Springer correspondence via Borel-Moore homology

Andreas,Erik,Gaëtan,Ziqian

12 sep 2024

Motivation

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Flag

Definition

A (full) **flag** in $\mathbb{C}^n$ is a sequence of subspaces $0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = \mathbb{C}^n$ with $\dim V_i = i$.
We denote by $\mathcal{F}\ell(n)$ the set of flags of $\mathbb{C}^n$.

Example

For $n = 2$, a flag is just a line in $\mathbb{C}^2$. Therefore, $\mathcal{F}\ell(2) = \mathbb{P}^1$.

The set of flags naturally sits inside a product of Grassmanians. It has the structure of a smooth projective algebraic variety.

Springer fibers

Definition

Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. The **Springer fiber** associated to A is the set $\mathcal{F}\ell(n)^A := \{V_\bullet \in \mathcal{F}\ell(n) | \ A(V_i) \subseteq V_i \text{ for all } i\}$.

Examples

- If $A = 0$ then $\mathcal{F}\ell(n)^A = \mathcal{F}\ell(n)$.
- If $A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}$ is a single Jordan block, then $\mathcal{F}\ell(n)^A$ consists of a single element.

Springer fibers

Fact

Each irreducible component of a Springer fiber has the same dimension $d(A)$.

We consider singular cohomology with rational coefficients, with top non-zero cohomology $H^{2d(A)}(\mathcal{F}\ell(n)^A)$.

Springer correspondence

Theorem (Springer correspondence for type A)

Let $n \in \mathbb{N}^*$.

- Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be a nilpotent matrix. The symmetric group S_n acts naturally on the vector space $H^{2d(A)}(\mathcal{F}\ell(n)^A)$. This action gives rise to an irreducible representation of S_n.
- Each irreducible representation of S_n is isomorphic to $H^{2d(A)}(\mathcal{F}\ell(n)^A)$ for some nilpotent matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, determined uniquely up to conjugation.

This theorem is particularly interesting since the action of S_n on $H^{2d(A)}(\mathcal{F}\ell(n)^A)$ doesn't come from an action on $\mathcal{F}\ell(n)^A$.

Lie algebras

Definition

A (complex) **Lie algebra** $\mathfrak{g}$ is a $\mathbb{C}$-vector space together with an antisymmetric bilinear map

$$[\bullet, \bullet] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which satisfies the *Jacobi identity*:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Lie algebras

Examples

- Abelian Lie algebras: for any vector space V, define the Lie bracket $[x, y] := 0 \ \forall x, y \in V$.
- Let A be an associative algebra. Define $[x, y] := xy - yx$. This gives to A the structure of a Lie algebra.
- If we consider $A = \mathrm{Mat}_{n\times n}(\mathbb{C})$ with the Lie bracket previously defined, we get the Lie algebra $\mathfrak{gl}_n$.
- $\mathfrak{sl}_n := \{x \in \mathfrak{gl}_n \mid \mathrm{Tr}(x) = 0\} \subseteq \mathfrak{gl}_n$.

Simple Lie algebras

Definition

An **ideal** I of $\mathfrak{g}$ is a subvector space such that $[x, y] \in I$ for all $x \in \mathfrak{g}, y \in I$.

Examples

- The trivial ideals: 0 and $\mathfrak{g}$.
- $\mathfrak{sl}_n \subseteq \mathfrak{gl}_n$ is an ideal.

Definition

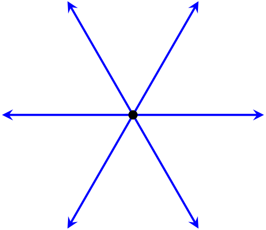
A non-abelian Lie algebra $\mathfrak{g}$ is **simple** if it contains no non-trivial ideal.

Example

The Lie algebra $\mathfrak{sl}_n$ is simple.

Root system and Weyl Group

To a simple Lie algebra, we can attach a geometric object called a *root system*, and a finite group W called the *Weyl group*. In the case of $\mathfrak{sl}_n$, this group is the symmetric group S_n.



Some subalgebras

Definition

Let $\mathfrak{g}$ be a simple Lie algebra. A maximal self-normalizing abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a **Cartan subalgebra**.

Definition

The **derived series** of $\mathfrak{g}$ is defined inductively by $D^0\,\mathfrak{g} = \mathfrak{g}$ and $D^{n+1}\,\mathfrak{g} = [D^n\,\mathfrak{g}, D^n\,\mathfrak{g}]$. The Lie algebra $\mathfrak{g}$ is **solvable** if $D^n\,\mathfrak{g} = 0$ for some n.

Definition

A maximal solvable subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is called a **Borel subalgebra**.

Some subalgebras

Example

In the case of $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$, we can take $\mathfrak{h}$ to be the subset of diagonal matrices and $\mathfrak{b}$ to be the subset of upper triangular matrices.

Simple Lie algebras and algebraic groups

Lie algebras also appears naturally when we study algebraic groups. Given G an algebraic group, the tangent space at the identity as naturally the structure of a Lie algebra $\mathfrak{g}$. This way, the group G acts naturally on the Lie algebra.
Using this construction, every simple Lie algebra appears as the Lie algebra of a simple algebraic group.

Example

The Lie algebra of GL_n is $\mathfrak{gl}_n$, the Lie algebra of SL_n is $\mathfrak{sl}_n$.

The flag variety

Definition

Let $\mathfrak{g}$ be a simple Lie algebra. We define the **flag variety**

$$\mathcal{F}\ell = \mathcal{F}\ell(\mathfrak{g}) := \{\mathfrak{b} \subseteq \mathfrak{g} \mid \mathfrak{b} \text{ is a Borel subalgebra}\}.$$

For $\mathfrak{g} = \mathfrak{sl}_n$, observe the bijection between the flags previously defined and the set of Borel subalgebras.
The flag variety $\mathcal{F}\ell$ has the structure of a projective complex algebraic variety.

Grothendieck-Springer space

Definition

Let $\mathfrak{g}$ be a Lie algebra. The **Grothendieck-Springer space** is defined as follows:

$$\tilde{\mathfrak{g}} := \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{F}\ell |\, x \in \mathfrak{b}\}.$$

We have a natural map

$$\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$$

which forgets the flag, i.e. $\pi(x, \mathfrak{b}) = x$.

Springer fibers

Definition

The fibers $\mathcal{F}\ell^x := \pi^{-1}(x)$ are called **Springer fibers**.

The Springer fiber $\mathcal{F}\ell^x$ is the set of Borel subalgebras which contain x:

$$\mathcal{F}\ell^x = \{\mathfrak{b} \in \mathcal{F}\ell |\ x \in \mathfrak{b}\}.$$