Talk4

Freitag, 14. Mai 2021 12:14

# Weight filtration

- $K$ fin. gen. field of char $0$, $G_K = \mathrm{Gal}(\bar{K}/K)$, $\ell$ prime

- $X/K$ smooth, geom. conn. variety (int, sep, loc. fin. type)

$$X \hookrightarrow \bar{X}$$

$\bar{X}$ simple normal crossing compactification (ex. by Hironaka)

<u>Recall</u> (from Grétar)

- $G$ pro-$\ell$ group i.e. $G$ profinite s.t. $G = \varprojlim_{\substack{N \trianglelefteq^{\mathrm{op}} G \\ G/N \ \ell\text{-group}}} G/N$

- $\mathbb{Z}_\ell[[G]] = \varprojlim_{\substack{G \twoheadrightarrow H \\ H \text{ finite}}} \mathbb{Z}_\ell[H]$ „compl. group ring"

- For $H$ finite qu. of $G$ have augm. map

$$\mathbb{Z}_\ell[H] \longrightarrow \mathbb{Z}_\ell$$
$$\sum a_i h_i \longmapsto \sum a_i$$

$\overset{\text{limit}}{\rightsquigarrow}$ $\mathbb{Z}_\ell[[G]] \xrightarrow{\varepsilon} \mathbb{Z}_\ell$ „augm. map" with

kernel $I$

- $\mathbb{Q}_\ell[[G]] := \varprojlim_n \left( \mathbb{Z}_\ell[[G]] / I^n \otimes \mathbb{Q}_\ell \right)$ „$\mathbb{Q}_\ell$-unipotent group ring"

with augm. ideal given by

$$\ker\left( \mathbb{Q}_\ell[[G]] \longrightarrow \mathbb{Q}_\ell \right)$$

$\rightarrow$ $I$ defines topology on $R[[G]]$

⌜ $\{$f.g. top. $\mathbb{Q}_\ell[[G]]$ modules$\}$

$\updownarrow$ 1:1

$\{$f.d. unip. cont. $\mathbb{Q}_\ell$-reps of $G\}$ ⌟

For $G = \pi_1^\ell(X_{\bar{k}}, \bar{x})$, $R \in \{\mathbb{Z}_\ell, \mathbb{Q}_\ell\}$ we have addit.

$$\pi_1^\ell(X_{\bar{k}}, \bar{x})^{ab} \otimes_{\mathbb{Z}_\ell} R = I(X) / I(X)^2$$

and

$$\pi_1^\ell(X_{\bar k}, \bar x)^{ab} \otimes_{\mathbb{Z}_\ell} \mathbb{Q} \cong H_1(X_{\bar k}, \mathbb{Q}_\ell)$$

($G_k$ equi. as long as $X$ has $k$-rational points)

## The weight filtration

$\bar x$ geom. point of $X$, $R \in \{\mathbb{Z}_\ell, \mathbb{Q}_\ell\}$

$$\mathcal{J}(\bar x) = \ker\left(R[[\pi_1^\ell(X_{\bar k}, \bar x)]] \to R[[\pi_1^\ell(\bar X_{\bar k}, \bar x)]]\right)$$
$$\| $$
$$\mathcal{J}$$

induced by $X \hookrightarrow \bar X$

**Def$^n$:** i) $R = \mathbb{Q}_\ell$

- $W^i = \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar k}, \bar x)]]$ $\quad i \geq 0$
- $W^{-1} = I$
- $W^{-2} = I^2 + \mathcal{J}$
- $W^{-l} = \sum_{\substack{a+b=l \\ a,b>0}} W^{-a} \cdot W^{-b}$ for $l > 2$

ii) $R = \mathbb{Z}_\ell$

- $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar k}, \bar x)]] \to \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar k}, \bar x)]]$

$i: \mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \to \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$

$$W^i\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \to i^{-1}(W^i\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]])$$

Remark:

$$W^{-i} = I \cdot W^{-i+1} + \mathcal{J} \cdot W^{-i+2}$$

Prop 1 $I^n \subset W^{-n}$, $W^{-2n-1} \subset I^n$

$\rightsquigarrow$ $W^\bullet$-adic top. = $I^\bullet$-adic top.

For $x \in X(k)$ with geom. point $\bar{x}$, we get $G_k \curvearrowright \pi_1^\ell(X_{\bar{k}}, \bar{x})$

Main theorem For all $\alpha \in \mathbb{Z}_\ell^\times$ sufficently close to 1, there exists $\sigma_\alpha \in G_k$ s.t

$\sigma_\alpha$ acts on $gr_W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ with $\alpha^i \cdot \mathrm{Id}$

For the proof need some lemmas.

Lemma 1: $\mathbb{F}$ finite field, $Y_{/\mathbb{F}}$ sm. geom. conn. variety

$\bar{Y}$ sm. compactific. w/ simple. nor. crossing boundary

Then

- $G_\mathbb{F}$ acts semisimply on $H^1(Y_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell)$
- $G_\mathbb{F}$-rep $H^1(Y_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell)$ is mixed of weight 1 and 2

where the 1 piece is

$\mathrm{Im}(H^1(\bar{Y}_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell) \to H^1(Y_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell))$

$\mathrm{Im}(H^1(\bar{F}_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell) \to H^1(\ldots_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell))$

<u>Proof:</u> $j: Y \hookrightarrow \bar{Y}$, $E_1, \ldots, E_n$ irred. cop. of $\bar{Y} \setminus Y$

- by Leray sequence for $Rj_*$ and observ. of Deligne we get a long exact sequence

$$0 \to H^1(\bar{Y}_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell) \to H^1(Y_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell) \to \bigoplus_i H^0(E_{i,\bar{\mathbb{F}}}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(-1)$$
$$\to H^2(\bar{Y}_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell) \to \ldots$$

- by Weil conj. $H^1(Y_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell)$ is pure of weight 1 + $G_{\mathbb{F}} \curvearrowright H^1(\bar{Y}_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell)$ semisimply by Tate.

- similar $V = \ker(\bigoplus H^0(E_{i,\bar{\mathbb{F}}}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(-1) \to H^2(\bar{Y}_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell))$ is pure of weight 2 + $G_{\mathbb{F}}$ acts semisimply

$\rightsquigarrow$ Have s.e.s.

$$0 \to H^1(\bar{Y}_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell) \to H^1(Y_{\bar{\mathbb{F}}}, \mathbb{Q}_\ell) \to V \to 0$$

which splits as outer terms have different weights. ∎

<u>Lemma 2:</u> For all $\alpha \in \mathbb{Q}_\ell^\times$ suff. close to 1 there exist $\sigma_\alpha \in G_K$

s.t. $\sigma_\alpha$ acts on $gr_W^{-i}(H_1(X_{\bar{k}}, \mathbb{Q}_\ell))$ via mult by $\alpha^i$

$[W^{-i} H_1(X_{\bar{k}}, \mathbb{Q}_\ell) = H_1(X_{\bar{k}}, \mathbb{Q}_\ell) \quad i \geq -1$

$W^{-2} H_1(X_{\bar{k}}, \mathbb{Q}_\ell) = \ker(H_1(X_{\bar{k}}, \mathbb{Q}_\ell) \to H_1(X_{\bar{k}}, \mathbb{Q}_\ell))$

$W^{i} H_1(X_{\bar{k}}, \mathbb{Q}_\ell) = 0 \quad i < -2]$

<u>Proof:</u> Consider nat. act.

$$\rho : G_k \to \mathrm{Gal}(H_1(X_{\bar{k}}, \mathbb{Q}_\ell))$$

<u>Step 1:</u> exist $\sigma_\alpha \in \overline{\mathrm{Im}(\rho)}$
<u>Step 2:</u> $\mathrm{Im}(\rho) \overset{\text{open}}{\subset} \overline{\mathrm{Im}(\rho)}$
$\Big\}$ $\Rightarrow$ conclusion

<u>Step 1:</u> · by result of Deligne about weight filtration on $H_1(X_{\bar{k}}, \mathbb{Q}_\ell)$ there exist $\gamma \in G_k$ s.t. $\gamma$ acts on $gr_W^{i}(H_1(X_{\bar{k}}, \mathbb{Q}_\ell))$ with weight $i$

· by definition have s.e.s.

$$0 \to gr_W^{-2}(H_1(X_{\bar{k}}, \mathbb{Q}_\ell)) \to H_1(X_{\bar{k}}, \mathbb{Q}_\ell) \to gr_W^{-1}(H_1(X_{\bar{k}}, \mathbb{Q}_\ell)) \to 0$$

As $\gamma$ acts with weight $i$ get splitting and hence a canonical $\gamma$-equivariant iso

$$gr_w^{-1}(H_1(X_{\bar{k}}, \mathbb{Q}_\ell)) \oplus gr_w^{-2}(H_1(X_{\bar{k}}, \mathbb{Q}_\ell)) \cong H_1(X_{\bar{k}}, \mathbb{Q}_\ell) \quad (*)$$

By Lemma 1 have finite $L/\mathbb{Q}_\ell$ s.t. $\gamma \curvearrowright H_1(X_{\bar{k}}, L)$ is diagonal.

Here we use the generalized notion for purity.

$\rightsquigarrow$ Choose basis $\{e_i\}$ of $\gamma$-eigenvectors of $H_1(X_{\bar{k}}, L)$

$\{e_1, \dots, e_r\}$ basis of $gr_w^{-2}$ $\qquad$ $\{e_{r+1}, \dots, e_n\}$ basis of $gr_w^{-1}$

i.e. $\gamma . e_i = \lambda_i e_i$ $\quad |\lambda_i| = q \quad 1 \leq i \leq r$

$\qquad |\lambda_i| = q^{\frac{1}{2}} \quad r+1 \leq i \qquad (**)$

- $T := \overline{\{\gamma^n\}}^{\circ} \subset \overline{\{\gamma^n\}}$ is a subtorus of the diag. torus

$D \in GL(H_1(X_{\bar{k}}, L))$ w.r.t. to $\{e_i\}$

- $T \hookrightarrow D$ ind. $X^*(D) \xrightarrow{\pi} X^*(T)$

where $\ker(\pi) = \{ \underline{a} \in \mathbb{Z}^n \mid \prod \lambda_i^{a_i} = 1 \}$

i.e. $T = \{ M \in D \mid \chi(M) = 1 \text{ for } \chi \in K \}$

But then $q^a \cdot Id_{gr_w^{-1}} \oplus q^{2a} \cdot Id_{gr_w^{-2}} \in T$ for all $a \in \mathbb{Z}$

Hence the closure $T'$ is given by

$$\alpha \cdot Id_{gr_w^{-1}} \oplus \alpha^2 Id_{gr_w^{-2}}, \ \alpha \in \mathbb{Q}_\ell^\times$$

is in $T$. As $(\ast\ast)$ is def. / $\mathbb{Q}_\ell$ $T' \subset \overline{I_{\mathbb{Z}}(p)}$

Step 2: - follows for $k$ number field by result of Bogomolov
- For general $K$ can reduce to number fields by argument of Ribet

Conclusion: $I_{\mathbb{Z}}(p) \cap T' \subset T'$ open and non-empty and hence contains non-empty open nbh of $1 \in T'$. $\square$

Theorem 3: $G_\alpha$ as in Lemma 2, $\alpha$ not a root of unity, $x \in X(k)$. Then $G_\alpha \curvearrowright \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/I^n$ semisimple for all $n$.

Proof of Main Theorem:

- by Prop 1 is enough to show claim for $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]/I^n$ with induced $W$-filtration
- $G_\alpha$ as in Lemma 2

Claim $G_\alpha$ is the desired one

Theorem 3 gives implicitly a $G_\alpha$ equ. splitting $s$ of

$$\begin{tikzcd} I/I^{n} \arrow[r] & I/I^{2} \arrow[l, bend right, "s"'] \end{tikzcd}$$

This induces naturally

$$\bigoplus_{i=0}^{n-1} \left(I/I^2\right)^{\otimes i} \xrightarrow{\oplus s^{\otimes i}} \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{K}}, \bar{x})]]/I^n$$

$$\cong$$

$$\bigoplus_{i=0}^{n-1} H_1(X_{\bar{K}}, \mathbb{Q}_\ell)^{\otimes i}$$

which is

- surj. by induction on $n$
- $G_\alpha$-equivariant as $s$ is

Then we are done by multiplication of $W$-filtration.

$\blacksquare$