MFO-ABSTRACT: ALGEBRAIC K-THEORY 2019
RECIPROCITY SHEAVES AND ABELIAN RAMIFICATION THEORY

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We report on our joint work with S. Saito [8] in which it is shown that the theory of reciprocity sheaves gives a unified picture of various classical abelian ramification phenomena.

1. Reciprocity sheaves, following Kahn-Saito-Yamazaki, see [2], [9]. Let k be a fixed perfect base field. In the following, a pair (X, D) consists of a separated finite type k-scheme X and an effective (possibly empty) Cartier divisor D on X, such that $X \setminus |D|$ is smooth. A *compactification* of (X, D) is a pair $(\overline{X}, \overline{D} + B)$, where $\overline{X}$ is a proper k-scheme and B and $\overline{D}$ are effective Cartier divisors such that $X = \overline{X} \setminus |B|$ and $D = \overline{D}_{|X}$. Given two pairs $\mathcal{X} = (X, D)$ and $\mathcal{Y} = (Y, E)$ we denote by $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y})$ the free abelian group with generators the integral closed subschemes $V \subset X \setminus |D| \times Y \setminus |E|$ which are finite and surjective over a component of $X \setminus |D|$ satisfying the property that the normalization of the closure $\tilde{V} \to X \times Y$ is proper over X and the inequality $D_{|\tilde{V}} \geq E_{|\tilde{V}}$ holds. We obtain a category $\underline{\mathbf{M}}\mathbf{Cor}$ with objects the pairs (X, D) and morphisms as defined above; the composition is induced by the usual composition of finite correspondences.

Let $\mathcal{X} = (\overline{X}, D)$ be a pair with $U = \overline{X} \setminus |D|$ and assume $\overline{X}$ is proper. For $S \in \mathbf{Sm}_k$ we define

$$h_0(\mathcal{X})(S) := \mathrm{Coker}(\underline{\mathbf{M}}\mathbf{Cor}((\mathbb{P}^1_S, \{\infty\}_S), \mathcal{X}) \xrightarrow{i_0^* - i_1^*} \mathbf{Cor}(S, U)).$$

This defines a presheaf with transfers $h_0(\mathcal{X})$ on $\mathbf{Sm}_k$. Let F be a presheaf with transfers on $\mathbf{Sm}_k$ and let $\mathcal{X} = (X, D)$ be a pair with $U = X \setminus |D|$. Set

$$\widetilde{F}(\mathcal{X}) := \left\{ a \in F(U) \;\middle|\; \begin{array}{l} \text{the Yoneda map } \mathbf{Cor}(U, -) \to F \text{ defined by } a \text{ factors via} \\ h_0(\overline{\mathcal{X}}), \text{ for some compactification } \overline{\mathcal{X}} \text{ of } \mathcal{X} \end{array} \right\}.$$

One can think of this as sections on U with poles on X controlled by D and some finite poles at infinity. If C is a proper smooth curve over a function field K, then $h_0(C, D)(K) = \mathrm{CH}_0(C, D)$ is the Chow group with modulus as defined by Serre; in this case we obtain a pairing

$$\widetilde{F}(C, D) \otimes_{\mathbb{Z}} \mathrm{CH}_0(C, D) \to F(K). \tag{1}$$

The assignment $\mathcal{X} \to \widetilde{F}(\mathcal{X})$ defines a presheaf on $\underline{\mathbf{M}}\mathbf{Cor}$. A *reciprocity presheaf* is a presheaf with transfers F on $\mathbf{Sm}_k$ such that for all $X \in \mathbf{Sm}_k$ we have

$$F(X) = \bigcup_{\overline{\mathcal{X}}} \widetilde{F}(\overline{\mathcal{X}}),$$

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where the union is over all compactifications of $(X, \emptyset)$. We say F is a *reciprocity sheaf* if it is a sheaf in the Nisnevich topology on $\mathbf{Sm}_k$.

2. Let F be a reciprocity sheaf. Denote by Φ the set of henselian discrete valuation rings of geometric type over k and by $\Phi_{\le n}$ the subset of those $L \in \Phi$ with $\mathrm{trdeg}(L/k) \le n$. For $L \in \Phi$ denote by $\mathcal{O}_L$ and $\mathfrak{m}_L$ the ring of integers and the maximal ideal, respectively. Set

$$\widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) := \widetilde{F}(\mathrm{Spec}\,\mathcal{O}_L, n \cdot \text{closed point}).$$

We define the *motivic conductor* $c^F = \{c_L^F : F(L) \to \mathbb{N}_0\}_{L\in\Phi}$ by

$$c_L^F(a) := \min\{n \ge 0 \mid a \in \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n})\}.$$

Definition 1 ([8, §4])**.** We say F has *level* $n \in [1, \infty]$ if for all $X \in \mathbf{Sm}_k$ and all $a \in F(\mathbb{A}^1_X)$ the condition $c^F_{k(x)(t)_\infty}(a_x) \le 1$, for all at most $(n-1)$-dimensional points $x \in X$, implies $a \in F(X)$. Here $k(x)(t)_\infty = \mathrm{Frac}(\mathcal{O}^h_{\mathbb{P}^1_{k(x)},\infty})$ and $a_x \in \widetilde{F}(k(x)(t)_\infty)$ denotes the pullback of a.

Theorem 1 ([8, Thm 4.15, Thm 4.29])**.** *(1) Let $X \in \mathbf{Sm}_k$ be connected, $a \in F(\mathbb{A}^1_X)$, and set $K := k(X)$.*

$$c^F_{K(t)_\infty}(a) \le 1 \Longrightarrow a \in F(X).$$

(2) Assume F has level $n \le \infty$. For $a \in F(X \setminus |D|)$ we have

$$a \in \widetilde{F}(X, D) \Longleftrightarrow \begin{array}{l}\text{there exists a compactification } (\overline{X}, \overline{D} + B) \text{ of } (X, D) \text{ such}\\ \text{that } c_L^F(\rho^* a) \le v_L(\rho^*(\overline{D} + B)), \text{ for all } \rho \in X(L) \text{ and all}\\ L \in \Phi_{\le n}.\end{array}$$

If F is a homotopy invariant sheaf and $a \in F(L)$, then $c_L^F(a) = 0$, if $a \in F(\mathcal{O}_L)$, and $c_L^F(a) = 1$, else. This implies:

Corollary 1. *Denote by $h^0_{\mathbb{A}^1}(F)$ the maximal $\mathbb{A}^1$-invariant subsheaf of F. Then $h^0_{\mathbb{A}^1}(F) = F^{c^F \le 1}$.*

We have the following general procedure to compute the motivic conductor: on any presheaf with transfers we define a general notion of conductor; the motivic conductor is the minimal conductor; one gets lower bounds for the motivic conductor by local symbol computations. Using this we show:

Theorem 2 ([8, Thm 5.2])**.** *Let G be a smooth commutative k-group. Then G is a reciprocity sheaf of level 1 and the motivic conductor is determined by the Rosenlicht-Serre modulus on curves* [10, III].

Theorem 3 ([8, Thm 6.4, Cor's 6.7, 6.8])**.** *Assume $\mathrm{char}(k) = 0$ and $q \ge 0$. The q-th differentials relative to k, Ω^q, is a reciprocity sheaf of level $q+1$ and for $L \in \Phi$ with local parameter t we have $\widetilde{\Omega^q}(\mathcal{O}_L, \mathfrak{m}^{-n}) = \frac{1}{t^{n-1}}\Omega^q_{\mathcal{O}_L}(\log)$; $h^0_{\mathbb{A}^1}(\Omega^q)(X) = H^0(\overline{X}, \Omega^q_{\overline{X}}(\log D))$, where $(\overline{X}, D)$ is an SNCD compactification of X; the closed forms $Z\Omega^q$ have level q.*

Corollary 2. *Let Y be a normal affine Cohen-Macaulay k-scheme, $\dim Y = d$. Then Y has rational singularities if and only if there exists an effective Cartier divisor D on Y whose support contains $Y_{\rm sing}$ such that the sheaf $Y_{\rm Zar} \ni U \mapsto \widetilde{\Omega^d}(U, D_{|U})$ is $(S2)$.*

Theorem 4 ([8, Thm 6.11, Cor 6.12])**.** *Assume $\mathrm{char}(k) = 0$ (as above). Denote by $\mathrm{MIC}_1(X)$ the group of isomorphism classes of integrable rank 1 connections on X. Then $X \mapsto \mathrm{MIC}_1(X)$ is a reciprocity sheaf of level 1; the motivic conductor of a rank 1 connection on $L \in \Phi$ is equal to its irregularity as defined in [4] (up to a shift by $+1$); $h^0_{\mathbb{A}^1}(\mathrm{MIC}_1)(X)$ are the regular singular rank 1 connections on X in the sense of Deligne.*

The pairing (1) for $F = \mathrm{MIC}_1$, was constructed before in [1, §4].

Theorem 5 ([8, Thm 8.8, Cor 8.10])**.** *Assume $\mathrm{char}(k) = p > 0$ and ℓ is a prime different from p. Let $\mathrm{Lisse}^1(X)$ be the group of isomorphism classes of $\overline{\mathbb{Q}}_\ell$-lisse rank 1 sheaves on X. Then $X \mapsto \mathrm{Lisse}^1(X)$ is a reciprocity sheaf of level 1; the motivic conductor is equal to the Artin conductor (defined via the Brylinski-Kato-Matsuda filtration cf. [3], [7]); $h^0_{\mathbb{A}^1}(\mathrm{Lisse}^1)(X)$ are the tamely ramified 1-dimensional $\overline{\mathbb{Q}}_\ell$-representations of $\pi_1^{\rm ab}(X)$ (defined using curve-tameness, see [5]).*

If we restrict to finite monodromy we obtain the pairing (1) for $F = H^1_{\text{ét}}(-, \mathbb{Q}/\mathbb{Z})$; this is the pairing from geometric class field theory in case K is a finite field.

It seems the following motivic conductor was not considered before.

Theorem 6 ([8, Thm 9.12])**.** *Assume $\mathrm{char}(k) = p > 0$. Let G be a commutative finite k-group. Denote by $H^1(G)(X) := H^1_{\rm fppf}(X, G)$ the group of isomorphism classes of G-torsors on X. Then $X \mapsto H^1(G)(X)$ is a reciprocity sheaf of level 2; it has level 1 if G has no infinitesimal unipotent part; the motivic conductor for $G = \alpha_p$ or for G without infinitesimal unipotent part is computed explicitly; if we write $G = G' \times G_u$ with G_u unipotent and G' without unipotent part, then $h^0_{\mathbb{A}^1}(H^1(G))(X) = H^1(G')(X) \oplus H^1(G_u)(\overline{X})$, where $\overline{X}$ is a smooth compactification of X.*

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