Springer correspondence via Borel-Moore homology

Andreas,Erik,Gaëtan,Ziqian

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Setup

Let X be a **nice** space i.e.

- X is a locally compact topological space
- X has the homotopy type of a finite CW-complex
- X admits a closed embedding into a C^∞-manifold M

(For example, any complex or real algebraic variety is a such nice space).

All ordinary (co)homology is taken with **rational** coefficients.

Definition :

We define the **Borel-Moore homology** $H_{\bullet}^{BM}(X)$ of X by the following equivalent definitions :

1) The Borel-Moore chain is a locally finite sums of singular simplices (i.e. finite sums when intersecting with compact subset).
2) Let $X \hookrightarrow M$ an embedding into a closed oriented n-manifold,
$$H_k^{BM}(X) := H^{n-k}(M, M - X)$$
3) Let $X_+ := X \cup \{\infty\}$ the one-point compacification,
$$H_k^{BM}(X) := H_k(X_+, \{\infty\})$$
4) $H_{\bullet}^{BM}(X)$ is the sheaf cohomology of the Verdier dualizing complex ω_X.

Poincaré duality

Theorem :

Let X be a smooth oriented manifold of dimension d. Then

$$H_k^{BM}(X) \simeq H^{d-k}(X)$$

Example

Example 1 :
Let X be compact.
Then

$$H_k^{BM}(X) \simeq H_k(X)$$

Example 2 :

$$H_k^{BM}(\mathbb{R}^n) = \begin{cases} 0 & \text{if } k \neq n \\ \mathbb{Q} & \text{otherwise} \end{cases}$$

Example

Example 3 :
Let $X = S^1 \times \mathbb{R}$.
Using Kunneth decomposition :

$$H_0^{BM}(X) \simeq H^2(X) = H^1(S^1) \otimes H^1(\mathbb{R}) = 0$$
$$H_1^{BM}(X) \simeq H^1(X) = \mathbb{Q}$$
$$H_2^{BM}(X) \simeq H^0(X) = \mathbb{Q}$$

Question : what about $X = S^1 \times \mathbb{R}/S^1 \times \{0\}$ **?**

Functoriality

Proposition :

Let $f : X \to Y$ be a map of spaces.

1) If f is **proper**, then f admits a pushforward :

$$f_* : H_\bullet^{BM}(X) \to H_\bullet^{BM}(Y)$$

2) If f is an **open embedding**, there is a restriction map :

$$f^! : H_\bullet^{BM}(Y) \to H_\bullet^{BM}(X)$$

Functoriality

Proposition :

Let $f : X \to Y$ be a map of spaces.

1) If f is an **oriented fibration** of relative complex dimension d, then f admits a pullback :

$$f^! : H_k^{BM}(Y) \to H_{k+2d}^{BM}(X)$$

2) If f is an **oriented embedding** of manifolds of codimension d, then f admits a pullback :

$$f^! : H_k^{BM}(Y) \to H_{k-2d}^{BM}(X)$$

Localization sequence

Let $\iota : Z \hookrightarrow X$ be a closed embedding and $j : U \hookrightarrow X$ the open embedding of the complement of Z.
In particular, ι of proper.

We get :

$$H_\bullet^{BM}(Z) \xrightarrow{\iota_*} H_\bullet^{BM}(X) \xrightarrow{j^!} H_\bullet^{BM}(U)$$

We would like also to have a long exact sequence as for ordinary homology.

Localization sequence

Let $X \hookrightarrow M$ an closed embedding into a smooth manifold. Then by definition,

$$H_k^{BM}(X) = H^{n-k}(M, M - X)$$

and

$$H_k^{BM}(Z) = H^{n-k}(M, M - Z)$$

There is also an open subset M' of M such that we have $U \hookrightarrow M'$. By the excision axiom,

$$H_k^{BM}(U) = H^{n-k}(M, M - U)$$

Localization sequence

Using the standard long exact sequence , we get finally :

Proposition :

For an open-closed decomposition $X = U \sqcup Z$,

$$\cdots H_{k+1}^{BM}(U) \xrightarrow{\partial} H_k^{BM}(Z) \xrightarrow{*} H_k^{BM}(X) \xrightarrow{j^!} H_k^{BM}(U) \xrightarrow{\partial} H_{k-1}^{BM}(Z) \cdots$$

Application

Example 4 :
Let $X = S^1 \times \mathbb{R}/S^1 \times \{0\}$ and $Z = \{pt\}$.
Using the previous long exact sequence, we get :

$$H_0^{BM}(X) = 0$$
$$H_1^{BM}(X) = \mathbb{Q}$$
$$H_2^{BM}(X) = \mathbb{Q}^2$$

One can compare with :

$$H^2(X) = 0$$
$$H^1(X) = 0$$
$$H^0(X) = \mathbb{Q}$$

Fundamental classes

Definition

Let X be a complex algebraic variety of complex dimension d. We call the **fundamental class** of X, the element

$$[X] := p^!(\mathbb{Q}) \in H_{2d}^{BM}(X)$$

where

$$p^! : H_0^{BM}(pt) \to H_{2d}^{BM}(X)$$

Remark :

If X is an algebraic variety of pure complex dimension d, then the fundamental classes of its irreducible components form a basis of $H_{2d}^{BM}(X)$.

Let X be a smooth d-manifold and $f : X \to Y$ a proper map.
Let $Z := X \times_Y X$.
Naturally, we have :

$$\begin{array}{ccccc} Z \times Z & \xleftarrow{\quad s \quad} & Z \times_X Z & \xrightarrow{\quad r \quad} & Z \\ \Vert & & \Vert & & \Vert \\ (X \times_Y X) \times (X \times_Y X) & \xleftarrow[(p_{12}, p_{23})]{} & X \times_Y X \times_Y X & \xrightarrow[p_{13}]{} & X \times_Y X \end{array}$$

with s oriented embedding of dimension d and r proper.

Definition

Definition

The convolution on $H_\bullet^{BM}(Z)$ is defined by :

$$* := r_* s^! : H_\bullet^{BM}(Z \times Z) \simeq H_\bullet^{BM}(Z) \otimes H_\bullet^{BM}(Z) \to H_{\bullet-2d}^{BM}(Z)$$

Proposition :

$H_{\bullet-2d}^{BM}(Z)$ is a graded associative algebra.

In particular, $H^{BM}_{\bullet-2d}(\mathrm{St}(\mathfrak{g}))$ and $H^{BM}_{\bullet-4m}(\mathrm{St}(\mathcal{N}))$ are graded algebras.

Definition

Let's denote :

$$A(\mathfrak{g}) := H^{BM}_{2d}(\mathrm{St}(\mathfrak{g}))$$

$$A(\mathcal{N}) := H^{BM}_{4m}(\mathrm{St}(\mathcal{N}))$$

Remark :

Under a similair convolution map, $H^{BM}_{2d(x)}(\mathcal{F}\ell^x)$ is natually a left (and right) $A(\mathcal{N})$-module.

Proposition :

$\mathrm{St}(\mathfrak{g})$ and $\mathrm{St}(\mathcal{N})$ are equidimensional of dimension d and $2m$, their irreducible components are indexed by the Weyl group W.

In particular, we can denote the bases of $A(\mathfrak{g})$ and $A(\mathcal{N})$ by :

$$\Lambda_w := [\overline{\mathrm{St}_w(\mathfrak{g})}]$$

$$T_w := [\overline{\mathrm{St}_w(\mathcal{N})}]$$

Big Steinberg

Theorem

$$\begin{array}{ccc} \mathbb{Q}[W] & \longrightarrow & A(\mathfrak{g}) \\ w \in W & \mapsto & \Lambda_w \end{array}$$

is an isomorphism of algebras.

Problem :

$$T_v * T_w \neq T_{vw}$$

in general, so we do not get an isomorphism of algebras by the same way.

Nilpotent Steinberg

Theorem

$$\mathbb{Q}[W] \simeq A(\mathcal{N})$$

Consequence

$A(\mathcal{N})$ is a semi-simple algebra.

Consequences

Lemma

Let $x \in \mathcal{N}$.
Let's write $A_x := H^{BM}_{2d(x)}(\mathcal{F}\ell^x)$ as $A(\mathcal{N})$-module,

$$(A_x)_R \simeq (A_x))^\vee_L$$

Idea :
Involution on $\mathrm{St}(\mathcal{N})$ gives an anti-involution $c \mapsto c^t$ on $A(\mathcal{N})$.
Under $\mathbb{Q}[W] \simeq A(\mathcal{N})$, it corresponds to $w \mapsto w^{-1}$.
Then

$$(A_x)^*_R \simeq (A_x)^{t,\vee}_R$$

Case $\mathrm{SL}_n(\mathbb{C})$

Theorem

Let $x \in \mathcal{N}$.
Then

1) A_x is a simple $A(\mathcal{N})$-module.
2) A_x and A_y are isomorphic if ans only if x and y are conjugated under $\mathrm{SL}_n(\mathbb{C})$.
3) $\{A_x \mid \bar{x} \in \mathcal{N}/\mathrm{SL}_n(\mathbb{C})\}$ is a coset representative of isomorphism classes of simple $A(\mathcal{N})$-module.

Idea :

- $$\mathbb{C} \otimes_{\mathbb{Q}} A(\mathcal{N}) \simeq \mathbb{C}[W] \simeq \bigoplus_{\alpha} \mathrm{End}_{\mathbb{C}}(E_\alpha)$$
  where E_α range over all simple $\mathbb{C}[W]$-module.

- $$A(\mathcal{N}) \simeq \mathrm{gr}A(\mathcal{N}) = \bigoplus_{\bar{x}} (A_x)_L \otimes (A_x)_R$$
  where $\bar{x}$ range over $\mathcal{N}/\mathrm{SL}_n(\mathbb{C})$.

  then,
  $$\mathbb{C} \otimes_{\mathbb{Q}} A(\mathcal{N}) \simeq \bigoplus_{\bar{x}} \mathrm{End}_{\mathbb{C}}(A_x)$$

- let's write $A_x = \sum_\alpha n_{\bar{x},\alpha} E_\alpha$
  then,

$$\bigoplus_\alpha \mathrm{End}_{\mathbb{C}}(E_\alpha) \simeq \bigoplus_{\bar{x}} \bigoplus_{\alpha,\beta} n_{\bar{x},\alpha} n_{\bar{x},\beta} \mathrm{Hom}_{\mathbb{C}}(E_\alpha, E_\beta)$$

  Therefore, $n_{\bar{x},\alpha} n_{\bar{x},\beta} = \delta_{\alpha,\beta}$ (the Kronecker symbol).

  There is an unique $n_{\bar{x},\alpha} \neq 0$ (and equals to 1) for each $\bar{x}$, and this α should be different for different $\bar{x}$.

Thanks :)