Statement of de Jong's conjecture

k finite field, char $k = p > 0$

X/k curve, $\bar{X} = X \otimes_k \bar{k}$.

Conjecture: Let $\ell \neq p$ a prime. Then $\forall$ lisse $\mathbb{F}_\ell((t))$-sheaves $\mathcal{E}$ the attached repr.

$$\pi_1(X) \longrightarrow GL(V_{\mathcal{E}})$$

has finite image.

<u>Rks</u> i) Conj. is similar to a conjecture of Deligne replacing $\overline{\mathbb{F}_\ell((t))}$ by $\overline{\mathbb{Q}_\ell}$.

ii) The above conj. is proved.

In general there is the following conjecture:

Let

- $F = \mathbb{F}_q((t))$ for some

finite set. IF / IFe.

- X normal variety.
- $\rho : \pi_1(X) \to GL(V)$ a finite-dim. repr.

[2,3,d]: Conjecture: $\rho(\pi_1(X))$ is finite.

~~Rks~~ 1) If

a) X not normal, then conj is not true,

b) $\ell \neq p$

b) ($\neq p$ ——)

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c) char $k = 0$ ——

(ii) For $\dim V = 1$

$\exists$ pseudo in $[d J]$.

<u>Application</u>: Get

· $\mathcal{O}/W(\mathbb{F})$ finite ext.

· Γ pro-finite grp

· $\mathcal{C}_{\mathcal{O}}$ = category of

complete noeth local

complete noeth. local

$\mathcal{O}$-alg. R with

$R/m_R = \mathbb{F}$.

- $\rho_0 : \Gamma \to GL_n(\mathbb{F})$ a cont. repr.
- $\varepsilon : \Gamma \to \mathcal{O}^\times$ lift of $\det(\rho_0)$

$\to Def(\Gamma, \rho_0, \varepsilon)$:

$\mathcal{C}_\mathcal{O} \longrightarrow Set$

$$R \longmapsto \Big\{ \rho : \Gamma \to GL_n(R) \;\Big|\; \rho \equiv \rho_0 \mod m_R, \; \det \rho = i \circ \varepsilon \Big\} \Big/ \sim$$

$$\rho \sim \rho' :\Longleftrightarrow \exists\, g \in \widetilde{GL}_n(R) : \rho' = g \cdot \rho \cdot g^{-1}.$$

Recall that Def! is representable if

a) If $T \in M_n(\mathbb{F})$ commutes with $\rho(g)$ $\forall g \in \Gamma$ then

$T = \lambda \cdot E_n$.

b) Φ_e is satisfied

Set $\Gamma = \pi_1(X), \pi_1(\tilde{X})$.

X smooth geom. conn. curve.

<u>Thm</u> [3, 5, dJ]. Suppose

i) Conj. 2.3 holds for X and n.

ii) $\rho_0|\pi_1(\tilde{X})$ is abs. irred.

iii) $\ell \nmid n$.

Then $\mathrm{Def}(\pi_1(\bar{X}), S_0, \varepsilon)$

is repr. by R_{univ}

and $\mathcal{O} \to R_{univ}$ is

a finite flat $\overset{\|}{R}$

complete intersection

homo.

$$\left(\begin{tikzcd} & \mathcal{O}[[x_1, \ldots, x_s]] \arrow[d, two heads] \\ \mathcal{O} \arrow[ur] \arrow[r] & R \end{tikzcd}\right)$$

Proof $\mathrm{Def}(\pi_1(\bar{X}), \ldots)$

is epim. of R:

ii) $\Rightarrow$ a)

b) follows from Katz-Lang finiteness thm.

(Ker $(\pi_1(X)^{ab} \to \mathrm{Gal}(\bar{k}/k))$ is finite.

$S = S_{univ}$, $\bar{S} = \bar{S}_{univ}$.

d) Since $(R, S, \pi_1(X))$

is a deformation of

So $\pi_1(\bar{X})$ we get

homo. of $\mathcal{O}$-alg.

$\varphi: \bar{R} \to R$ and

an $g' \in GL_n(R)$ s.t.

$\varphi(\bar{s}(g)) = (g')^{-1} \cdot s(g) \cdot g'$

$\forall g \in \pi_1(\bar{X})$.

Replace s by $g' \cdot s g'^{-1}$

is still univ.

b) Let $F \in \pi_1(X)$ be a preimage of $\mathrm{Frob} \in \mathrm{Gal}(\bar{k}/k)$.

Let $h_1 \in \underline{GL_n(\tilde{R})}$ s.t. $h_1 \bmod m_{\tilde{R}}$

$= \rho_0(F)$

Define

$$\bar{\rho}^F : \pi_1(\bar{X}) \to GL_n(\bar{R})$$

$$\gamma \mapsto h_1 \cdot \bar{\rho}(F^{-1} \cdot \gamma F) \cdot h_1^{-1}$$

This is a deform. of $\rho_0|\pi_1(\bar{X})$.

$\to$ Get a map $\underline{\bar{\Phi}} : \bar{R} \to \bar{R}$ and

$h_2 \in GL_n(\bar{R})$ s.t.:

$$\Phi(\bar{s}(g)) = h_2^{-1}\, \bar{s}^F(g) \cdot h_2 \quad \forall g$$

<u>Lemma</u>: i) $\Phi \in \mathrm{Aut}_0(\bar{R})$,

ii) $\psi \circ \Phi = \psi$.

<u>Pf</u> i) Consider $\bar{s}^{F^{-1}}$.

ii) $s^F \sim s$ (take $g = s(F$

c) Lemma: $R = \mathcal{O}[[X_1, \dots, X_d]]$

$s \subset N$.

Pf. By [Manin] need to show that

$$H^2(\pi_1(\bar{X}), \mathfrak{sl}_n(A)) \overset{?}{=} 0$$

$$\mathrm{Ad} \circ \varrho_0 \Big|_{X \cdot U(\pi_1)}$$

Have $H^2(\pi_1(\bar{X}), \mathfrak{sl}_n(A))$

$\cong H^2_{\acute{e}t}(\bar{X}, \mathcal{F})$

$\cdots H_{et}(\cdots$

where $\mathcal{F} = \mathcal{F}_{\mathrm{shv}}(\mathbb{R})$ ← a locally const. sheaf.

1. Case $\bar{X}$ affine

$\Rightarrow H^2_{et}(\cdots) = 0$

2. Case $\bar{X}$ projective

Poincaré - duality $\Rightarrow$

$H^2_{et}(\bar{X}, \mathcal{F}) = H^0_{et}(\bar{X}, \mathcal{F}^\vee)$

But $\mathcal{F} \cong \mathcal{F}^\vee$ via the Killing-form on $\mathfrak{sl}(H)$. $(\ell \nmid n)$

But $H^0_{et}(\bar{X}, \mathcal{F})$

$= \mathfrak{sl}(H)^{\pi_1(\bar{X})} \otimes \mathbb{C}$

as ρ_0 is abs. irred.

$(\mathfrak{gl}_n(\mathbb{F}) = \mathrm{End}(\mathbb{F}^n)$,

use Schur's lemma).

d) Set $I_\phi = (X_1 - \Phi(x_1), \ldots,$

$\overline{R} \supset X_s - \Phi(x_s))$

Lemma $\psi : \overline{R} \to R$

identifies R with $\overline{R}/I_\phi$.

Pf ...

Pf of φ surjective:

For this it suffices to see that

$$m\bar{n}/m\bar{n}^2 + m_0\cdot\bar{n} \longrightarrow mn/mn^2 + m_0 n$$

is surjective. The dual of is

$$H^1(\Pi_1(X), \mathrm{Hom}(\mathbb{F}))$$

$$\to H^1(\mathrm{Th}(\bar{X}), \mathrm{sh}(A)).$$

The kernel of this map is $H^1(\mathrm{Gal}(\bar{k}/k), H^0(\mathrm{Th}(\bar{X}), \mathrm{sh}(A))$

$$\| \\ C$$

ii) Construct a deform. of S_0 over $\bar{R}/I_{\Phi}$:

$$\mathcal{O} \hookrightarrow R \longrightarrow R/I_\Phi$$

cf. [AG].

e) <u>Lemma:</u>

$$\widehat{R}$$

$$\mathcal{O} \longrightarrow \underline{\mathcal{O}[[x_1 .. x_s]]} \big/ (f_1, \dots, f_s)$$

$$f_1 = x_1 - \Phi(x_1), \quad \dots, \quad f_s = x_s - \Phi(x_s)$$

is a finite flat complete intersection map

$\Leftrightarrow \dim R/_{\pi_0 R} = 0$

Pf: "$\Leftarrow$" [Matsumura, Comm. Alg 16.13] $\Rightarrow$ $(\pi_0, f_1, \dots, f_s)$ is a regular sequence $\Rightarrow$

$(\bar{f}_1, \dots, \bar{f}_s)$ regular sequence in $\mathcal{O}/_{\pi_0}[[x_1, \dots, x_d]]$.

$\Rightarrow$ [Mats, 20.F]

R flat / $\mathcal{O}$.

Furstenberg follows from Nakayama.

$\Rightarrow$ "Always true"

1) Lemma: $\dim R/\Pi_\alpha R \neq 0$.

Pf.: Suppose that $\exists$

$R/\Pi_\alpha R \twoheadrightarrow A$

dim $A = 1$.

W.l.o.g. A integral,

$A' :=$ normalisation

$\cong \mathbb{F}'[[t]]$, where

$\mathbb{F}'/\mathbb{F}$ finite.

Consider

$(R \to A') \circ S$

This is a deformation S'

of $S_0 \otimes \mathbb{F}'$.

Assumption i) of Thm 3.5,

$\Rightarrow S'(\pi_1(\bar{X}))$ finite

By the following lemma

hence $S'_{\pi_1(\bar{X})} = S_0 \otimes A'_{\pi_1(\bar{X})}$

which shows with

$\boxed{\text{another lemma.}}$ that

d (d_j)

$k \to A'$ factorises

over $\mathbb{F}'$ $\not{\!\!\!\,\,}$ contradiction!

Lemma: Let G be a finite grp, k any field. Let

$$\rho : G \to Gl_n(k[[t]])$$

be a repr. If

$\rho_0 := \rho \bmod (t)$

is also used, then.

$g \cong g_0 \oplus h[[t]]$.

RE. cf. [YdJ] D.