Informal Group Disobedience
GRK 2240 Retreat 2024

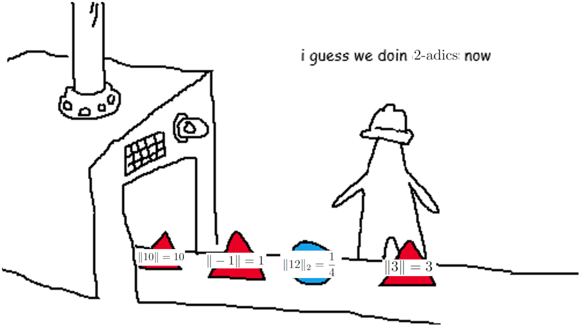
Alexander Ziegler

September 20, 2024

Heinrich Heine Universität Düsseldorf
BERGISCHE UNIVERSITÄT WUPPERTAL
RUHR UNIVERSITÄT BOCHUM

The p-adic numbers

TODO: Put a nice, short and quippy definition of $\mathbb{Q}_p$ here.[1]



[1] This was funny during the talk because I just explained the p-adics verbally but the slide just by itself is rather unhelpful.

Recollection of terminology

Every p-adic number field K comes equipped with a valuation

$$v\colon K \to \mathbb{Z} \cup \{\infty\}$$

and a (discrete) valuation ring

$$\mathcal{O}_K = v^{-1}(\{x \geq 0\}).$$

This ring is local with the maximal ideal $v^{-1}(\{x > 0\})$ admitting a generator π_K which we call uniformizer and a finite field called residue class field

$$\mathcal{O}_K/(\pi_K).$$

Unramified extensions

Definition

We call a finite extension $L|K$ unramified if π_K is a uniformizer of L. This is the case if and only if the corresponding extension of residue class fields has the same degree as $L|K$.

Heuristically unramified extensions are completely determined by the data on residue class fields and thus simple to understand.[2]

Fact

Unramified extensions are your friends.

Ramified extensions with degree coprime to p are okay-ish.

All other finite extensions are horrible.

[2] One should mention Hensel's lemma here but I can't. I am most likely not sitting next to your screen.

Theorem (Kronecker-Weber)

The maximal abelian extension of $\mathbb{Q}$ is the composite of all cyclotomic extensions of $\mathbb{Q}$ i.e. $\mathbb{Q}^{ab}$ can be constructed by adjoining all roots of unity to $\mathbb{Q}$.

It is Hilbert's unsolved twelfth problem to find an analogue of this theorem for arbitrary number fields. There is a solution for quadratic imaginary extensions of $\mathbb{Q}$ by adjoining "torsion points" of an elliptic curve and its j-invariant.

Formal modules

Definition

A formal R-module is a formal group law F over R with a homomorphism of rings

$$R \to \mathrm{End}_R(F), a \mapsto [a]_F(X)$$

such that $[a]_F(X) \equiv aX$ modulo terms of degree at least 2.

Lubin-Tate modules

Definition

Take K to be a p-adic number field with valuation ring $\mathcal{O}_K$ and uniformizer π_K. A Lubin-Tate module is a formal module F over $\mathcal{O}_K$ such that

$$[\pi_K]_F(X) \equiv X^{p^n} \bmod \pi$$

with $p^n := (\mathcal{O}_K : \pi_K \mathcal{O}_K)$.

Fact

Lubin-Tate modules exist for all choices of K and π_K. One can show that Lubin-Tate modules are completely determined by π_K and K up to a sensible notion of an isomorphism.

Definition

The π_K^n-torsion group of a Lubin-Tate module given by a uniformizer π_K is defined as

$$F(n) := \ker([\pi_K^n]_F)$$

where we interpret the underlying formal module as an $\mathcal{O}_K$-module that acts on all elements of positive valuation in $\overline{K}$ the algebraic closure of K.

Definition

Denote by L_n the field $K(F(n))$ obtained by adjoining the π_K^n-torsion points of F. We call it a Lubin-Tate extension.

An analogue of Kronecker-Weber

Theorem

Let K be a p-adic number field with maximal abelian extension K^{ab}. One can identify

$$K^{ab} = K^{nr} L_{\pi_K}$$

where K^{nr} is the maximal unramified extension of K and L_{π_K} the composite of all Lubin-Tate extensions L_n given by a uniformizer π_K.

Motivating Lubin-Tate theory

Asked 8 years, 11 months ago Modified 8 years, 11 months ago Viewed 5k times

42

The Lubin-Tate theory gives an amazingly clean and streamlined way of constructing the subfield (usually denoted) $F_\pi \subset F^{\mathrm{ab}}$ for a local field F fixed by the Artin map associated to the prime element π (i.e. such that $F^{\mathrm{ab}} = F_\pi \cdot F^{\mathrm{un}}$ with the usual notations). The idea to consider 1-dim. formal groups over the ring of integers $\mathcal{O}_F$ is a deus ex machina for me, and I wonder if anyone can explain Lubin-Tate's motivation to consider such a thing?

Related, on page 50 of J.S. Milne's online notes on the class field theory, he offers the speculation that the motivation comes from complex multiplication of elliptic curves and how one might try to get an analogue of the theory for local fields. But this requires again that it is somehow natural to consider formal groups as an analogue which I think still needs a motivation.

> What is the motivation to consider formal groups a la Lubin-Tate theory? Is there a way to motivate their construction?

nt.number-theory algebraic-number-theory class-field-theory local-fields

Share Cite Improve this question Follow

asked Oct 13, 2015 at 14:45
Stiofán Fordham

Lubin, Tate and Lubin-Tate theory

Maybe @lubin can comment... – Igor Rivin Oct 13, 2015 at 15:50

The prehistory explains a lot, but there is still a jump, that, perhaps, can only be explained by Tate's great originality. – anon Oct 13, 2015 at 16:09

"Tate's great originality"? Is this why this is called the "LUBIN-Tate" theory? Geez... – Igor Rivin Oct 13, 2015 at 20:23

It's true that it was Lubin who found the formal groups that do everything for you, but it was Tate who understood all the implications of their existence, and put everything together in the paper you're referring to. – Lubin Oct 16, 2015 at 0:59

Connections to cohomology theories

Definition

The Lubin-Tate formal group law of height n for the degree n unramified extension of $\mathbb{Q}_p$ with uniformizer p gives rise to the Johnson-Wilson cohomology theory $\tilde{K}(n)$ also called integral lifts of Morava K-theory.
In this sense Morava K-theory, denoted by $K(n)$, is $\tilde{K}(n)/p$.

Ohhh there is some time left

Fact

For chromatic level 1 $K(1)$ is one of p summands that make up complex K-theory modulo p. The other summands are suspensions of $K(1)$.

Fact

Take $K(0)$ to be singular cohomology with rational coefficients. The Morava K-theories are the fields of cohomology theories in the sense that all "modules" over them are free.