Oberseminar 28.04.21

Part 1:

Recall two ways to think about $\pi_1^{ét}(X, \bar{x})$.

i) Finite étale covers $Y \to X$

$Fét_X$. Fixed geom. point

$\bar{x} : \mathrm{Spec}(\Omega) \to X$, Ω is alg. closed

$Fib_{\bar{x}} : Fét_X \to Set$

$$Y \longmapsto Y_{\bar{x}}$$

$$\pi_1^{ét}(X, \bar{x}) = \mathrm{Aut}(Fib_{\bar{x}})$$

ii) Special Galois étale covers

$$Y \to X$$

This means $\mathrm{Aut}_X(Y)$ acts

transitively on $Y_{\bar{x}}$

$$\pi_1^{ét}(X, \bar{x}) = \varprojlim \mathrm{Aut}_X(Y)$$

$\exists$ $\tilde{Y}$ universal cover

s.t. $\pi_1^{ét}(X, \bar{x}) = \mathrm{Aut}_X(\tilde{Y})$

- If $\mathrm{char}(k) = 0$ and $\bar{k} \subset \Omega$ extension of alg. closed fields then
$$\pi_1^{ét}(X_{\bar{k}}, \bar{x}) = \pi_1^{ét}(X_\Omega, \bar{x})$$
(X some nice scheme over k)

We have a fundamental SES often called Homotopy exact seq.

<u>Prop:</u> Let X be quasi-compact and geom. integral over a field k. Fix an alg. closure $\bar{k} \supset k$ and let k_s/k be the separable closure inside $\bar{k}$ write $\bar{X} := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k_s)$ and

let $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$ be a geom point

Then we have a SES

$$\boxed{1 \to \pi_1^{et}(\bar{X}, \bar{x}) \to \pi_1^{et}(X, \bar{x}) \to Gal(k_s/k) \to 1}$$

induced by the maps $\bar{X} \to X$ and $X \to Spec(k)$.

In general let

$$1 \to N \to G \to \Gamma \to 1$$

be a SES of profinite groups.

- N is normal

$$\Rightarrow G \longrightarrow Aut(N)$$

by conjugation

$$N \longrightarrow Inn(N)$$

$\Rightarrow$ in the quotient

$$\Gamma \twoheadrightarrow Out(N) = Aut(N)/Inn(N)$$

In our case we get an action of $Gal(k_s/k)$ by outer autom. on $\pi_1^{et}(\bar{X}, \bar{x})$.

• We have a GAGA type theorem

<u>Theorem</u>: Let X be a connected scheme of finite type $/\mathbb{C}$

then the functor

$$(Y \to X) \longmapsto (Y^{an} \longrightarrow X^{an})$$

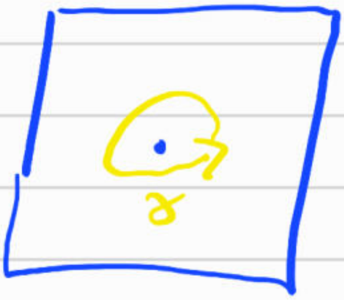
induces an eq. of categories of finite étale covers of X and the finite topological covers of X^{an}. For any $\mathbb{C}$-point $\bar{x} \in X$

this induces an isom.

$$\widehat{\pi_1^{top}(X^{an}, \bar{x})} \simeq \pi_1^{et}(X, \bar{x})$$

<u>Example</u>: Take $\mathbb{G}_m$. In the analytic setting we get

$$\pi_1^{top}(\mathbb{G}_m, x) = \mathbb{Z}$$



But $\mathbb{G}_m = \mathrm{Spec}(\mathbb{C}[t, t^{-1}])$

the fin. étale covers are

$$Y_n = \mathrm{Spec}(\mathbb{C}[t^{1/n}, t^{-1/n}])$$

$$\mathrm{Aut}_{\mathbb{G}_m}(Y_n) = \mathbb{Z}/n\mathbb{Z}$$

$$\rightsquigarrow \pi_1^{\text{ét}}(\mathbb{G}_m, \bar{x}) = \hat{\mathbb{Z}}$$

- What I've said so far gives if X is a scheme over $\mathbb{Q}$ then we have a SES

$$1 \to \widehat{\pi_1^{\text{top}}(X^{an}, x)} \to \pi_1^{\text{ét}}(X, \bar{x}) \to \mathrm{Gal}(\mathbb{Q}) \to 1$$

$\Rightarrow$ have an action of $\mathrm{Gal}(\mathbb{Q})$ on $\widehat{\pi_1^{\text{top}}(X^{an}, x)}$

Example: $\pi_1^{top}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}) = F_2$

$\rightsquigarrow$ action of $\mathrm{Gal}(\bar{\mathbb{Q}})$ on $\widehat{F_2}$

$\varprojlim_{N \text{ is of finite index}} F_2/N$

There is an equivalence of categories

$\{$ cont. f.d. $\mathbb{Q}_\ell$-rep^{ns} of $\pi_1^{et}(X, \bar{x})\}$

$\simeq$

$\{ \mathbb{Q}_\ell$-local systems on $X\}$

- X is connected, locally Noeth. and ℓ is invertible in X

- I'm not going to say what a $\mathbb{Q}_\ell$-local system.

- Work of Bhatt - Scholze

pro-étale topology

⇝ · cohom

· fundam. grp

$$\widehat{\pi_1^{proet}} = \pi_1^{\acute{e}t}$$

But "very often / usually"

$$\pi_1^{proet} \neq \pi_1^{\acute{e}t}$$

· The correspondence

$$\{\pi_1^{proet} - \text{rep}^{\text{ns}}\}$$

$$\sim \{\mathbb{Q}_\ell - \text{local systems}\}$$

$$\neq$$

"honest local systems"

Part 2

We fix some notation

- k is a f.g. field of char. 0
  fix $\bar{k}$ an alg. closure of k.

- X is a smooth, geom. connected variety $/k$

  Non example $k = \mathbb{Q}$

  $$X = \operatorname{Spec}\left(\frac{\mathbb{Q}[T]}{T^2 - 2}\right)$$

  $X_{\bar{\mathbb{Q}}}$ is two points

- $\bar{X}$ is a simple normal crossing compactification of X

  $X \hookrightarrow \bar{X}$ open subscheme

  $D = \bar{X} \setminus X$ locally looks like an intersection of coordinate hyperplanes

$\overline{X}$ is proper $/k$

Example:

$$X = \mathbb{A}^n_k \;, \quad \overline{X} = \mathbb{P}^n_k$$

Definition: Let ℓ be a prime and Γ a group. The pro-ℓ completion of Γ is defined as

$$\hat{\Gamma}_{(\ell)} = \varprojlim_{\substack{N \trianglelefteq \Gamma \\ \Gamma/N \text{ is an } \ell\text{-group}}} \Gamma/N$$

Standard example:

$$\Gamma = \mathbb{Z} \;, \quad \hat{\Gamma}_{(\ell)} = \mathbb{Z}_\ell .$$

G a semi-simple alg. group

$G(\mathbb{Z}_\ell)$ is not (in general) pro-ℓ group

but has an open subgroup that

is

Let $\bar{x} : \bar{k} \to X$ be a geom point. We will be considering the geom. fundam. group
$\pi_1^{et}(X_{\bar{k}}, \bar{x})$

We write $\pi_1^{\ell}(X_{\bar{k}}, \bar{x})$ for the pro-ℓ completion of $\pi_1^{et}(X_{\bar{k}}, \bar{x})$.

<u>Paths</u>: Let $\bar{x} : \operatorname{Spec}(\bar{k}) \to X$, $\bar{y} : \operatorname{Spec}(\bar{k}) \to X$

a path from $\bar{x}$ to $\bar{y}$ is an isom. of functors $\mathrm{Fib}_{\bar{x}} \to \mathrm{Fib}_{\bar{y}}$.

The set of all paths $\bar{x}$ to $\bar{y}$ is denoted
$$\pi_1^{et}(X; \bar{x}, \bar{y})$$

We have a left action of $\pi_1^{et}(X_{\bar{k}}, \bar{x})$ on $\pi_1^{et}(X_{\bar{k}}, \bar{x}, \bar{y})$ given by
$$\varphi * t = t \circ \varphi^{-1}$$

this makes $\pi_1^{et}(X_{\bar{k}}; \bar{x}, \bar{y})$ a left $\pi_1^{et}(X_{\bar{k}}, \bar{x})$-torsor

Define

$$\pi_1^\ell(X_{\bar k};\bar x,\bar y)$$

$$:= \pi_1^\ell(X_{\bar k},\bar x) \times^{\pi_1^{ét}(X_{\bar k},\bar x)} \pi_1^{ét}(X_{\bar k};\bar x,\bar y)$$

because we have a quot. map

$$\pi_1^{ét}(X_{\bar k},\bar x) \to \pi_1^\ell(X_{\bar k},\bar x)$$

so $\pi_1^\ell(X_{\bar k};\bar x,\bar y)$ is somthing like

elements (h,s), $h\in \pi_1^\ell(X_{\bar k},\bar x)$

$s\in \pi_1^{ét}(X_{\bar k};\bar x,\bar y)$ s.t.

$$(h,s)\sim (gh,gs)$$

for all $g\in \pi_1^{ét}(X_{\bar k},x)$

$\pi_1^\ell(X_{\bar k};\bar x,\bar y)$ is also a right $\pi_1^\ell(X_{\bar k},\bar y)$ torsor

- **Definition**:

$$\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}};\bar{x},\bar{y})]] := \varprojlim_{\pi_1^\ell(X_{\bar{k}};\bar{x},\bar{y}) \twoheadrightarrow H} \mathbb{Z}_\ell[H]$$

H finite

- $\mathbb{Z}_\ell[H]$ is the free $\mathbb{Z}_\ell$-mod on H

if $\bar{x}=\bar{y}$ then $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}},\bar{x})]]$ is called the pro-ℓ group ring or the ℓ-adic group ring.

- $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}};\bar{x},\bar{y})]]$ is a $\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}},\bar{x})]]$-$\mathbb{Z}_\ell[[\pi_1^\ell(X_{\bar{k}},\bar{y})]]$-bimodule.

- We have an equivalence

$\{$ f.g. $\mathbb{Z}_\ell[[\bar\pi_1^\ell(X_{\bar k},\bar x)]]$-mod$\}$

and $\{$cont. rep^n of $\bar\pi_1^\ell(X_{\bar k},\bar x)$ on f.g. free $\mathbb{Z}_\ell$-modules.$\}$

- Augmentation ideal:

For each finite quot. H we have an augment. map

$$\mathbb{Z}_\ell[H] \to \mathbb{Z}_\ell$$
$$h \longmapsto 1$$

get an augm. map in limit

$$\mathbb{Z}_\ell[[\bar\pi_1^\ell(X_{\bar k},\bar x)]] \to \mathbb{Z}_\ell$$

The augm. ideal is the kern of this map. Call it $\mathcal{I}(\bar x)$.

We get an $\mathcal{I}$-adic filtration

Let

$$\mathcal{I}(\bar x,\bar y)^n = \mathcal{I}(\bar x)^n \cdot \mathbb{Z}_\ell[[\bar\pi_1^\ell(X_{\bar k},\bar x,\bar y)]]$$

$\implies$ get $\mathcal{I}$-adic topology on

$\mathbb{Z}_\ell[\![\pi_1^\ell(X_{\bar{k}}, \bar{x}, \bar{y})]\!]$

- Top. properties are subtle
- $\mathcal{I}$-adic top. is coarser than profinite topology.

- Pro unipotent groupoids

Define $\mathbb{Q}_\ell[\![\pi_1^\ell(X_{\bar{k}}; \bar{x}, \bar{y})]\!]$

as $\varprojlim_n \left(\mathbb{Z}_\ell[\![\pi_1^\ell(X_{\bar{k}}; \bar{x}, \bar{y})]\!]/\mathcal{I}(\bar{x}, \bar{y})^n \otimes \mathbb{Q}_\ell\right)$

By abusing notation a bit, we write

$$\mathcal{I}(\bar{x}, \bar{y})^n := \ker\Big(\mathbb{Q}_\ell[\![\pi_1^\ell(X_{\bar{k}}, \bar{x}, \bar{y})]\!] \longrightarrow \mathbb{Z}_\ell[\![\pi_1^\ell(X_{\bar{k}}; \bar{x}, \bar{y})]\!]/\mathcal{I}(\bar{x}, \bar{y})^n \otimes \mathbb{Q}_\ell\Big)$$

and we topologize $\mathbb{Q}_\ell[\![\pi_1^\ell(X_{\bar{k}}, \bar{x}, \bar{y})]\!]$ with the $\mathcal{I}$-adic top.

If $\bar{x} = \bar{y}$ this is the pro-unip. group ins

($\mathbb{Q}_\ell$ - Mal'cev Hopf algebra)

Why pro-unipot.

We have an equiv. of categories

$$\{ \text{f.g top } \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]\text{-mod} \}$$

$$\simeq$$

$$\{ \text{f.d. unipot. continuous } \mathbb{Q}_\ell\text{-rep}^{\text{n}} \text{ of } \pi_1^\ell(X_{\bar{k}}, \bar{x}) \}$$

Example: suppose $\pi_1^\ell(X_{\bar{k}}, \bar{x})$ is a free pro-ℓ group on generators $\gamma_1, \dots, \gamma_m$

Then the map $\gamma_i \mapsto T_i + 1$ induces an isom.

$$R[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]] \to R\langle\langle T_1, \dots, T_m \rangle\rangle$$

When $R\langle\langle \cdots \rangle\rangle$ is non comm. power serie and R is $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$

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If $n=1$, so $\mathcal{I}_1^\ell(X_{\bar{\eta}},\bar{x}) = \mathbb{Z}_\ell$

f_n be the Weierstrass polynomial

$f_n = \underbrace{(T+1)^{\ell^n} - 1}$

(Weierstrass poly.

$T^s + a_{s-1}T^{s-1} + \cdots + a_0$

s.t. $a_{s-1}, \ldots, a_0 \in \ell\mathbb{Z}_\ell$)

injection

$\mathbb{Z}_\ell[T] \hookrightarrow \mathbb{Z}_\ell[[T]]$

induce an isom

$\mathbb{Z}_\ell[T]/(f_n) \xrightarrow{\sim} \mathbb{Z}_\ell[[T]]/(f_n)$

Consider the map

$$\varphi : \mathbb{Z}_\ell[[T]] \to \mathbb{Z}_\ell[[\mathbb{Z}_\ell]]$$

$$T \mapsto \gamma - 1$$

(γ some top. gen. of $\mathbb{Z}_\ell$)

$$\mathbb{Z}_\ell[[T]]/f_n \xrightarrow{\sim} \mathbb{Z}_\ell[T]/(f_n)$$

$$\xrightarrow{\psi} \mathbb{Z}_\ell[\underbrace{\mathbb{Z}_\ell/\Gamma_n}]$$

where Γ_n is the unique subgp of $\mathbb{Z}_\ell$ of index ℓ^n

ψ is given by

$$T \bmod f_n \mapsto \gamma - 1 \bmod \Gamma_n$$

$$f_{n+1} = f_n \cdot \left((T+1)^{\ell^n(\ell-1)} + \dots + (T+1)^{\ell^n} + 1 \right)$$

$\Rightarrow$ commutative diagr

$$\begin{tikzcd}
\mathbb{Z}_\ell[[T]]/f_{n+1} \arrow[r, "\sim"] \arrow[d, two heads] & \mathbb{Z}_\ell[T]/f_{n+1} \\
\mathbb{Z}_\ell[[T]]/f_n \arrow[r, "\sim"] & \mathbb{Z}_\ell[T]/f_n
\end{tikzcd}$$

$\rightsquigarrow$ gives an isom. in the limit

$$\begin{tikzcd}
\varprojlim_n \mathbb{Z}_\ell[T]/f_n \arrow[r, "\sim"] \arrow[d, phantom, "\cong"] & \mathbb{Z}_\ell[[\mathbb{Z}_\ell]] \\
\mathbb{Z}_\ell[[T]] &
\end{tikzcd}$$

<u>Proposition:</u> Let $R = \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ and $I(\bar{x})$ be the augm. ideal of $R \otimes \pi_1^\ell(X_{\bar{k}}, \bar{x})$. Then

(1) The map $g \mapsto g-1$ induces a canonical isom.

$$\pi_1^\ell(X_{\bar{k}}, \bar{x})^{ab} \otimes_{\mathbb{Z}_\ell} R \xrightarrow{\sim} I(\bar{x})/I(\bar{x})^2$$

(2) Composition w. any elem. of $\pi_1^\ell(X_{\bar{k}}, \bar{x}_1, \bar{x}_2)$ induces an isom

$$I(\bar{x}_1)/I(\bar{x}_1)^2$$

$$\simeq I(\bar{x}_1, \bar{x}_2)/I(\bar{x}_1, \bar{x}_2)^2$$

$$\simeq I(\bar{x}_2)/I(\bar{x}_2)^2$$

Ind. of the choice of element.