Grothendieck's quasi unipotent Monodromy thm

§1 Local systems from geometry

- K cdvf $\hookrightarrow K^{sep}$

$G_K = \mathrm{Gal}(K^{sep}/K)$

$\mathcal{O}_K$, $\kappa = \mathcal{O}_K / \mathfrak{m}_K$

Fix ℓ prime , $\ell \neq \mathrm{char}(\kappa)$

$I_K = \mathrm{Ker}(G_K \longrightarrow G_\kappa)$

inertia group

- Balls

X noeth scheme

$\bar{x} = \mathrm{Spec}\, k \longrightarrow X$ geom pt

$x = \mathrm{Im}(\bar{x}) \in X$ (i.e. $k = k(x)^{sep}$)

Take $\mathcal{U} \subset X$ affine s.t $\bar{x} \to \mathcal{U} \subset X$

$$X_{(\bar{x})} = \varprojlim_{\substack{\bar{x} \to V \\ \downarrow \text{ affine étale} \\ \bar{x} \to \mathcal{U}}} V = \mathrm{Spec}\, \underbrace{\mathcal{O}_{X,\bar{x}}}_{\text{strict henselian}}$$

$$\mathcal{O}_{X,\bar{x}} = \mathcal{O}_{X,x}^{sh}$$

Assume $x, \eta \in X$, $x \in \overline{\{\eta\}} \subset X$

(x is a specialization of η)

$\rightsquigarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,\eta}$

$\rightsquigarrow \mathcal{O}_{X,\bar{x}} \longrightarrow \mathcal{O}_{X,\bar{\eta}}$

$\rightsquigarrow X_{(\bar{\eta})} \longrightarrow X_{(\bar{x})}$

geom local systems

- $f : X \to S$ proper smooth, $X \to S$ smooth, $\ell \in \mathcal{O}_S^\times$

Have:

$$R^i f_{\text{ét}*}(\mathbb{Z}_\ell) = \lim_{\leftarrow n} R^i f_{\text{ét}*}(\mathbb{Z}/\ell^n\mathbb{Z})$$

is a <u>local system</u>, i.e.

a) $\bar{t} = \operatorname{Spec} k \to S$ geom pt

$$\Rightarrow \quad R^i f_{\text{ét}*}(\mathbb{Z}_\ell)_{\bar{t}} = H^i_{\text{ét}}(X_{\bar{t}}, \mathbb{Z}_\ell)$$

$X_{\bar{t}} = X \times_S \bar{t}$

$R^i f_{\text{ét}*}(\mathbb{Z}_\ell)_{\bar{t}} = H^i_{\text{ét}}(X_{(\bar{t})}, \mathbb{Z}_\ell)$, $X_{(\bar{t})} = X \times_S S_{(\bar{t})}$

finite type $\mathbb{Z}_\ell$-mod

and

b) If t is a specialization of η

then $S_{(\bar{\eta})} \to S_{(\bar{t})}$ induces

$$H^i_{\text{ét}}(X_{(\bar{t})}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i_{\text{ét}}(X_{(\bar{\eta})}, \mathbb{Z}_\ell)$$

The representation

Assume S connected and fix $\bar{s}_0 \to S$

set $H := H^i_{et}(X_{\bar{s}_0}, \mathbb{Z}_\ell)$

let K cdvf with

$$\bar{\eta} = \operatorname{Spec} K^{sep} \longrightarrow S$$

obtain an action of G_K on H as follows:

Take $\sigma \in G_K$, choose

$$S_{(\bar\xi)} \to S_{(\bar\eta)}, \quad S_{(\bar\xi)} \to S_{(\bar s_0)} \quad \text{specializ.}$$

$$\begin{array}{ccc}
H = H_{\bar s_0} & \dashrightarrow & H_{\bar s_0} = H \\
\wr\wr & & \wr\wr \\
H_{\bar\xi} & & H_{\bar\xi} \\
\wr\wr & & \wr\wr \\
H_{\bar\eta} = H^i_{et}(X_{\bar\eta}) & \xrightarrow[\cong]{\sigma^*} & H^i_{et}(X_{\bar\eta})
\end{array}$$

$$\to \quad G_K \longrightarrow \operatorname{Aut}_{\mathbb{Z}_\ell}(H)$$

Assume

$$\begin{array}{ccc} \bar{\xi} & \longrightarrow & S \\ \downarrow & \nearrow & \\ \operatorname{Spec} \mathcal{O}_K & & \\ \uparrow \text{cl pt} & & \\ \bar{s} = \operatorname{Spec} \bar{k} & & \end{array}$$

$H := H^i(X_{\bar{\xi}_0})$

then

$$\begin{array}{ccc} H^i(X_{\bar{\xi}}) & \xrightarrow{\sigma} & H^i(X_{\bar{\xi}}) \\ \text{s.a.} \wr\wr & & \wr\wr \\ H^i(X_{\bar{s}}) & \xrightarrow{\bar{\sigma}} & H^i(X_{\bar{s}}) \end{array}$$

$\sigma \in G_K$ and $\bar{\sigma}$ = image of σ in G_k

The $I_K = \operatorname{Ker}(G_K \to G_k)$

acts trivially

→ In this case the action $G_K \to \operatorname{Aut}(H)$

factors via

$$\pi_1(\operatorname{Spec} \mathcal{O}_K, \bar{\xi}) = \frac{G_K}{I_K} \longrightarrow \operatorname{Aut}(H)$$

→ they assemble together to give a repr $\pi_1(S, \bar{\xi}) \longrightarrow \operatorname{Aut}(H)$

Question:

What happens if $\operatorname{Spec} \bar{K} \to S$ does not extend to $\operatorname{Spec} \mathcal{O}_{\bar{K}} \to S$?

§2 Statement of g.unip monodromy thm

Constr: $K \hookrightarrow K^{sep}$, $\mathcal{O}_K$, κ
cdvf

$I \subset G = G_K$

$K^{ur} = (K^{sep})^I =$ max'l unramified extn.

pick $t \in \mathcal{O}_K$ uniformizer
($\to$ uniformizer of $\mathcal{O}_{K^{ur}}$)

$K^\ell = K^{ur}(\sqrt[\ell^n]{t} \mid n \geq 1)$

$\Rightarrow$ For all L/K^ℓ fin:

$[L : K_\ell]$ prime to ℓ

$\rightsquigarrow$ obtain

$$I^{\ell'} \subset I \subset G \to \frac{G}{I} = G_k$$

$I^{\ell'} = \mathrm{Gal}(K^{sep}/K^{\ell})$ is a profinite group, $I = \mathrm{Gal}(K^{sep}/K^{ur})$

and

$$\frac{I}{I^{\ell'}} = \mathrm{Gal}\left(K^{\ell}/K^{ur}\right) \xrightarrow{\simeq} \varprojlim_n \mu_{\ell^n}(k^{sep}) = \mathbb{Z}_\ell(1)$$

⌜ choose a comp system $(t_n) \in K^{\ell}$ with ⌝

$$t_0 = t, \quad t_{n+1}^{\ell} = t_n$$

$$\mathrm{Gal}\left(\frac{K^{\ell}}{K^{ur}}\right) \ni \sigma \longmapsto \left(\frac{\sigma(t_n)}{t_n} \bmod m_K\right)_n$$

⌟

⌞

Thm (Grothendieck's quasi-unip Thm)

$f : X \to S$ sm prop, $\ell \in \mathcal{O}_S^*$
(↑ noeth)

K cdvf

$\bar{\eta} = \operatorname{Spec} K^{sep} \to S$, $H = H^i_{et}(X_{\bar{\eta}}, \mathbb{Z}_\ell)$

gives $\rho : G \to \operatorname{Aut}_{\mathbb{Z}_\ell}(H)$, $I^{\ell'} \subset I \subset G$

Then:

(1) $| \rho(I^{\ell'}) | < \infty$

(2) $\tau \in \frac{I}{I^{\ell'}}$ top generator

($\hat{=}$ comp system of ℓ^n roots of 1 in $\mathbb{Z}_\ell(1)$)

$\Rightarrow \rho(\tau)$ is quasi-unipotent

i.e. $\exists\, n, m$ $\left((\rho(\tau))^n - \mathrm{id}_H \right)^m = 0$

$\Longleftrightarrow$ eigenvalues of $\rho(\tau)$ are roots of 1

§3 cm pf

(1) to show
$$|\rho(I^{\ell'})| < \infty$$

$H = H_{free} \oplus \underbrace{H_{tors}}_{fin}$

$\Rightarrow$ suff to show
image of $\rho(I^{\ell'})$ in $Aut_{\mathbb{Z}_\ell}(H_{free})$
$= Gl_n(\mathbb{Z}_\ell)$

is finite

Have s.e.s
$$0 \to \underbrace{\ker}_{\text{pro } \ell\text{-gp}} \to Gl_n(\mathbb{Z}_\ell) \to Gl_n(\mathbb{Z}/\ell^2\mathbb{Z}) \to 0$$

and every cts hom
$I^{\ell'} \to$ pro-ℓ is trivial
[it is a limit of (fin ℓ'-gp) $\to$ (fin-ℓ-gp)]

$\Rightarrow |\rho(I^{\ell'})| \leq |Gl_n(\mathbb{Z}/\ell^2)| < \infty$

on (2) [$\rho(\tau)$ quasi-unip]

First case:

$$K \leftarrow \mathcal{O}_K \rightarrow \kappa$$

cdvf

Assume

(*) No finite extension of κ contains all $\sqrt[\ell^n]{1}$ $\forall n$

⌈ Note: κ f.g / prime field

$\Rightarrow$ κ has only fin many ℓ-power roots of 1

$\Rightarrow$ (*)

$\rightarrow$ does not cover the case over $\mathbb{C}$ ⌋

[In this case we proof in general

$$\rho : G_K \longrightarrow GL_n(\mathbb{Z}_\ell) \quad \text{cts}$$

$\Rightarrow \forall \tau \in I$: $\rho(\tau)$ is quasi-unip.

pf of 1st case (following Serre–Tate)

Know $|\rho(I^{\ell'})| < \infty$

$|GL_n(\mathbb{Z}/\ell^2)| < \infty$

$\longrightarrow$ after replacing K by a finite ext K'/K

we can assume

$$\rho : \frac{G}{I^{\ell'}} \longrightarrow GL_n(\mathbb{Z}_\ell)$$

and $\rho \equiv 1 \mod \ell^2$

(in this case we show ρ unipotent)

Two maps

$G \longrightarrow G_{\mathcal{K}}$, $\sigma \longmapsto \bar{\sigma}$

$$\theta : \frac{I}{I^{\ell'}} \xrightarrow{\simeq} \mathbb{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}(\mathcal{K}^{sep})$$

$\circlearrowleft G_{\mathcal{K}}$

Let $\sigma \in G$, $\tau \in I$

$$\frac{(\sigma\tau\sigma^{-1})(t_n)}{t_n} = \sigma\left(\frac{\tau\sigma^{-1}(t_n)}{\sigma^{-1}(t_n)}\right)$$

$$\Rightarrow \theta(\sigma\tau\sigma^{-1}) = \bar{\sigma}(\theta(\tau))$$

Note $\alpha \in \mathbb{Z}_\ell(1)$, $\sigma \in G_{\mathcal{K}}$

$$\Rightarrow \quad \bar{\sigma}(\alpha) = \alpha^{\chi(\sigma)}$$

where $\chi : G_K \to G_{\mathcal{K}} \to \mathbb{Z}_\ell^\times$

(cyclotomic char)

Thus

$$\theta(\sigma \tau \sigma^{-1}) = \bar{\sigma}\, \theta(\tau) = \theta(\tau^{\chi(\sigma)})$$

$$\Rightarrow \quad \sigma \tau \sigma^{-1} = \tau^{\chi(\sigma)} \text{ in } \frac{I}{I^{\ell}}$$

θ isom

$$\Rightarrow \quad \boxed{S\, \rho(\tau)\, S^{-1} = \rho(\tau)^{\chi(\sigma)} \text{ in } GL_n(\mathbb{Z}_\ell)} \quad (2*)$$

where $S = \rho(\sigma)$

$$\rho(\tau) \equiv 1 \bmod \ell^2$$

$$\Rightarrow \quad A = \log \rho(\tau) \in \ell^2 M_n(\mathbb{Z}_\ell)$$

$(2*) \Rightarrow$ A and $\chi(\sigma) A$ are conjugate

Take

$\lambda_1, \ldots, \lambda_n$ eigenvalues of A

$s_i(A)$ = i-th elem symmetric polynomial in $\lambda_1, \ldots, \lambda_n$

$$\Rightarrow \quad s_i(A) = s_i(\chi(\sigma) A) = \chi(\sigma)^i \, s_i(A)$$

$$\Rightarrow \quad (1 - \chi(\sigma)^i) \cdot s_i(A) = 0$$

$(*) \Rightarrow$ we can choose $\sigma \in G$ s.t.

$\chi(\sigma)$ is not some root of 1

[since: (ζ_{ℓ^n}) comp system of ℓ^n of 1]

$$(\sigma(\zeta_{\ell^n}))_n = (\zeta_{\ell^n}^{\chi(\sigma)})_n$$

if $\chi(\sigma)$ is an N-th root of 1

$$\Rightarrow \quad \sigma^N(\zeta_{\ell^n}) = \zeta_{\ell^n} \quad \forall n$$

if this true for σ

$\Rightarrow \exists$ fin field extn of K which cont all ζ_{ℓ^n} ↯ to $(*)$]

$\Rightarrow \quad s_i(A) = 0 \quad \forall i$

$\Rightarrow$ all $\lambda_i = 0 \Rightarrow A$ nilpotent.

$\Rightarrow \quad \rho(\tau) = \exp(A)$ unip.

$\square$ 1. case.

<u>general case</u> K general, $t \in \mathcal{O}_K$ loc param.

Néron <u>desingularization</u>

can write $\mathcal{O}_K = \varinjlim_i A_i$

with A_i local henselian

essentially sm over $\begin{cases} \mathbb{F}_p[[t]] & \text{equichar } p \\ \mathbb{Q}[[t]] & \text{equichar } 0 \\ W(\mathbb{F}_p)[[t]] & \text{mixed char} \end{cases}$

and $K = \varinjlim A_i[1/t]$

$$\begin{array}{ccc} X_{\bar{K}} & & X_{\overline{A_{i_0}[1/\ell]}} \\ \downarrow & \text{spreads out to} & \downarrow \quad \text{some } i_0 \\ \operatorname{Spec} \bar{K} & & \operatorname{Spec} \overline{A_{i_0}[1/\ell]} \end{array}$$

$\to$
$$\begin{array}{ccc} G_K & \longrightarrow & \operatorname{Aut}_{\mathbb{Z}_\ell}(H) \\ & \searrow \quad \nearrow & \\ & \pi_1(A_{i_0}[1/\ell]) & \end{array}$$

<u>Have</u> $A = A_{i_0}$, $\kappa_A = A/m$ satisfies $(*)$

(it is fin gen over a prime field)

$$0 \to I \to \pi_1(A[1/\ell]) \to \pi_1(A) = G_{\kappa_A} \to 0$$

$\exists\ I^{\ell'}$ as above

s.t. $\theta : \frac{I}{I^{\ell'}} \cong \varprojlim \mu_{\ell^m}(\kappa^{sep})$

(Abhyankar's lemma)

$\to$ same proof as in the first case works $\square$