deformation ring for $V_{\mathbb{F}}$.*

Remark 3.6. (i) The proposition for framed deformations holds when G doesn't satisfy Φ_p, but $R^{\square}_{V_{\mathbb{F}}}$ might not be noetherian.

(ii) When a finite dimensional G-representation $V_{\mathbb{F}}$ over $\mathbb{F}$ satisfies $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, we call $V_{\mathbb{F}}$ Schur. Schur's lemma says that if a finite dimensional G-representation over $\mathbb{F}$ is absolutely irreducible (remains irreducible after base change to the algebraic closure of $\mathbb{F}$), then it is Schur.

Proof of Proposition 3.4 (i). Let $\dim_{\mathbb{F}} V_{\mathbb{F}} = n$, so we can write $V_{\mathbb{F}}$ as the continuous G-representation $\overline{\rho} : G \to GL_n(\mathbb{F})$. Since G is a profinite group, we may write $G = \varprojlim_\alpha G/H_\alpha$, where H_α are open normal subgroups of G with finite index and $H_\alpha \subseteq \ker(\overline{\rho})$. Now for each α, G/H_α is a finite group and has a group representation of the form $\langle g_1, ..., g_s | r_1(g_1, ..., g_s), ..., r_t(g_1, ..., g_s)\rangle$. We define the following $W(\mathbb{F})$-algebra:

$$\mathcal{R} := W(\mathbb{F}) \left[X_k^{i,j} | k = 1, ..., s, i, j = 1, ..., n \right] / \mathcal{I}$$

where $\mathcal{I} := (r_1(X_1, ..., X_s), ..., r_t(X_1, ..., X_s))$ is the ideal generated by the relations. We have the reduction map defined by

$$\begin{array}{ccc} \mathcal{R} & \longrightarrow & \mathbb{F} \\ X_k^{i,j} & \longmapsto & (i,j) - \text{th entry of } \overline{\rho}(g_k) \end{array}$$

Let $\mathcal{J} := \ker(\mathcal{R} \to \mathbb{F})$ be the kernel and we may take the $\mathcal{J}$-adic completion of $\mathcal{R}$. Define $R^{\square}_\alpha := \widehat{\mathcal{R}}^{\mathcal{J}}$. Let the matrix $\mathbf{X}_k$ be the image of X_k in $GL_n(R^{\square}_i)$. Then there exists an unique representation $\rho^{\square}_\alpha$ defined by the following:

$$\begin{array}{ccc} \rho^{\square}_\alpha : G/H_\alpha & \longrightarrow & GL_n(R^{\square}_\alpha) \\ g_k & \longmapsto & \mathbf{X}_k \end{array}$$

In particular, for each framed deformation $\rho_A : G \to GL_n(A)$ together with its restriction $\rho_A^\alpha : G/H_\alpha \to GL_n(A)$ to G/H_α, there exists a unique map $\phi : R^{\square}_\alpha \to A$

$$\begin{array}{ccc} \phi : GL(R^{\square}_\alpha) & \longrightarrow & GL(A) \\ X_k & \longmapsto & \rho_A^\alpha(g_k) \end{array}$$

such that $\rho_A^\alpha = \phi \circ \rho^{\square}_\alpha$. Thus the pair $\left(R^{\square}_i, \rho^{\square}_i\right)$ is universal for all i. By the profinite structure of G, we already have an inverse system determined by the index set I of i's. Thus we may take the projective limit

$$\left(R^{\square}_{V_{\mathbb{F}}}, \rho^{\square}_{V_{\mathbb{F}}}\right) := \varprojlim_i \left(R^{\square}_i, \rho^{\square}_i\right)$$

It is clear that $\left(R^{\square}_{V_{\mathbb{F}}}, \rho^{\square}_{V_{\mathbb{F}}}\right)$ satisfies the desired universal properties. We are left to show that $R^{\square}_{V_{\mathbb{F}}}$ is Noetherian. By construction, $R^{\square}_{V_{\mathbb{F}}}$ is a complete local ring. Denote its maximal ideal by $\mathfrak{m}$. It suffices to show that $\mathfrak{m}/(\mathfrak{m}^2, p)$ is a finite dimensional $\mathbb{F}$-vector space. $\square$

We postpone the proof for (ii) until Section 6.

4. The tangent space

Let $\mathbb{F}[\epsilon] = \mathbb{F}[X]/(X^2)$ be the ring of dual numbers over $\mathbb{F}$.

Definition 4.1. *We call $D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$ the Zariski tangent space of $D_{V_{\mathbb{F}}}$. Similarly, we call $D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$ the Zariski tangent space of $D^{\square}_{V_{\mathbb{F}}}$.*

Remark 4.2. By representability from Proposition 3.4, we have an isomorphism of $\mathbb{F}$-vector spaces

$$D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) = \mathrm{Hom}_{W(\mathbb{F})}(R_{V_{\mathbb{F}}}, \mathbb{F}[\epsilon]) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_R/(p, \mathfrak{m}_R^2), \mathbb{F}) = t_{R_{V_{\mathbb{F}}}}$$

where $t_{R_{V_{\mathbb{F}}}}$ is the tangent space of the local ring $R_{V_{\mathbb{F}}}$.

Lemma 4.3. *Denote the G-representation $End_{\mathbb{F}} V_{\mathbb{F}}$ by $adV_{\mathbb{F}}$.*

(i) There exists a canonical isomorphism

$$D_{V_{\mathbb{F}}}\left(\mathbb{F}[\epsilon]\right) \xrightarrow{\sim} H^1(G, \mathrm{ad}V_{\mathbb{F}})$$

(ii) If G satisfies Φ_p, then $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ is a finite dimensional $\mathbb{F}$-vector space.

(iii) $\dim_{\mathbb{F}} D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) = \dim_{\mathbb{F}} D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) + n^2 - h^0(G, \mathrm{ad}V_{\mathbb{F}})$

Proof. (i) Let $W_{\mathbb{F}}$ be any $\mathbb{F}$-module with G-action, then we have isomorphisms functorial in $W_{\mathbb{F}}$

$$\mathrm{End}_{\mathbb{F}}(W_{\mathbb{F}})^G \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}[G]}(W_{\mathbb{F}}, W_{\mathbb{F}})$$

Take the first derived functors of $\mathrm{End}_{\mathbb{F}}(-)^G$ and $\mathrm{Hom}_{\mathbb{F}[G]}(-,-)$ and we obtain an isomorphim

$$\mathrm{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}}) \xrightarrow{\sim} H^1(G, \mathrm{ad}V_{\mathbb{F}})$$

Take $V_{\mathbb{F}[\epsilon]} \in D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$. Since $V_{\mathbb{F}[\epsilon]}/\epsilon V_{\mathbb{F}[\epsilon]} \cong V_{\mathbb{F}}$, we have an exact sequence

$$0 \longrightarrow \epsilon V_{\mathbb{F}[\epsilon]} \longrightarrow V_{\mathbb{F}[\epsilon]} \longrightarrow V_{\mathbb{F}} \longrightarrow 0$$

Via $\epsilon \mapsto 1$ we have the identification $\epsilon V_{\mathbb{F}[\epsilon]} \cong V_{\mathbb{F}}$. Thus $V_{\mathbb{F}[\epsilon]}$ is an extension of $V_{\mathbb{F}}$ by $V_{\mathbb{F}}$, and $V_{\mathbb{F}[\epsilon]} \in \mathrm{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}}) \cong H^1(G, \mathrm{ad}V_{\mathbb{F}})$. Conversely, take $E \in \mathrm{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}}) \cong H^1(G, \mathrm{ad}V_{\mathbb{F}})$, we have an exact sequence equivariant under continuous G-action:

$$0 \longrightarrow V_{\mathbb{F}} \longrightarrow E \longrightarrow V_{\mathbb{F}} \longrightarrow 0$$

Identify $V_{\mathbb{F}}$ on the left with $\epsilon V_{\mathbb{F}[\epsilon]}$ via $1 \mapsto \epsilon$. Thus E is a $\mathbb{F}[\epsilon]$-module with continuous G-action, that reduce to $V_{\mathbb{F}}$ by the map on the right. Hence $E \in D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$.

(ii) Let G be a profinite group that satisfies Φ_p, and fix $\overline{\rho} : G \to GL(V_{\mathbb{F}})$. Take $G' = \ker(\overline{\rho})$, which is an open subgroup in G with finite index.

Let $r : GL(V_{\mathbb{F}[\epsilon]}) \to GL(V_{\mathbb{F}})$ be the reduction morphism. Notice that $\ker(r)$ is a pro-p group. Let $(V_{\mathbb{F}[\epsilon]}, \rho_{\mathbb{F}[\epsilon]})$ be a deformation of $\overline{\rho}$. Then $\overline{\rho} = r \circ \rho_{\mathbb{F}[\epsilon]}$ and similarly for their restrictions to G'. Since $\mathrm{Im}(\rho_{\mathbb{F}[\epsilon]}|_{G'}) \subseteq \ker(r)$, we know that $G'/\ker(\rho_{\mathbb{F}[\epsilon]}|_{G'})$ is a pro-p group. Then there exists a maximal pro-p quotient G'/H of G', through which $\rho_{\mathbb{F}[\epsilon]}|_{G'}$ factorizes.

Since G satisfies Φ_p, G'/H is topologically finitely generated. Thus G/H is topologically finitely generated. We know that all deformations $\rho_{\mathbb{F}[\epsilon]}$ factorizes through G/H, so we may conclude that $D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$ is a finite dimensional $\mathbb{F}$-vector space.

An alternate proof for this uses the inflation-restriction exact sequence

$$0 \to H^1(G/G', \mathrm{ad}V_{\mathbb{F}}^{G'}) \to H^1(G, \mathrm{ad}V_{\mathbb{F}}) \to H^1(G', \mathrm{ad}V_{\mathbb{F}})^{G/G'} \to \cdots$$

Now $H^1(G/G', \mathrm{ad}V_{\mathbb{F}}^{G'})$ is clearly a finite dimensional $\mathbb{F}_p$-vector space, and $H^1(G', \mathrm{ad}V_{\mathbb{F}})^{G/G'}$ is finite dimensional due to G satisfying Φ_p. Here we use the equivalent condition that $\mathrm{Hom}(G', \mathbb{F}_p)$ is a finite dimensional $\mathbb{F}_p$-vector space.

(iii) Fix $V_{\mathbb{F}}$ with a fixed basis and a deformation $V_{\mathbb{F}[\epsilon]}$. The set of $\mathbb{F}[\epsilon]$-basis lifting the basis of $V_{\mathbb{F}}$ has $\mathbb{F}$-dimension n^2. Let β, β' be two bases of $V_{\mathbb{F}[\epsilon]}$ lifting the fixed basis of $V_{\mathbb{F}}$. Compare them on the short exact sequence

$$0 \to \epsilon V_{\mathbb{F}[\epsilon]} \to V_{\mathbb{F}[\epsilon]} \to V_{\mathbb{F}} \to 0$$

Let $\iota : (V_{\mathbb{F}[\epsilon]}, \beta) \to (V_{\mathbb{F}[\epsilon]}, \beta')$. Then $\iota \mod \epsilon$ is the identity in $\mathrm{ad}V_{\mathbb{F}}$, and $\iota(\beta) \cong \beta'$ corresponds to elements in $(\mathrm{ad}V_{\mathbb{F}})^G$. Therefore, $\dim_{\mathbb{F}} D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) = \dim_{\mathbb{F}} D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) + n^2 - h^0(G, \mathrm{ad}V_{\mathbb{F}})$. Furthermore, we may conclude that fibres of

$$D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) \longrightarrow D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$$

are $\mathrm{ad}V_{\mathbb{F}}/(\mathrm{ad}V_{\mathbb{F}})^G$-torsors.

□

Corollary 4.4. *Lemma 4.3 (ii) shows that in Proposition 3.4, $R_{V_{\mathbb{F}}}$ and $R^{\square}_{V_{\mathbb{F}}}$ are Noetherian.*

5. Traces

Theorem 5.1 (Mazur '87, Carayol '91)**.** *Suppose that $V_{\mathbb{F}}$ is absolutely irreducible and $A \in \mathfrak{AR}_{W(\mathbb{F})}$. Let V_A and V'_A be two deformations such that*

$$\mathrm{tr}(\sigma|V_A) = \mathrm{tr}(\sigma|V'_A)$$

for all $\sigma \in G$. Then V_A and V'_A are isomorphic deformations.

6. Representability revisited

Here we present the proof of Proposition 3.4 (ii) by Mark Kisin as in [Kis, §3]. See [Kis09, Appendix A] for more details on the language of groupoids.
Recall that $\mathfrak{AR}_{W(\mathbb{F})}$ is the category of finite local Artinian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$.

Definition 6.1. *We define $D_{V_{\mathbb{F}}}$ as a groupoid over $\mathfrak{AR}_{W(\mathbb{F})}$ as follows.*

(i) Objects: For all $A \in \mathfrak{AR}_{W(\mathbb{F})}$, $D_{V_{\mathbb{F}}}(A)$ is a category whose objects are pairs (V_A, ι) such that
- *V_A is a free A-module with a continuous G-action,*
- *$\iota : V_{\mathbb{F}} \xrightarrow{\sim} V_A \otimes_A \mathbb{F}$ is a G-equivariant isomorphism of $\mathbb{F}$-vector spaces.*

The morphisms in $D_{V_{\mathbb{F}}}(A)$ are A-linear isomorphisms compatible with the G-action and the datum (V_A, ι_A).

(ii) Let $A \to A'$ be a morphism in $\mathfrak{AR}_{W(\mathbb{F})}$. A cover $(V_A, \iota_A) \to (V_{A'}, \iota_{A'})$ of $A \to A'$ in the 2-category $D_{V_{\mathbb{F}}}$ consists of an equivalence class $[\alpha]$, where $[\alpha] : V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$ is an A'-linear isomorphism, compatible with the G-action as well as the datum (V_A, ι_A) and $(V_{A'}, \iota_{A'})$. Two morphisms $[\alpha], [\alpha'] : V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$ are equivalent if they differ by multiplying an element of $A'^{\times}$.

Now let $\widehat{\mathfrak{AR}}_{W(\mathbb{F})}$ be the category of complete local Noetherian $W(\mathbb{F})$-algebras. The opposite category $(\widehat{\mathfrak{AR}}_{W(\mathbb{F})})^{\circ}$ is then equivalent to the category of formal spectra of $W(\mathbb{F})$-algebras.

Definition 6.2. *An equivalence relation $R \rightrightarrows X$ in $(\widehat{\mathfrak{AR}}_{W(\mathbb{F})})^{\circ}$ consists of a pair of morphisms $R \to X$ and $R \to X$ such that*

(i) $R \to X \times X$ is a closed embedding,
(ii) For all $T \in \widehat{\mathfrak{AR}}_{W(\mathbb{F})}$, $R(T) \subseteq (X \times X)(T)$ is an equivalence relation.

Consider the $W(\mathbb{F})$-group scheme PGL_n. Let $\mathrm{id} : \mathrm{PGL}_n \to \mathrm{Spec}\mathbb{F}$ be the composition of the reduction map $\mathrm{Spec}W(\mathbb{F}) \to \mathrm{Spec}\mathbb{F}$ and the identity section of the counit map $e : \mathrm{Spec}W(\mathbb{F}) \to \mathrm{PGL}_n$. Now $\ker(\mathrm{id})$ is a closed subscheme of PGL_n. Thus we may take the formal completion of the $\mathrm{PGL}_n/W(\mathbb{F})$ along $\ker(\mathrm{id})$. Denote this formal completion by $\widehat{\mathrm{PGL}_n}$, and this is a formal $W(\mathbb{F})$-scheme. Thus $\widehat{\mathrm{PGL}_n}$ is a group object in the category $(\widehat{\mathfrak{AR}}_{W(\mathbb{F})})^{\circ}$.
Recall from Proposition 3.4 (i) that $D^{\square}_{V_{\mathbb{F}}}$ is representable by $R^{\square}_{V_{\mathbb{F}}}$. Take the formal spectrum $X_{V_{\mathbb{F}}} := \mathrm{Spf}R^{\square}_{V_{\mathbb{F}}}$, and so $D^{\square}_{V_{\mathbb{F}}}$ is representable by $X_{V_{\mathbb{F}}}$.

Proposition 6.3 (Proposition 3.4)**.** *(ii) If $End_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, then $D_{V_{\mathbb{F}}}$ is pro-representable by a complete local Noetherian $W(\mathbb{F})$-algebra $R_{V_{\mathbb{F}}}$.*

Proof of Proposition 3.4 (ii). For $A \in \mathfrak{AR}_{W(\mathbb{F})}$, $\widehat{\mathrm{PGL}_n}(A)$ acts on each deformation V_A via conjugation. This action is functorial, and so we have a $\widehat{\mathrm{PGL}_n}$-action on $X_{V_{\mathbb{F}}}$. We would like to construct the quotient $X_{V_{\mathbb{F}}}/\widehat{PGL_n}$. To start, we want to consider the equivalence relation on $X_{V_{\mathbb{F}}}$ coming form the $\widehat{\mathrm{PGL}_n}$-action. We have an equivalence relation in $(\widehat{\mathfrak{AR}}_{W(\mathbb{F})})^{\circ}$:

$$\begin{array}{ccc} X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}_n} & \rightrightarrows & X_{V_{\mathbb{F}}} \\ (x,g) & \longmapsto & (x, gx) \end{array}$$

Indeed, by assumption $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, $\widehat{\mathrm{PGL}_n}$ acts freely on $X_{V_{\mathbb{F}}}$. Thus the induced map

$$\begin{array}{ccc} X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}_n} & \longrightarrow & X_{V_{\mathbb{F}}} \times X_{V_{\mathbb{F}}} \\ (x,g) & \longmapsto & (x, gx) \end{array}$$

is a closed immersion. Then we apply the theorem from [SGA3, VIIb, Thm 1.4] to the equivalence relation $\widehat{\mathrm{PGL}_n} \times X_{V_{\mathbb{F}}} \rightrightarrows X_{V_{\mathbb{F}}}$. We formulate it for our specific case as below.

Theorem 6.4. *Let $R \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_{V_{\mathbb{F}}}$ be an equivalence relation in $(\widehat{\mathfrak{AR}}_{W(\mathbb{F})})^{\circ}$ such that d_0 is flat. Then the quotient $X_{V_{\mathbb{F}}}/R$ exists in $(\widehat{\mathfrak{AR}}_{W(\mathbb{F})})^{\circ}$, and the canonical projection of $X_{V_{\mathbb{F}}}$ on $X_{V_{\mathbb{F}}}/R = \mathrm{Coker}(d_0, d_1)$ is surjective and flat. The morphism $R \to X_{V_{\mathbb{F}}} \times_{X_{V_{\mathbb{F}}}/R} X_{V_{\mathbb{F}}}$ induced by d_0 and d_1 is an isomorphism.*

Now we have obtained a quotient $X_{V_{\mathbb{F}}}/\widehat{PGL_n}$ in $(\widehat{\mathfrak{UR}}_{W(\mathbb{F})})^{\circ}$, which is the formal spectrum of a ring $R_{V_{\mathbb{F}}} \in \widehat{\mathfrak{UR}}_{W(\mathbb{F})}$. Now it follows from the construction of the quotient that for all $A \in \mathfrak{UR}_{W(\mathbb{F})}$, the category $D_{V_{\mathbb{F}}}(A)$ is determined by the functors $D_{V_{\mathbb{F}}}(R_{V_{\mathbb{F}}}) \to D_{V_{\mathbb{F}}}(A)$. Thus the groupoid $D_{V_{\mathbb{F}}}$ is representable by $R_{V_{\mathbb{F}}}$. □

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