The theorem of Beilinson-Bernstein

Mini course at GRK retreat 2024

(with Andreas Bode & Julian Reichardt)

1. Talk: Representation theory of $\mathfrak{sl}_2$

$$\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C}) = \{ A \in \mathrm{Mat}_2(\mathbb{C}) : \mathrm{tr}\, A = 0 \}$$

a Lie algebra over $\mathbb{C}$, $[A,B] = AB - BA$.

<u>Properties:</u> (i) $\mathfrak{g} = \mathbb{C}f + \mathbb{C}h + \mathbb{C}e$,

$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(ii) $\mathfrak{g}$ is non-commutative, e.g.

$[h,f] = -2f$, $[h,e] = 2e$, $[e,f] = h$.

(iii) $\mathfrak{g}$ is simple, i.e. has no proper ideals.

(if $I \neq 0$, take $v = af + bh + ce$ in $I \setminus \{0\}$

and apply commutator relations to it ...)

Def: A **representation** of $\mathfrak{g}$ is a $\mathbb{C}$-vector space V together with a linear map

$$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$$

s.t. $\rho[x,y] = \rho x \cdot \rho y - \rho y \cdot \rho x \quad \forall x,y \in \mathfrak{g} \qquad (*)$

Examples: (i) $V_{triv} := \mathbb{C}$, $\rho = 0$ (triv. rep.)

(ii) $V_{std} := \mathbb{C}^2$, $\rho(x)(v) := x \underset{\substack{\uparrow \\ \text{matrix mult.}}}{\cdot} v$ (standard rep.)

(iii) $V_{ad} := \mathfrak{g}$, $\rho(x)(y) := [x,y]$ (adjoint rep.)

$\forall x,y \in \mathfrak{g}$.

(Verification of $(*)$ amounts to Jacobi-identity

$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$)

We have obvious notions of sub/quotient representation, irreducible rep., direct sum, (iso)morphism of representations etc.

Say that a rep. (V, ρ) of $\mathfrak{g}$ is <u>finite-dimensional</u> if $\dim_{\mathbb{C}} V < \infty$.

Here is the most important result on such reps.

<u>Weyl's theorem</u>: Any finite-dimensional rep. of $\mathfrak{g}$ is a direct sum of irreducible reps.

How can we determine the irreducible finite-dim. representations of $\mathfrak{g}$?

Let V be an irreducible finite-dim. representation.

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \circlearrowright V$ is diagonalizable endomorphism

$$\Rightarrow V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha \quad , \quad h(v) = \alpha v \quad \forall v \in V_\alpha$$

How does e, f act on V_α ?

Fundamental identity:

$$h e(v) = e h(v) + [h, e](v) = (\alpha + 2) e(v)$$

$\Rightarrow e : V_\alpha \to V_{\alpha+2}$. Similarly, $f(V_\alpha) \subset V_{\alpha-2}$

Choose $v \overset{*}{\in} V_n$ for $n \in \mathbb{C}$ with $V_{n+2} = 0$.

Let $m \in \mathbb{N}$ minimal for $f^m(v) = 0$.

Using $[e,f] = h$ + some induction $\Rightarrow$

$$e f^m(v) = m(n-m+1) f^{m-1}(v)$$

Hence: * $n-m+1 = 0$, so $n \in \mathbb{N}$

* $\{v, f(v), f^2(v), \dots\}$ spans V

* eigenvalues of h are $n, n-2, \dots, -n+2, -n$

* $\dim V_i = 1 \ \forall i$ and $\dim V = n+1$.

Let's check this for our examples V_{triv}, V_{std}, V_{ad} :

(i) $V_{triv} = (V_{triv})_0$

(ii) $V_{std} = (V_{std})_{-1} \oplus (V_{std})_1$:

with $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2$ have $h(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

so $(V_{std})_1 = \mathbb{C}x$. Similarly, $(V_{std})_{-1} = \mathbb{C}y$

for $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$.

(iii) $V_{ad} = \underbrace{(V_{ad})_{-2}}_{\overset{\shortparallel}{\mathbb{C}f}} \oplus \underbrace{(V_{ad})_0}_{\overset{\shortparallel}{\mathbb{C}h}} \oplus \underbrace{(V_{ad})_2}_{\overset{\shortparallel}{\mathbb{C}e}}$

From here, it is not so hard anymore to show:

<u>Classification theorem:</u>

Let $\mathrm{Sym}^n \mathbb{C}^2 := \langle x^n, x^{n-1}y, \dots, xy^{n-1}, y^n \rangle_{\mathbb{C}} \subset \mathbb{C}[x,y]$

$\mathfrak{g}$-action: $\forall A \in \mathfrak{g}$ define inductively

$A(f \cdot g) := A(f)g + fA(g) \quad \forall f, g \in \mathbb{C}[x,y]$

Then $\mathrm{Sym}^n \mathbb{C}^2$ is an irreducible $\mathfrak{g}$-representation

and $n \longmapsto \mathrm{Sym}^n \mathbb{C}^2$ is a bijection

$$\mathbb{N} \xrightarrow{\cong} \{\text{Irred. fin-dim. } \mathfrak{g}\text{-rep.}\} / \cong$$

What about infinite-dimensional representations?

Let $T(\mathfrak{g}) = \bigoplus_{d\geq 0} T(\mathfrak{g})^d$, $T(\mathfrak{g})^d = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (d times)

be the tensor algebra of $\mathfrak{g}$.

<u>Def</u>: The <u>universal enveloping algebra</u> $\mathcal{U}(\mathfrak{g})$

is the quotient of $T(\mathfrak{g})$ by $\langle x\otimes y - y\otimes x - [x,y] \rangle$
$\forall\, x,y\in\mathfrak{g}$

Note: (i) $\mathcal{U}(\mathfrak{g})$ is an associative f.g. $\mathbb{C}$-algebra

(ii) elements are "polynomials" in e, h, f; $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{g}) = \infty$

(iii) $\mathcal{U}(\mathfrak{g})$ is non-commutative, the basic relation is $x \cdot y - y \cdot x = [x, y]$
$\forall\, x, y \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$.

(iv) By construction, have equivalence

$$\{\mathfrak{g}\text{-representations}\} \xleftrightarrow{\sim} \{\mathcal{U}(\mathfrak{g})\text{-modules}\}$$

An interesting element in $\mathcal{U}(\mathfrak{g}) \setminus \mathfrak{g}$ is

$$\Omega := h^2 + 2h + 4fe$$

Let $\mathfrak{Z}(\mathfrak{g}) :=$ center of $\mathcal{U}(\mathfrak{g})$

<u>Fact</u>: $\mathfrak{Z}(\mathfrak{g}) = \mathbb{C}[\Omega]$

(for "$\subseteq$" need so-called Harish-Chandra isom.)

A fundamental result on irreducible reps is:

<u>Schur's lemma</u>: Ω acts on any irreducible $\mathfrak{g}$-representation V as a scalar in $\mathbb{C}$.

Proof: $\mathrm{End}_{\mathfrak{g}}(V)$ is an associative division algebra over $\mathbb{C}$

(since $\ker f$, $\operatorname{im} f$ are subrepresentations of V for $f \in \operatorname{End}_{\mathfrak{g}}(V)$), so $\Omega \in \operatorname{End}_{\mathfrak{g}} V = \mathbb{C}$. $\square$

<u>Example</u>: $V = \operatorname{Sym}^n \mathbb{C}^2$, $n \geq 0$

$x^n \in V_n$ with $e(x^n) = n x^{n-1} e(x) = 0$

$$\Rightarrow \Omega(x^n) = h^2(x^n) + 2h(x^n) + 4fe(x^n)$$

$$= n^2 x^n + 2n x^n = (n^2 + 2n) x^n$$

$\Rightarrow \Omega$ acts on V by $n^2 + 2n$.

Back to infinite-dimensional representations:

$\mathfrak{b} := \mathbb{C}h + \mathbb{C}e \subseteq \mathfrak{g}$ subalgebra $\rightsquigarrow \mathcal{U}(\mathfrak{b})$

Given $\lambda \in \mathbb{C}$ get homomorphism

$$\mathcal{U}(\mathfrak{b}) \longrightarrow \mathbb{C}, \quad h^j e^k \longmapsto \lambda^j$$

Form $\mathcal{U}(\mathfrak{g})$-module

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$$

(a „<u>Verma module</u>" of weight λ)

Some properties of Verma modules:

* $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathbb{C}f) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b})$ (as bimodules)

$\Rightarrow M(\lambda) = \mathcal{U}(\mathbb{C}f) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ as $\mathcal{U}(\mathbb{C}f)$-module

$\Rightarrow M(\lambda)$ has $\mathbb{C}$-basis $\{v_i\}_{i \geq 0}$ s.t.

$$h v_i = i f^{i-1} [h, f] \otimes 1 + f^i \lambda \otimes 1 = (\lambda - 2i) v_i$$

Similarly,
$$e v_i = (\lambda - i + 1) v_{i-1}$$

$$f v_i = (i+1) v_{i+1}$$

$(v_i := f^i \otimes 1)$

* $M(\lambda) = V_\lambda \oplus V_{\lambda-2} \oplus \cdots$, all of dim 1

with „highest weight" λ

* $N(\lambda) := \sum_{N \subsetneq M(\lambda)} N$ is proper submodule, so

is unique maximal submodule of $M(\lambda)$

* $M(\lambda)$ has unique irreducible quotient

$$L(\lambda) := M(\lambda) / N(\lambda).$$

With some more work, can show (<u>Verma's theorem</u>):

* $M(\lambda)$ is irreducible iff $\lambda \notin \mathbb{N}$

  (so $L(\lambda) = M(\lambda)$ is infinite-dim.)

* $L(\lambda) \cong \mathrm{Sym}^{\lambda}\mathbb{C}^2$ if $\lambda \in \mathbb{N}$.

  (in this case $N(\lambda) \cong L(-\lambda-2)$)

* In general, $M(\lambda)$ has finite length as $\mathcal{U}(\mathfrak{g})$-module.

Note that $\Omega \circlearrowright M(\lambda)$ as $\lambda^2 + 2\lambda$

(as in the fin.-dim. case: take $v_0 = 1 \otimes 1 \in V_\lambda$

and compute $\Omega(v_0)$)

$\Rightarrow$ in the subcategory („block") of all

$\mathfrak{g}$-representations with trivial central character

$\Omega = 0$ (i.e. in $\mathrm{Mod}(\mathcal{U}(\mathfrak{g})/\Omega)$)

have 2 Vermamodules $M(0)$ and $M(-2)$,

and their irreducible quotients

$$M(0) \twoheadrightarrow L(0) = V_{triv} \quad \text{and} \quad M(-2) = L(-2)$$

of dim ∞.

3 final remarks:

(i) The $M(\lambda)$, $\lambda \in \mathbb{C}$, generate a subcategory $\mathcal{O} \subseteq \mathrm{Mod}(\mathcal{U}(\mathfrak{g}))$, whose simple objects are given by the $L(\lambda)$.

(ii) Category $\mathcal{O}$ admits a duality $N \longrightarrow N^\vee$,

e.g. $M(\lambda)^\vee$ is given as

$$M(\lambda)^\vee := \bigoplus_{\alpha \in \mathbb{C}} \left(M(\lambda)_\alpha \right)^* \subset M(\lambda)^*$$

with $\mathfrak{g}$ - action : $x \in \mathfrak{g}$, $f \in M(\lambda)^\vee$, $m \in M(\lambda)$

$$(xf)(m) := f({}^t x\, m)$$

$M(\lambda)^\vee$ now <u>contains</u> $L(\lambda)$ as unique irreducible subobject.

(iii) The precise prediction of the multiplicities $[L(\mu):M(\lambda)]$, $\mu, \lambda \in \mathbb{C}$ (in the general case of a classical Lie algebra) became known as <u>Kazhdan-Lusztig conjecture</u> (K.-L., 1970s).

(Proved ~1980 by Beilinson-Bernstein & independently Brylinski-Kashiwara).