PAC fields of characteristic zero are C1

after J. Kollár

Dzoara, Vivien, Kay

GRK2240 Retreat, Möhnesee 2024

C1 fields

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- C1 = quasi-algebraically closed

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- more generally: fraction field of excellent henselian DVR's with algebraically closed residue field (Lang)

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- $\exists\ k : \mathrm{Br}(k) = 0$ and k not C1 (Ax, works in any char)

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$$\mathrm{Br}(k) = 0 \text{ in all cases}$$

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- PAC $=$ pseudo algebraically closed (Ax in 1968)

Theorem, Kollár, Fried-Jarden

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In particular

$$k \text{ PAC} \implies \mathrm{Br}(k) = 0 \qquad \text{(only trivial SBV}/k\text{)}$$

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- $\mathrm{Spec}(\prod_p \mathbb{F}_p) = \coprod_p \mathrm{Spec}\, \mathbb{F}_p \sqcup X$
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  $\rightsquigarrow \kappa(\mathfrak{p})$ is PAC of char 0, for $\mathfrak{p} \in X$
- same with infinite products of PAC fields
- $K =$ function field of a smooth var$/\mathbb{Q}$ or $\bar{\mathbb{F}}_p$
  $\implies (K^{\mathrm{sep}})^{<\sigma_1,\ldots,\sigma_n>}$ is PAC for almost all $\sigma_i \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$

Conjecture, Ax 1968

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wrong without "perfect" (as k C1 implies $[k : k^p] \leq p$)

Theorem 1, Kollár 2007

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Consequence of:

Theorem 2

k=any field of char 0 $\Longrightarrow$

every hypersurface $H \subset \mathbb{P}^n_k$ of degree $\leq n$ has a geometrically integral closed subvariety $Y \subset H$

Remarks:

- any H is geom conn! If it is smooth, it is also geom integral

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$$H_{\mathbb{C}} = \{X_0 + X_1 - iX_2 = 0\} \cup \{X_0 + X_1 + iX_2 = 0\}$$

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  $\Rightarrow \{f = 0\} \subset \mathbb{P}^2_{\mathbb{Q}}$ has deg 3 and no geom int subvariety!

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$$F = g^{-1}(\{v = 0\}) \text{ contains geom int subvar } \Rightarrow \text{ Thm 2} \Rightarrow \text{ Thm 1}$$

Proof of Theorem 3

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- Basics of divisors
- Resolution of singularities
- Connectedness result

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- $D_1 \sim_{\mathbb{Q}} D_2$ if $\exists m > 0$ st $mD_1 \sim mD_2$

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- $D_3 = (xy(x + y) = 0) \subset \mathbb{A}^2_k$

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Theorem (H. Hironaka 1964)

There exists a resolution of singularities, i.e.

$$\pi : X' \longrightarrow X$$

where

- X' sm
- π succ blow-up in $Z \cup X \backslash X_{\mathrm{reg}}$
- $\pi^{-1}(Z \cup X \backslash X_{\mathrm{reg}})_{\mathrm{red}}$ sncd.

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  - $\pi^{-1}(V_Z)_{\mathrm{red}}$ sncd

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  - V_1 sm
  - $\pi^{-1}(V_Z)_{\mathrm{red}}$ sncd
  - π is an iso over V_U

- $L := \left((n+1-d)p_1^* H_0 + p_2^* Z\right)\Big|_V$ ample $(d \leq n)$

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Theorem 3

$$g^{-1}(x) \supset \text{geom int closed subvar}, \quad \forall\, x \in \mathbb{P}^1(k)$$

Connectedness result:

- Y sm, proj var $/k$
- C sm curve, $c \in C(k)$
- $f : Y \longrightarrow C$ dominant with geom conn snc fibers
- D $f-$vertical $\mathbb{Q}$-div with $-(K_Y + D)$ ample

Theorem 4

$$f^{-1}(c) \supseteq \text{irre cpt, which is geom conn}$$

Proof of Theorem 4

From now fix the situation

- $\mathrm{char}(k) = 0$
- Y sm, proj var$/k$
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Connectedness Theorem

$f^{-1}(c) \cap \mathrm{Supp}(D_{\geq 1})$ is geom conn.

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- If $A = 0$, nothing to show $\rightsquigarrow$ assume $A \neq 0$

Claim

$f_*\mathcal{O}_Y(B) \twoheadrightarrow f_*\mathcal{O}_A(B|_A)$

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- have a short exact sequence

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$\Rightarrow$ $R^1 f_*\mathcal{O}_Y(B-A) = 0$ by Kawamata-Viehweg Vanishing

Theorem (Kawamata-Viehweg Vanishing)

- M ample on Y
- $\Delta = \sum a_i P_i$ sncd on Y, P_i distinct, $0 < a_i < 1$
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Note: False in positive characteristic

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- apply f_* to get $f_*\mathcal{O}_A \hookrightarrow f_*\mathcal{O}_A(B|_A)$

Recall: want to show

$$f^{-1}(c) \cap A \text{ is conn}$$

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- $f_*\mathcal{O}_A \hookrightarrow f_*\mathcal{O}_A(B|_A) \twoheadleftarrow f_*\mathcal{O}_Y(B)$ $\leftarrow$ locally free of rank 1

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$$A = \coprod_{i=1}^{r} f^{-1}(c_i) \cap A =: \coprod_{i=1}^{r} A(c_i)$$

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- If $f^{-1}(c) \cap A = \emptyset \rightsquigarrow$ ok, because connected

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$$\mathcal{O}_{C,c} \twoheadrightarrow (f_*\mathcal{O}_{A(c)})_c = H^0(A(c), \mathcal{O}_{A(c)}) = \sum_{j=1}^{m} H^0(A(c)_j, \mathcal{O}_{A(c)_j}) = k^n$$

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$\Rightarrow$ $m = n = 1 \Rightarrow A(c)$ is connected $\square$

Theorem 4

$D = \sum a_i P_i$ f-vertical $\mathbb{Q}$-divisor with $-(K_Y + D)$ ample.
$\Rightarrow f^{-1}(c) \supseteq$ irred comp, which is geom conn over k

Proof:

$c \in C(k)$

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$\Rightarrow$ $D + \lambda f^*G$ f-vertical $\forall \lambda \in \mathbb{Q} \Rightarrow$ sncd

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- set $D' := D + \lambda_0 f^*G$, where $\lambda_0 := \min \left\{ \frac{1-a_i}{e_i} \mid f(P_i) = c \right\}$

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  - $\rightsquigarrow$ $\exists$ an irred comp E with coeff 1 in D'

Claim

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- $m :=$ multiplicity of E in f^*G, consider

$$D'' := D' - \frac{\varepsilon}{m} f^*G + \varepsilon E$$

for $0 < \varepsilon \ll 1$

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$\Rightarrow$ fiber of $E = D''_{\geq 1}$ over c is geom conn by the Connectedness Theorem $\square$

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$$\Downarrow$$

Theorem 3

$g : V = \{uf + vh = 0\} \subset \mathbb{P}^n_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k$
$\Rightarrow g^{-1}(x) \supset$ geom int closed subvar, $\forall x \in \mathbb{P}^1(k)$

$$\Downarrow$$

Theorem 2

every hypersurface $H \subset \mathbb{P}^n_k$ of degree $\leq n$ has a geometrically integral closed subvariety $Y \subset H$

Theorem 1, Kollár 2007

Every PAC field of char 0 is C1