Arithmetic representations

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1. Arithmetic representations
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2. Finiteness of \pi_1-reps: from Deligne to Litt

1.1 Arithmetic representations

Def. k = f.g. field, X/k sm sep geo conn curve. $\ell \nmid \operatorname{char} k$.

A cts rep $\rho : \pi_1(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}}_\ell)$

Can be replaced by any fin ext $E/\mathbb{Q}_\ell$ or the completion

is called arithmetic if $\exists\, k'/k$ fin ext, and

$$\tilde{\rho} : \pi_1(X_{k'}, \bar{x}) \to GL_N(\overline{\mathbb{Q}}_\ell)$$

a cts rep s.t. $\rho \simeq$ a subquot of $\tilde{\rho}|_{\pi_1(X_{\bar{k}}, \bar{x})}$

Continuity: In the sense that $\exists\, E/\mathbb{Q}_\ell$ fin ext s.t. ρ comes from some cts $\pi_1 \to GL_n(E)$.

$\leadsto$ arith reps are those "coming from arith monodromy rep's up to fin field extension of the base field".

1.2. Example reps coming from geometry.

Easist case: $f : Y \to X_{\bar{k}}$ sm proper. Then $\forall\, i \geq 0$, any subquot of the geo. monod. rep

$$\begin{array}{ccc} Y & & X_{\bar{k}} \\ \downarrow & & \downarrow \\ \tilde{Y} & \to & X_{k'} \end{array}$$

$$\pi_1(X_{\bar{k}}, \bar{x}) \longrightarrow GL((R^i f_* \overline{\mathbb{Q}}_\ell)_{\bar{x}})$$

is arithmetic.

$$\pi_1(X_{\bar{k}}, \bar{x}) \searrow \pi_1(X_{k'}, \bar{x}) \nearrow GL((R^i f_* \overline{\mathbb{Q}}_\ell)_{\bar{x}})$$

1.3 Def (Rep's arising from geometry).

Suppose $\boxed{\operatorname{char} k = 0}$

A cts rep $\rho : \pi_1(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}}_\ell)$ is said to arise from geometry

A cts rep $\rho : \pi_1(X_{\bar k}, \bar x) \longrightarrow GL_n(\overline{\mathbb{Q}_\ell})$ is said to arise from geometry if $\exists\ F = \bar F \supset k$, a sm proper F-morphism $f : Y \to U$

$$\begin{tikzcd} Y \arrow[d, "\text{sm proper } f"'] & \\ U \arrow[r, hook, "\iota"] & X_F \end{tikzcd} \qquad \boxed{\begin{tikzcd} Y \arrow[r, hook] & \bar Y \arrow[d, "\text{sm proper}"'] & \bar Y \setminus Y \arrow[l, hook'] \arrow[dl] \\ & X_F & \end{tikzcd}}$$

s.t. $\rho \simeq$ geo monodromy rep asso. to a lisse subquot of $R^i(\iota \circ f)_* \overline{\mathbb{Q}_\ell}$.

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when char $k = p$

Consider "tame" representations only.

1.4 Prop Rep's arising from geometry are arithmetic.

Pf. Suppose $\rho : \pi_1(X_{\bar k}, \bar x) \to GL_n(\overline{\mathbb{Q}_\ell})$ asing from geo.
i.e. $\exists\ F = \bar F \supset k$ & a smooth proper f.

$$\begin{tikzcd} Y \arrow[d, "f"'] & \\ U \arrow[r, hook, "\iota"] & X_F \end{tikzcd}$$

s.t. $\rho \simeq$ monod rep asso. to a lisse subq of $R^i(\iota \circ f)_* \overline{\mathbb{Q}_\ell}$.

Spreading-out $\rightsquigarrow \exists$ f.g. k-alg $R \subset F$, R-sch $\mathcal{Y}$ & $\mathcal{U}$ with

$$s : Y \xrightarrow{\sim} \mathcal{Y}_F \qquad t : U \xrightarrow{\sim} \mathcal{U}_F$$

and a sm proper $g : \mathcal{Y} \to \mathcal{U}$ an open embedding $\tilde\iota : \mathcal{U} \hookrightarrow X_R$

s.t.

$$\begin{tikzcd} & \text{generic fiber} & \overline{\mathbb{Q}_\ell} & \text{closed fiber} \\ Y \arrow[r, "s"] \arrow[d, "f"'] & \mathcal{Y}_F \arrow[r] \arrow[d, "g_F"] & \mathcal{Y} \arrow[d, "g"] & \mathcal{Y}_z \arrow[l] \arrow[d, "g_z"] \\ U \arrow[r, "t"] \arrow[d, hook, "\iota"'] & \mathcal{U}_F \arrow[r] \arrow[d, hook, "\tilde\iota_F"] & \mathcal{U} \arrow[d, hook, "\tilde\iota"] & \mathcal{U}_z \arrow[l] \arrow[d, hook, "\tilde\iota_z"] \\ X_F & X_F \arrow[r] & X_R & X_z \arrow[l] \end{tikzcd}$$

$k(z)/k$ fin ext.

Specializing: Pick a closed pt $z \in \operatorname{Spec} R$. Consider

$$R^i(\iota \circ f)_* \overline{\mathbb{Q}_\ell} \longrightarrow R^i(\tilde\iota \circ g)_* \overline{\mathbb{Q}_\ell} \longrightarrow R^i(\tilde\iota \circ g)_{z*} \overline{\mathbb{Q}_\ell}.$$

$\cdots \longrightarrow K(\mathrm{Loc})_{*}\overline{\mathbb{Q}}_\ell \longrightarrow K(\mathrm{Loc})_{Z*}\overline{\mathbb{Q}}_\ell .$

$$\begin{tikzcd}
\boxed{\pi_1(X_{\bar k}, \bar x)} \arrow[r] \arrow[rrd, bend left] & \pi_1(X, \bar x) & \\
\pi_1(X_F, \bar x) \arrow[u, "\wr"] \arrow[ur] \arrow[r] & \pi_1(X_R, \bar x) & \pi_1(X_z, \bar x) \arrow[l] \arrow[ul]
\end{tikzcd}$$

1.5. Semisimple representations

Prop $\rho : \pi_1(X_{\bar k}, \bar x) \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ cts ss. rep. Then

ρ is arith $\iff$ $\exists\, k'/k$ fin ext, cts rep $\tilde\rho : \pi_1(X_{k'}, \bar x) \to GL_n(\overline{\mathbb{Q}}_\ell)$

s.t. $\tilde\rho\big|_{\pi_1(X_{\bar k}, \bar x)} \simeq \rho$.

1.6. Another perspective

ρ cts. ss. The

ρ is arith $\iff$ $\exists$ open subgp $H \subset G_k$ s.t.

$$\mathrm{Tr}(\rho \circ \psi_h(\sigma)) = \mathrm{Tr}(\rho(\sigma)) \quad \forall \sigma \in \pi_1(X_{\bar k}, \bar x),$$
$$h \in H \subset G_k .$$

where $\psi_h : \pi_1(X_{\bar k}, \bar x) \xrightarrow{\sim} \pi_1(X_{\bar k}, \bar x)$ is the action of $h \in G_k$ on π_1^{geo}.

1.7 Rigidity lemma

$k = \mathbb{F}_q$, $\bar k = \overline{\mathbb{F}}_q$. $\ell \nmid \mathrm{char}\, k$.

Thm
- C/k sm affine curve
- $x \in C(k)$ rat pt. $\bar x \in C(\bar k)$ asso. geo pt.
- $\rho : \pi_1(C_{\bar k}, \bar x) \to GL_n(\overline{\mathbb{Q}}_\ell)$ cts. $\underline{\underline{\text{irr}}}$ $\underline{\underline{\text{arith}}}$ rep
- A = loc. Artinian $\overline{\mathbb{Q}}_\ell$-alg with res fied $\overline{\mathbb{Q}}_\ell$
- $\tilde\rho : \pi_1(C_{\bar k}, \bar x) \longrightarrow GL_n(A)$ is a deform of ρ to A.

[i.e. $\tilde\rho \otimes_A \overline{\mathbb{Q}}_\ell \underset{\pi_1^{geo}\text{-equiv.}}{\simeq} \rho$ as rep's.]

such that

$$\mathrm{Tr}(\tilde\rho \circ \psi_{\varphi_x}) = \mathrm{Tr}(\tilde\rho)$$

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$$\underset{\substack{\uparrow \\ \text{``Frob'' at } x.}}{\varphi_x}$$

where

$$\psi_{\varphi_x} : \pi_1(C_{\bar k}, \bar x) \to \pi_1(C_{\bar k}, \bar x)$$

is the map induced by $\varphi_x \in G_k$.

<u>Then</u> $\tilde\rho \simeq \rho \otimes_{\overline{\mathbb{Q}}_\ell} A : \pi_1(C_{\bar k}, \bar x) \to GL_n(A)$.

The proof is a "game of the theory of weights".

<u>Def</u> (Weil groups) • $\underset{\underset{\mathbb{Z}}{\shortparallel}}{W(k)} \subset G_k$ gen by $\varphi_x^{-1} \in \underset{\underset{\hat{\mathbb{Z}}}{\shortparallel}}{G_k}$ the "geometric Frob".

• $W(C) = \pi_1(C, \bar x) \times_{G_k} W(k)$

•

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(C_{\bar k}, \bar x) & \to & W(C) & \to & W(k) & \to & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_1(C_{\bar k}, \bar x) & \to & \pi_1(C, \bar x) & \to & G_k & \to & 1 \end{array}$$

<u>Def</u> (Weight) $V \in \mathrm{Rep}^{cts}_{\overline{\mathbb{Q}}_\ell}(G_k)$. Then V is <u>ι-pure of weight w</u> if

$$\left| \begin{array}{l} \text{each eigenvalue} \\ \text{of } \varphi_x^{-1} \curvearrowright V \end{array} \right|_\iota = q^{\frac{w}{2}}$$

where $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ some iso of fields (not of top fields).

<u>Example</u> (Tate twist) $X = \mathrm{Spec}\, \mathbb{F}_q$. Define the lisse sheaf

$$\overline{\mathbb{Q}}_\ell(1) = \varprojlim_n \mu_{\ell^n} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$$

Pick geo pt $\mathrm{Spec}\, \overline{\mathbb{F}}_q \xrightarrow{\bar x} \mathrm{Spec}\, \mathbb{F}_q$. Then $(\overline{\mathbb{Q}}_\ell(1))_{\bar x} \in \mathrm{Rep}^{cts}_{\overline{\mathbb{Q}}_\ell}(G_{\mathbb{F}_q})$

The φ_x^{-1}-action is induced from

$$\mu_{\ell^n}(\overline{\mathbb{F}}_q) \longrightarrow \mu_{\ell^n}(\overline{\mathbb{F}}_q)$$

$$\mu_{\ell^n}(\overline{\mathbb{F}}_q) \longrightarrow \mu_{\ell^n}(\overline{\mathbb{F}}_q)$$
$$\alpha \longmapsto \alpha^{q^{-1}}$$

$\rightsquigarrow$ gives a cyclotomic character $G_{\mathbb{F}_q} \to \mathbb{Z}_\ell^\times$ $\quad \varphi_x^{-1} \mapsto q^{-1}$

$\rightsquigarrow$ $(\overline{\mathbb{Q}}_\ell(1))_{\bar{x}}$ is ι-pure of weight -2 for any $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$.

Pf of Rigidity lemma

Step 1 Extending geo rep to Weil rep.

$$0 \to \pi_1^{geo} \longrightarrow W(C) \underset{\iota}{\overset{s}{\rightleftarrows}} \underset{W(k)}{\mathbb{Z}} \longrightarrow 0.$$

$x \in C(k) \Rightarrow \exists$ splitting s and hence $W(C) = \pi_1^{geo} \rtimes \mathbb{Z}$.

Write $g := s(1)$. $\psi_g : \pi_1^{geo} \to \pi_1^{geo}$ $\;h \mapsto ghg^{-1}$ is the restriction of

$$\widehat{\psi}_g : \pi_1^{arith} \to \pi_1^{arith}, \quad a \mapsto gag^{-1}$$

and $\widehat{\psi}_g$ is inner automorphism.

ρ is arith $\rightsquigarrow$ after enlarging k by a finite ext, can assume ρ is iso to (a subquot of) the restriction of some $\rho' : \pi_1^{arith} \to GL_n(\overline{\mathbb{Q}}_\ell)$.

- $\mathrm{Aut}(\pi_1^{arith}) \curvearrowright$ the set of iso classes of irr reps appearing as subquot of $\widetilde{\rho}$ by precomposing.

$\mathrm{Inn}(\pi_1^{arith}) \curvearrowright$ acts trivially by def of rep iso's.

In part, $\widehat{\psi}_g \in \mathrm{Inn}(\pi_1^{arith}) \curvearrowright$ acts trivially.

Hence, it's restriction $\psi_g \in \mathrm{Out}(\pi_1^{geo})$ sends an irr rep to the same class (i.e. can use the same matrix from $\widehat{\psi}_g \in \mathrm{Inn}(\pi_1^{arith})$).

$\rightsquigarrow \rho \xmapsto{\psi_g} \rho \circ \psi_g : \pi_1^{geo} \to GL_n(\overline{\mathbb{Q}}_\ell)$

$\rho \simeq \rho \circ \psi_g$ i.e. $\exists$ a matrix $M \in GL_n(\overline{\mathbb{Q}}_\ell)$ s.t.

$\rho \simeq \rho\circ\psi_g$ i.e. $\exists$ a matrix $M\in GL_n(\overline{\mathbb{Q}}_\ell)$ s.t.

$$\rho\circ\psi_g(h) = M\rho(h)M^{-1}, \quad \forall h\in\pi_1^{geo}.$$

ρ is <u>irr</u> $\overset{\text{Schur's lemma}}{\Longrightarrow}$ $\mathrm{End}_{\pi_1^{geo}\text{-equiv}}(\rho)$ is 1-dim'l $\overline{\mathbb{Q}}_\ell$-vs.

<u>Def</u> $\rho' : W(C) = \pi_1^{geo}\rtimes\mathbb{Z} \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$

$$(h, m) \mapsto M^m\rho(h) \quad \cancel{(?) \text{ or } M^m\rho(h)M^{-m}}$$

It is this one by comments of Julian

To conclude: $\rho : \pi_1^{geo} \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ extends to some

$$\tilde{\rho} := \rho' : W(C) \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$$

and the space of all such ext's is a 1-dim'l $\overline{\mathbb{Q}}_\ell$-vs.

<u>Step 2</u> Weight analysis

<u>Thm (Deligne)</u> The geo. monod gp of any rk <u>1</u> lisse sheaf over a normal geo conn $\mathbb{F}_q$-sch is finite.

<u>Thm (Lafforgue, VII 6(2))</u> X sm curve $/\mathbb{F}_q$.

$\mathcal{F}$ lisse, <u>irr</u> rank r. <u>determinant is a character of finite order.</u>

<u>Then</u> $\forall x\in X_{(0)}$ closed pt, the roots of

$$\det(Id - T\varphi_x^{-1} ; \mathcal{F}_{\bar{x}})$$

are alg numbers with complex norm 1 wrt any embeding $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$

In other words: $\mathcal{F}_{\bar{x}} \in \mathrm{Rep}^{cts}_{\overline{\mathbb{Q}}_\ell}(G_k)$ is ι-pure of weight 0 $\forall x\in X_{(0)}$, $\forall \iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$

By Deligne: $\det(\rho) : \pi_1^{geo} \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ has finite image.

By twisting possibly a character of $W(k) = \mathbb{Z} \to \overline{\mathbb{Q}}_\ell^\times$

$$\det(\rho') : W(C) = \pi_1^{geo}\rtimes\mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$

is a character of finite order.

By Lafforgue: $\rho: W(C) \to GL_n(\overline{\mathbb{Q}}_\ell)$ has weight 0.

Hence $\rho \otimes \rho^\vee : W(C) \to GL_{n^2}(\overline{\mathbb{Q}}_\ell)$ has weight 0.

Note: $\rho \otimes \rho^\vee$ is unaffected with different choices of the extension ρ' of ρ

Step 3 Small deformations

(A, m_A) loc. Artin $\overline{\mathbb{Q}}_\ell$-alg, res field $\overline{\mathbb{Q}}_\ell$.

$I \subset A$ non-zero ideal

[E.g. $A = \overline{\mathbb{Q}}_\ell[T]/T^2$, $I = m_A = (T)$]

$0 \to I \to A \to B \to 0$ $\qquad (I^2 = 0)$

small extension of local Artin $\overline{\mathbb{Q}}_\ell$-alg's with res. field $\overline{\mathbb{Q}}_\ell$.

Let $\tilde{\rho}: \pi_1(C_{\bar{k}}, \bar{x}) \to GL_n(A)$ be a deform as in the thm

i.e.

$$\tilde{\rho} \otimes_A \overline{\mathbb{Q}}_\ell \simeq \rho \quad \& \quad Tr(\tilde{\rho} \circ \psi_{\varphi_x}) = Tr(\tilde{\rho})$$

Then

$$\tilde{\rho} \otimes_A B \simeq \rho \otimes_{\overline{\mathbb{Q}}_\ell} B$$

i.e. $\tilde{\rho}$ & $\rho \otimes_{\overline{\mathbb{Q}}_\ell} A$ are both deformations of $\rho \otimes_{\overline{\mathbb{Q}}_\ell} B$ to A.

Prop. The space of deformations of $\rho \otimes_{\overline{\mathbb{Q}}_\ell} B$ is

$$H^1(\pi_1^{geo}, \rho \otimes \rho^\vee) \otimes_A I.$$

Prop (Smooth) affine curves over sep closed field are $K(\pi, 1)$.

Def ($K(\pi,1)$) A scheme X is a $K(\pi,1)$ if $\forall$ lcc sheaf $\mathcal{F}$

$$H^i(\pi_1^{arith}, \mathcal{F}_{\bar{x}}) = H^i(X, \mathcal{F}) \quad \forall i.$$

Pf. Since $k = k^{sep} \rightsquigarrow$ coh dim of X is 1.

- $H^0(X, \mathcal{F}) = \mathcal{F}_{-}^{\pi_1^{arith}} = H^0(\pi^{arith}, \mathcal{F})$

- $H^0(X, \mathcal{F}) = \mathcal{F}_{\bar{x}}^{\pi_1^{arith}} = H^0(\pi_1^{arith}, \mathcal{F}_{\bar{x}})$.
- $H^1(\pi_1^{arith}, \mathcal{F}_{\bar{x}}) \overset{!}{=} H^1(X, \mathcal{F})$

<u>My trial</u> Hochschild-Serre ss.

Since X affine $\rightsquigarrow$ have univ. cover $\tilde{X}$.

$$H^p(\pi_1^{arith}(X), H^q(\tilde{X}, \mathcal{F}|_{\tilde{X}})) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

$\mathcal{F}$ is lcc $\rightsquigarrow$ $\mathcal{F}|_{\tilde{X}}$ is also lcc

since $\pi_1^{arith}(\tilde{X}) = 0$ $\rightsquigarrow$ $\mathcal{F}|_{\tilde{X}}$ is constant.

<u>Aim</u> $\underline{H^i(\tilde{X}, \mathcal{F}|_{\tilde{X}}) = 0} \quad \forall i > 0$

If we have this

$$\boxed{\Rightarrow H^p(\pi_1^{arith}(X), \mathcal{F}_{\bar{x}}) = H^p(X_{et}, \mathcal{F}) \quad \forall p.}$$

<u>Thm</u> (Carayol) (A, m_A, F) local ring, $R = A$-alg.

$\rho, \rho' : R \to GL_n(A)$ two reps.

Suppose
- residue rep $\bar{\rho} : R \otimes F \to GL_n(F)$ is abs irr
- $Tr(\rho) = Tr(\rho') : R \to A$

<u>Then</u> $\rho \simeq \rho'$

$$[\tilde{\rho}] - [\rho \underset{\overline{\mathbb{Q}}_\ell}{\otimes} A] \in H^1(\pi_1^{geo}, \rho \otimes \rho^\vee) \underset{A}{\otimes} I \overset{k(\pi,1)}{\overset{\downarrow}{=}} H^1(C_{\bar{k}}, \rho \otimes \rho^\vee) \underset{A}{\otimes} I.$$

<u>Obs 1</u>
- $\rho : W(C) \to GL_n(\overline{\mathbb{Q}}_\ell)$ abs. irr.
- $\tilde{\rho} \circ \psi_g$ and $\tilde{\rho}$ have the same trace

$\overset{\text{Carayol}}{\Longrightarrow}$ $\tilde{\rho} \circ \psi_g \simeq \tilde{\rho}$

i.e. $\tilde{\rho} : \pi_1^{geo} \to GL_n(A)$ is φ_x-inv.

Obs 2   Step 1: $\rho \circ \psi_g \simeq \rho$ (bc ρ is arith)

$$\Rightarrow \left(\rho \underset{\mathbb{Q}}{\otimes} A\right) \circ \psi_g \simeq \rho \underset{\mathbb{Q}_\ell}{\otimes} A$$

i.e. $\rho \underset{\mathbb{Q}_\ell}{\otimes} A$ is also φ_x-inv.

$$\Rightarrow [\tilde{\rho}] - [\rho \otimes A] \in \left(H^1(C_{\bar{k}}, \rho \otimes \rho^\vee) \underset{A}{\otimes} I\right)^{\varphi_x}.$$

Recall (Step 2) $\rho \otimes \rho^\vee$ has weight 0.

Thm (Deligne) $H^1_c(C_{\bar{k}}, \rho \otimes \rho^\vee)$ has weight $\{0, 1\}$.

Poincaré duality: $H^1_c(C_{\bar{k}}, \mathcal{F}^\vee)(1)^\vee = H^1(C_{\bar{k}}, \mathcal{F})$

$\Rightarrow H^1(C_{\bar{k}}, \rho \otimes \rho^\vee)$ has weight $\{1, 2\}$.

$\Rightarrow H^1(C_{\bar{k}}, \rho \otimes \rho^\vee) \underset{\mathbb{Q}_\ell}{\otimes} I$ has weight $\{1, 2\}$.

But $G_k \curvearrowright \left(H^1(C_{\bar{k}}, \rho \otimes \rho^\vee) \underset{\mathbb{Q}_\ell}{\otimes} I\right)^{\varphi_x}$ as id

it can only have weight 0.

$$\Rightarrow \left(H^1(C_{\bar{k}}, \rho \otimes \rho^\vee) \otimes I\right)^{\varphi_x} = 0$$

$$\Rightarrow \tilde{\rho} \simeq \rho \underset{\mathbb{Q}_\ell}{\otimes} A \qquad \blacksquare$$