Springer correspondence via Borel-Moore homology

Andreas,Erik,Gaëtan,Ziqian

12 sep 2024

Example in type A

Nontrivial example of Springer fibre in type A_2

Let $\mathfrak{g} = \mathfrak{sl}_3$ with flag variety $\mathcal{F}\ell(3)$. Consider the nilpotent

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}.$$

Any flag preserved by A lies in the following two families

$$\{\langle \varepsilon_1 \rangle \subset \langle \varepsilon_1, \lambda\varepsilon_2 + \mu\varepsilon_3 \rangle \subset \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \lambda, \mu \in \mathbb{C}\},$$
$$\{\langle \lambda\varepsilon_1 + \mu\varepsilon_2 \rangle \subset \langle \varepsilon_1, \varepsilon_2 \rangle \subset \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle : \lambda, \mu \in \mathbb{C}\}.$$

These families are isomorphic to $\mathbb{P}^1$ and intersect in a single point corresponding to the (standard) flag $\langle \varepsilon_1 \rangle \subset \langle \varepsilon_1, \varepsilon_2 \rangle \subset \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$. Hence, one can visualize $\mathcal{F}\ell(3)^A$ as two spheres S^2 glued together at a single point.

Example in type A

Example continued

This space has two irreducible components, and so $H^2(\mathcal{F}\ell(3)^A)$ has dimension 2. In fact, $H^2(\mathcal{F}\ell(3)^A)$ we will see later that it is the unique 2-dimensional irreducible representation of S_3 ! Moreover, this action is not coming from an S_3-action on $\mathcal{F}\ell(3)^A$.

Springer resolution

Let $\mathfrak{g}$ be any simple Lie algebra, and denote by $\mathcal{N}$ the variety of nilpotent elements (i.e. $x \in \mathfrak{g}$ such that $\operatorname{ad} x \colon \mathfrak{g} \to \mathfrak{g}$ is nilpotent). In the case of $\mathfrak{g} = \mathfrak{sl}_n$, the variety $\mathcal{N}$ consists of the nilpotent matrices in the usual sense : the ones with characteristic polynomial equal to t^n.

Springer resolution

Define

$$\tilde{\mathcal{N}} := \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{F}\ell : x \in \mathfrak{b}\}.$$

The projection $\mu \colon \tilde{\mathcal{N}} \to \mathcal{N}$ is called the *Springer resolution*.

Note that the map is similar to the Grothendieck-Springer space $\pi \colon \tilde{\mathfrak{g}} \to \mathfrak{g}$. More on this later.

Springer resolution

Proposition

The map $\mu\colon \tilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities, i.e.

- The variety $\tilde{\mathcal{N}}$ is smooth,
- The map μ is proper,
- There is a dense open subset $U \subset \mathcal{N}$ such that the restriction to $\mu^{-1}(U)$ is an isomorphism.

Sketch proof

For the first part, note that the projection $\tilde{\mathcal{N}} \to \mathcal{F}\ell$ is a vector bundle over a smooth variety. For the second part, the fibers of μ are the Springer varieties, which are compact as they are closed subsets of the (compact) flag variety. For the last part, regular nilpotent elements are contained in a unique Borel subalgebra.

Example of Springer resolution

Example in $\mathfrak{sl}_2$

Consider $\mathfrak{g} = \mathfrak{sl}_2$. Let $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g}$ and note that $P_A(t) = t^2 + \det(A)$. Hence,

$$\mathcal{N} = \{A \in \mathfrak{g} : \det(A) = 0\}$$

and it can be identified with a singular conic in $\mathbb{C}^3$. The space $\tilde{\mathcal{N}}$ can be identified with the set of pairs $(x, \mathfrak{b})$ where $x \in \mathcal{N}$ and $\mathfrak{b}$ is a line that contains x. Remember that for each regular nilpotent element of $\mathfrak{g}$, there is a unqiue Borel subalgebra that contains it. As a consequence, the Springer resolution $\mu \colon \tilde{\mathcal{N}} \to \mathcal{N}$ crushes the zero section $\mathbb{P}^1 \cong \mathcal{F}\ell \subset \tilde{\mathcal{N}}$ to a point.

Draw picture

Steinberg varieties

In the previous talk, we saw the *Grothendieck-Springer space*

$$\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{F}\ell : x \in \mathfrak{b}\}$$

with the projection $\pi \colon \tilde{\mathfrak{g}} \to \mathfrak{g}$.

Definition

We define the *big Steinberg variety* $\mathrm{St}(\mathfrak{g})$ and the *small/nilpotent Steinberg variety* $\mathrm{St}(\mathcal{N})$ as the pullbacks

$$\begin{array}{ccc} \mathrm{St}(\mathfrak{g}) & \longrightarrow & \tilde{\mathfrak{g}} \\ \downarrow & & \downarrow \pi \\ \tilde{\mathfrak{g}} & \xrightarrow{\pi} & \mathfrak{g} \end{array} \qquad \begin{array}{ccc} \mathrm{St}(\mathcal{N}) & \longrightarrow & \tilde{\mathcal{N}} \\ \downarrow & & \downarrow \mu \\ \tilde{\mathcal{N}} & \xrightarrow{\mu} & \mathcal{N}. \end{array}$$

Steinberg varieties

Explicitly,

$$\mathrm{St}(\mathfrak{g}) = \{(x, \mathfrak{b}_1, \mathfrak{b}_2) \in \mathfrak{g} \times \mathcal{F}\ell^2 : x \in \mathfrak{b}_1 \cap \mathfrak{b}_2\},$$
$$\mathrm{St}(\mathcal{N}) = \{(x, \mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{N} \times \mathcal{F}\ell^2 : x \in \mathfrak{b}_1 \cap \mathfrak{b}_2\}.$$

Fact (Bruhat decomposition)

There is a bijection between W and the G-orbits of $\mathcal{F}\ell \times \mathcal{F}\ell$.

For $w \in W$, denote by Y_w the corresponding G-orbit. For $(\mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{F}\ell \times \mathcal{F}\ell$, we write $\mathfrak{b}_1 \xrightarrow{w} \mathfrak{b}_2$ and say that $\mathfrak{b}_1$ is *in relative position* w to $\mathfrak{b}_2$ if $(\mathfrak{b}_1, \mathfrak{b}_2) \in Y_w$.

Decomposition of Steinberg varieties

For $w \in W$, we define

$$\mathrm{St}_w(\mathfrak{g}) := \{(x, \mathfrak{b}_1, \mathfrak{b}_2) \in \mathrm{St}(\mathfrak{g}) : \mathfrak{b}_1 \xrightarrow{w} \mathfrak{b}_2\},$$
$$\mathrm{St}_w(\mathcal{N}) := \{(x, \mathfrak{b}_1, \mathfrak{b}_2) \in \mathrm{St}(\mathcal{N}) : \mathfrak{b}_1 \xrightarrow{w} \mathfrak{b}_2\}.$$

By the Bruhat decomposition, we obtain stratifications

$$\mathrm{St}(\mathfrak{g}) = \bigsqcup_{w \in W} \mathrm{St}_w(\mathfrak{g}), \quad \mathrm{St}(\mathcal{N}) = \bigsqcup_{w \in W} \mathrm{St}_w(\mathcal{N}).$$

Decomposition of Steinberg varieties

Proposition

The projections

$$s_{\mathfrak{g},w} : \mathrm{St}_w(\mathfrak{g}) \to Y_w$$
$$s_{\mathcal{N},w} : \mathrm{St}_w(\mathcal{N}) \to Y_w$$

carry the structure of a vector bundle : over a point $(\mathfrak{b}_1, \mathfrak{b}_2) \in Y_w$, the fibres of $s_{\mathfrak{g},w}$ and $s_{\mathcal{N},w}$ are $\mathfrak{b}_1 \cap \mathfrak{b}_2$ and $[\mathfrak{b}_1, \mathfrak{b}_1] \cap [\mathfrak{b}_2, \mathfrak{b}_2]$ respectively.

Decomposition of Steinberg varieties

Proposition

The projections

$$s_{\mathfrak{g},w} : \mathrm{St}_w(\mathfrak{g}) \to Y_w$$
$$s_{\mathcal{N},w} : \mathrm{St}_w(\mathcal{N}) \to Y_w$$

carry the structure of a vector bundle : over a point $(\mathfrak{b}_1, \mathfrak{b}_2) \in Y_w$, the fibres of $s_{\mathfrak{g},w}$ and $s_{\mathcal{N},w}$ are $\mathfrak{b}_1 \cap \mathfrak{b}_2$ and $[\mathfrak{b}_1, \mathfrak{b}_1] \cap [\mathfrak{b}_2, \mathfrak{b}_2]$ respectively.
Furthermore, $\mathrm{St}_w(\mathfrak{g})$ and $\mathrm{St}_w(\mathcal{N})$ are smooth connected varieties of dimensions $\dim \mathfrak{g}$ and $\dim(\mathcal{N})$ respectively. It follows that $\mathrm{St}(\mathfrak{g})$ and $\mathrm{St}(\mathcal{N})$ are equidimensional.

Example of Steinberg varieties

Example of $\mathfrak{g} = \mathfrak{sl}_2$

In this case $W \cong S_2 = \{e, s\}$ and $\mathcal{F}\ell \cong \mathbb{P}^1$. We have

$$Y_e = \{(\mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{F}\ell^2 : \mathfrak{b}_1 = \mathfrak{b}_2\} \cong \mathcal{F}\ell$$

$$Y_s = \{(\mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{F}\ell^2 : \mathfrak{b}_1 \neq \mathfrak{b}_2\}$$

$$\mathrm{St}_e(\mathfrak{g}) \cong \tilde{\mathfrak{g}}$$

$$\mathrm{St}_s(\mathfrak{g}) = \{(x, \mathfrak{b}_1, \mathfrak{b}_2) \in \mathfrak{g} \times \mathcal{F}\ell^2 : x \in \mathfrak{b}_1 \cap \mathfrak{b}_2, \mathfrak{b}_1 \neq \mathfrak{b}_2\}$$

$$\mathrm{St}_e(\mathcal{N}) \cong \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{F}\ell : x \in \mathfrak{b}\} = \tilde{\mathcal{N}}$$

$$\mathrm{St}_s(\mathcal{N}) = \{(x, \mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{N} \times \mathcal{F}\ell^2 : x \in \mathfrak{b}_1 \cap \mathfrak{b}_2, \mathfrak{b}_1 \neq \mathfrak{b}_2\}$$

We want to check the dimensions of the components. The orbit Y_s has dimension 2.

Example of Steinberg varieties

Example of $\mathfrak{g} = \mathfrak{sl}_2$ continued

We have already seen that $\tilde{\mathcal{N}}$ is indeed 2-dimensional, and we have that $\dim \tilde{\mathfrak{g}} = \dim \mathfrak{g} = 3$.

The dimensions of $\mathrm{St}_s(\mathfrak{g})$ and $\mathrm{St}_s(\mathcal{N})$ follow from the fact that two distinct flags $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are 2-dimensional vector spaces in a vector space of dimension 3, hence the intersection has dimension 1. It follows that $\dim \mathrm{St}_s(\mathfrak{g}) = 1 + 2 = 3$

As distinct Borel subalgebras have distinct nilpotent radicals, $[\mathfrak{b}_1, \mathfrak{b}_1]$ and $[\mathfrak{b}_2, \mathfrak{b}_2]$ are two distinct lines and so their intersection is trivial. It follows that $\dim \mathrm{St}_s(\mathcal{N}) = 0 + 2 = 2$.