Introduction

Let X be a "nice" connected top. space, $x \in X$

Def $\pi_1^{top}(X,x) := \left\{ \begin{array}{l} \gamma : [0,1] \to X \mid \\ \gamma(0) = \gamma(1) = x \end{array} \right\} \Big/ \sim$

(based homotopy)

This is the most basic non-abelian invariant of a topological space.

Cor. of Hurewicz: $\left(\pi_1^{top}(X,x)\right)^{ab} \simeq H_1(X,\mathbb{Z})$

If X has the homotopy type of a finite CW complex, then $\pi_1^{top}(X,x)$ is finitely presented group.

Def Let R be a comm. ring w/ unit. An R-local system on X is a locally constant sheaf of finite R-modules, that is locally free.

Examples

- X any nice top space.

  $\underline{R}$ is an R-local system

- $X \simeq \mathbb{C}^\times$, coordinate z, $R \simeq \mathbb{C}$

  $$\mathcal{F}(U) = \left\{ f \in C^\infty(U, \mathbb{C}) \;\middle|\; \frac{df}{dz} - f = 0 \right\}$$

  This has a global solution:

  $$e^z$$

  $\rightsquigarrow$ constant rank 1 - local system

- $X \simeq \mathbb{C}^\times$, coord. z, $R = \mathbb{C}$

  $$\mathcal{G}(U) := \left\{ g \in C^\infty(U, \mathbb{C}) \;\middle|\; \frac{d^2g}{dz^2} + \frac{1}{z}\frac{dg}{dz} = 0 \right\}$$

  2 solutions: $\log z$, 1

  $\rightsquigarrow$ non-trivial rank 2 local system!

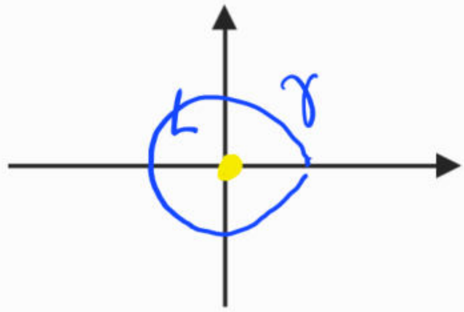
Fact: $\exists$ "natural" equivalence of categories:

$$\left\{\begin{array}{l}\text{local systems}\\ \text{of } R \text{ modules}\\ \text{on } X\end{array}\right\} \longleftrightarrow \underbrace{\left\{ \mathrm{Rep}_R\ \pi_1(X,x) \right\}}_{\text{reps on finite free } R\text{-modules}}$$

$$\text{"} \longrightarrow \text{"}$$

parallel transport around loops

$$\mathcal{L} \longmapsto \left\{ \begin{array}{l} \mathbb{Z} \longrightarrow GL_2(\mathbb{C}) \\ [\gamma] \longmapsto \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} \end{array} \right\}$$



Key to equivalence:

$$\begin{array}{c} \tilde{X} \\ \downarrow \\ X \end{array} \quad \text{universal cover} \quad \left(\pi_1(\tilde{X}) = \{e\} \right)$$

$$\Rightarrow \quad \pi_1(X) \simeq \underbrace{\mathrm{Aut}(\tilde{X}/X)}_{\text{Deck transformations}}$$

From now on, I use the term "local system" to refer to a rep of π_1

We say 2 local systems

$\rho_1 : \pi_1(X) \to GL_n(K)$, $\rho_2 : \pi_1(X) \to GL_n(K)$

are equivalent if $\exists\, M \in GL_n(K)$

w/ $\underline{Ad_M \circ \rho_1 = \rho_2}$

Def Let $B/\mathbb{C}$ be sm., conn., quasi-proj.
Let L be a $\mathbb{C}$-local system on B^{an}.
We say L is of geometric origin or motivic if $\exists$

- $\mathcal{U} \subseteq B$ Zariski open, dense
- $\begin{array}{c} \mathcal{Y} \\ \downarrow \pi \\ \mathcal{U} \end{array}$ sm proj. morphism
- $i \geq 0$ s.t.

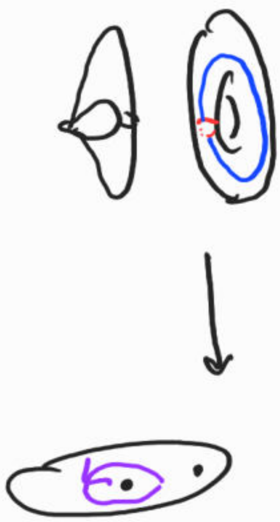
$L\big|_{\mathcal{U}}$ a subquotient of $R^i \pi_* \mathbb{C}$.

Example

$$\begin{array}{ccc} \mathcal{E} & \hookrightarrow & zy^2 = x(x-1)(x-\lambda) \\ \downarrow \pi & & \cap\!| \\ & & \mathbb{P}^2_{[x:y:z]} \\ \mathbb{P}^1 \setminus \{0,1,\infty\} \ni \lambda & & \end{array}$$

$R^1\pi_* \mathbb{Z}$ gives a representation of

$$\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \to GL_2(\mathbb{Z})$$



(local picture around 0)

(red cycle is pinched)

$$\gamma \longmapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{"Dehn twist"}$$

Can compute explicitly

$$\underbrace{F_2}_{\text{free on 2 letters}} \simeq \pi_1^{top}(\mathbb{P}^1 \setminus \{0,1,\infty\}) \simeq \Gamma(2) := \{A \in SL_2(\mathbb{Z}) \text{ w } A \equiv I_2 \bmod 2\}$$

Questions

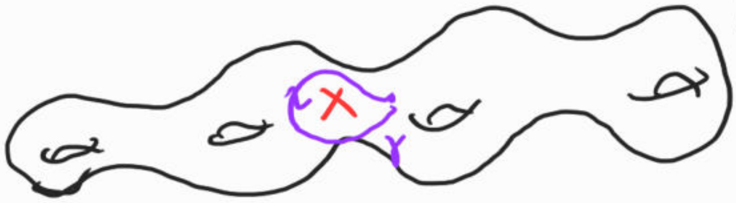
- How do we characterize local systems of geo. origin?
- "How many" motivic local systems are there?
- If there are "many" motivic local systems, so what?

Briefly explain some of the thms we cover.

Thm (Grothendieck) If $\mathbb{L}$ is of geometric origin, it has quasi-unipotent monodromy @ ∞.

"Example"

L is local system on X of geo. origin.

X : 

$$L \longleftrightarrow \rho : \pi_1(X) \to GL_n(\mathbb{C})$$

$$\Psi \quad [\gamma]$$

$\rho([\gamma])$ well defined up to conjugacy in $GL_n(\mathbb{C})$

Grothendieck's quasi-unipotent monodromy theory: eigenvalues of $\rho([\gamma])$ are all roots of 1.

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Motivation for next results.

Thm (Mazur): $E/\mathbb{Q}$ be an elliptic curve. $P \in E(\mathbb{Q})_{tors}$. Then

$$\boxed{12P = 0}$$

Thm (Merel) For every $d \geq 1$, $\exists N = N(d)$ s.t. if

- $K/\mathbb{Q}$ deg d # field
- E/K is an elliptic curve
- $P \in E(K)_{tors}$

$$\rightsquigarrow NP = 0$$

Geometric analog: replace # field w/ $\mathbb{C}$ function field, or by a curve.

Conj (Geometric torsion) $\dim(X) \geq 1$

Let $X/\mathbb{C}$ be smooth, connected, $n \geq 1$. $\exists N = N(X, n)$ s.t. if

- $\begin{matrix} A \\ \downarrow \\ X \end{matrix}$ is an abelian scheme of dim n w/ no fixed part.

- $\sigma : X \longrightarrow A$ is a torsion section

Then $N\sigma = 0$ (0 section)

($N\sigma$: group law on the abelian scheme)

Exercise prove for $n=1$, i.e., elliptic curves. (Hint: genus of modular curves $\longrightarrow \infty$)

Thm (Litt)

Let $X/\mathbb{C}$ have dim ≥ 1. $\exists\, N = N(X)$ s.t. if

$$\rho : \pi_1(X) \to GL_n(\overline{\mathbb{Z}})$$

is a $\overline{\mathbb{Z}}$ - local system of geo. origin and $M > N$ w/

ρ mod M trivial

$\Rightarrow$ ρ is trivial!

Thm (Litt) Let $X \geq \mathbb{C}$ have dim ≥ 1, fix $r \geq 1$. Then $\exists$ finitely many

$$\rho : \pi_1^{top}(X^{an}) \to GL_r(\mathbb{Q}_\ell)$$

that come from alg. geometry.

False if we replace $\mathbb{Q}_\ell$ w/ $\overline{\mathbb{Q}}_\ell$!!

We will also discuss a conjecture of de Jong, about reps of $\pi_1(X)$, where $X/\mathbb{F}_q$ is a curve.

Work of Drinfeld $\Rightarrow$ dJ's conj $\Rightarrow$ hard Lefschetz for Perverse sheaves $/\mathbb{C}$.

Time permitting, we will discuss related work of Esnault-Kerz.

"Hidden structure"

Let $X|\mathbb{Q}$ be a smooth variety.
Does $\pi_1^{top}(X^{an})$ have any "extra structure"?

① $(\pi_1^{top}(X))^{unip}$ has a mixed Hodge structure

② Let $\Pi := \widehat{\pi_1^{top}(X^{an})}$ ← profinite completion
$\exists\ k|\mathbb{Q}$ f.g. field s.t.

$$G_k := \mathrm{Gal}(\bar{k}/k) \to \mathrm{Out}(\Pi)$$

This remembers a tremendous amount about X.
(thm of Mochizuki)

Étale Fundamental Gps

X normal connected scheme.

A morphism $Y \to X$ is *finite étale* if

- It is finite
- It is flat
- It is unramified

If $X/\mathbb{C}$, $Y \to X$ is finite étale iff $Y^{an} \to X^{an}$ is a finite covering map.

Examples

- $\mathbb{G}_m/\mathbb{C} \longrightarrow \mathbb{G}_m/\mathbb{C}$ is finite étale
  $x \longmapsto x^3$

- $\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ is NOT finite étale
  $x \longmapsto x^3$

Category $F\acute{E}t_X$

Objects $\begin{array}{c} Y \\ \downarrow \\ X \end{array}$ finite étale covers

morphisms
$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

Let $x \longrightarrow X$ be a geometric point.

Then $\exists$ a "fiber functor"

$$F_x : F\acute{E}t_X \longrightarrow \text{Sets}$$
$$\begin{array}{c} Y \\ \downarrow \\ X \end{array} \longmapsto Y_x$$

Def $\pi_1^{\acute{e}t}(X, x) := \underbrace{\mathrm{Aut}(F_x)}_{\text{profinite gp}}$

$$\Rightarrow \quad F\acute{E}t_X \simeq \underbrace{\pi_1^{\acute{e}t}(X,x)\text{-sets}}$$

finite sets equipped w/ a continuous action of $\pi_1^{\acute{e}t}(X,x)$

Useless def!

Examples

- k be a field, pick $\bar{k}$

$x = \mathrm{Spec}(k)$ $\qquad\qquad \bar{x} = \mathrm{Spec}(\bar{k})$

$$\pi_1(x, \bar{x}) = \mathrm{Gal}(\bar{k}/k)$$

Generalizes the Galois correspondence!

finite separable extensions$/k$ $\quad \ell/k \longleftrightarrow$ open subgroups of G_k

$\updownarrow$

finite sets w/ continuous, transitive G_k action.

- X integral (noetherian) normal.

$K = \kappa(\eta)$

generic point of X

Pick $\overline{K}$ of K, which yields

$$\overline{\eta} := \operatorname{Spec}(\overline{K}) \longrightarrow \operatorname{Spec}(K) = \eta \longrightarrow X$$

Covariant-ness:

$$\underbrace{\pi_1(\eta, \overline{\eta})} \twoheadrightarrow \pi_1(X, \overline{\eta})$$

$$\operatorname{Gal}(\overline{K}/K) \twoheadrightarrow \pi_1(X, \overline{\eta})$$

Q: Which quotient is it?!

For simplicity, $X = \operatorname{Spec} A$.

$$A \hookrightarrow K$$

normal domain

A normal domain $\Rightarrow$ localizing at ht 1 primes $\mathfrak{p}$, $A_{(\mathfrak{p})}$, yield dvrs

$$\begin{tikzcd} \bar{A} \arrow[r, hook] \arrow[d, dash] & \bar{K} \arrow[d, dash] \\ A^{unr} \arrow[r, hook] \arrow[d, dash] & K^{unr} \arrow[d, dash] \\ A \arrow[r, hook] & K \end{tikzcd}$$

the maximal extension that is unramified over all $\forall A$

$\rightsquigarrow \pi_1(X, \bar{\eta}) \simeq \mathrm{Gal}(K^{unr}/K)$

$=$

- $X = \mathrm{Spec}(\mathbb{C}[t, t^{-1}])$

  finite étale covers are of the form

  $\mathrm{Spec}(\mathbb{C}[t^{1/n}, t^{-1/n}])$

$$\left(\begin{array}{ccc} \mathbb{C}^\times & \longrightarrow & \mathbb{C}^\times \\ z & \longmapsto & z^n \end{array} \right)$$

$$\Rightarrow \pi_1\left(\mathrm{Spec}(\mathbb{C}[t,t^{-1}]), \text{base pt}\right)$$

$$\simeq \widehat{\mathbb{Z}}$$