Appendix A.

THE DE RHAM-WITT COMPLEX AND ADDITIVE CHOW GROUPS OVER A FIELD: THE CHARACTERISTIC 2 CASE

KAY RÜLLING

The main theorems of [Rül07b] (see also [Rül07a]) were stated only for fields of characteristic $\neq 2$. This originates in the use of [HM04, Thm 4.2.8] which was only for odd primes. In this appendix it is explained that thanks to [Cos08] the results of [Rül07b] extend directly to the characteristic 2 case. I thank Amalendu Krishna and Jinhyun Park for the opportunity to detail this extension here.

A.1. Let $(\mathbb{W}_S\Omega^\bullet_A)_S$ denote the big de Rham Witt complex from [Hes15], where A is a ring and S is running through all truncation sets. It comes with the maps d, F_n, V_n, restriction, and multiplication. For $m \geq 1$ we set $\mathbb{W}_m\Omega^\bullet_A := \mathbb{W}_{\{1,\ldots,m\}}\Omega^\bullet_A$ and for a fixed prime p we denote the p-typical de Rham-Witt complex by $W_n\Omega^\bullet_A := \mathbb{W}_{\{1,p,\ldots,p^{n-1}\}}\Omega^\bullet_A$. Note that in general $d\circ d$ is not zero in $\mathbb{W}_S\Omega^\bullet_A$. Denote by $\mathbb{W}_S\Omega^\bullet_{A/\mathbb{Z}}$ the Witt complex from [Hes15, Rem 4.8] (with $k = \mathbb{W}(\mathbb{Z})$); it is the initial object in the category of Witt complexes with $\mathbb{W}(\mathbb{Z})$-linear differential. Note that $\mathbb{W}_S\Omega^\bullet_{A/\mathbb{Z}}$ is always a dga, in particular we have $d\circ d = 0$. Furthermore, if A contains a field, then $\mathbb{W}_S\Omega^\bullet_{A/\mathbb{Z}} = \mathbb{W}_S\Omega^\bullet_A$ see [Hes15, Rem 4.2, c)]; if A is an $\mathbb{F}_p$-algebra, then the p-typical de Rham-Witt complex is the one from Bloch-Deligne-Illusie. Also note in case A is an $\mathbb{F}_p$-algebra or a $\mathbb{Q}$-algebra, then we have a decomposition

$$\mathbb{W}_S\Omega^\bullet_A = \prod_{(j,p)=1} \mathbb{W}_{\mathcal{P}\cap S/j}\Omega^\bullet_A, \tag{A.1.1}$$

as in [Rül07b, Thm 1.11]. (Indeed, in this case the construction of the V-complex in [Rül07b, Prop 1.2] goes through and the same proof as in [Rül07b, Thm 1.11] shows that it is the initial object in the category of Witt complexes as in [Hes15] and that it decomposes as in (A.1.1).)

Theorem A.2 ([Rül07b, Thm 3.20] for $\mathrm{char}(k) \neq 2$)**.** *Let k be a field. Then there is an isomorphism*

$$\mathbb{W}_m\Omega^n_k \xrightarrow{\simeq} \mathrm{CH}^{n-1}(\mathbb{A}^1_k|(m+1)\cdot\{0\}, n), \quad m \geq 1,$$

where the right hand side is the additive Chow groups of Bloch-Esnault. Furthermore, via this isomorphism, the maps d, F_n, V_n, restriction and multiplication on the de Rham-Witt side correspond to $\mathcal{D}$, $\mathcal{F}_n$, $\mathcal{V}_n$, restriction, and $*$, on the Chow side, as defined in [Rül07b, Def-Prop 3.9].

Thanks to [Cos08], which was not at disposal when [Rül07b] was written, the proof of *loc. cit.* goes through, also for $p = 2$. We will explain this in more detail in the following.

A.3. We fix a prime p. Let A be a $\mathbb{Z}_{(p)}$-algebra and denote by $A[x]$ the polynomial ring in the variable x. Then the group $W_n\Omega^q_{A[x]/\mathbb{Z}}$ (resp .$W_n\Omega^q_{A[x,1/x]/\mathbb{Z}}$) is freely

The author is supported by the DFG Heisenberg Grant RU 1412/2-2.

generated by elements of the following type:

$$\begin{aligned} &a[x]^j, && a\in W_n\Omega^q_{A/Z},\, j\ge 0\,(\text{resp. } j\in\mathbb{Z}),\\ &b[x]^{j-1}d[x], && b\in W_n\Omega^{q-1}_{A/Z},\, j\ge 1\,(\text{resp. } j\in\mathbb{Z}),\\ &V^s(a[x]^j), && a\in W_{n-s}\Omega^q_{A/Z},\, s\in\{1,\dots,n-1\},\, j\ge 1 \text{ with } (j,p)=1\\ & && (\text{resp. } j\in\mathbb{Z}\setminus p\mathbb{Z}),\\ &dV^s(b[x^j]), && b\in W_{n-s}\Omega^{q-1}_{A/Z},\, s\in\{1,\dots,n-1\},\, j\ge 1 \text{ with } (j,p)=1\\ & && (\text{resp. } j\in\mathbb{Z}\setminus p\mathbb{Z}).\end{aligned}$$

For $A[x]$ and p odd this is [HM04, Thm 4.2.8], for $p=2$ this follows from [Cos08, Thm 4.3] (one has to observe that the functor P constructed in this references sends a $\mathbb{W}(\mathbb{Z})$-linear p-typical Witt complexes over A to a $\mathbb{W}(\mathbb{Z})$-linear p-typical Witt complex over $A[x]$ and that it preserves surjections); the result for $A[x,1/x]$ is deduced from this as in [Rül07b, Thm 2.1], where the reference to [Rül07b, Prop 1.18] should be replaced by [Hes15, Thm C].

Theorem A.4 ([Rül07b, Thm 2.6] for $\mathrm{char}(k)\neq 2$)**.** *Let L/k be a finite field extension. Then there exists a trace map*

$$\mathrm{Tr}_{L/k}:\mathbb{W}_S\Omega^\bullet_L\to\mathbb{W}_S\Omega^\bullet_k,$$

which satisfies the properties (i) - (v) from [Rül07b, Thm 2.6]*, furthermore* [Rül07b, Prop 2.7] *holds.*

Proof. The proof is the same as in *loc. cit.*, once we made the following remarks: [Rül07b, Lem 1.20] holds for any $\mathbb{F}_p$-algebra A with the same proof; [Rül07b, Lem 2.3] also holds for $p=2$, this follows from [Ill79, I, Prop 3.2, 3.4] and a limit argument; [Rül07b, Prop 2.4] holds with the same proof also for $p=2$ once the reference to [Rül07b, Thm 2.1] is replaced by A.3 above. □

A.5. Let p be a prime and A a $\mathbb{Z}_{(p)}$-algebra. For a finite truncation set S we define

$$\mathrm{Fil}_{S,j}:=\mathrm{Ker}(\mathbb{W}_S\Omega^\bullet_{A[[t]]/\mathbb{Z}}\to\mathbb{W}_S\Omega^\bullet_{A[[t]]/(t^j)/\mathbb{Z}}),\quad j\ge 1,$$

and

$$\mathbb{W}_S\hat{\Omega}^\bullet_{A((t))/\mathbb{Z}}=\varprojlim_j\mathbb{W}_S\hat{\Omega}^\bullet_{A((t))/\mathbb{Z}}/\mathrm{Fil}_{S,j}.$$

Then any element in $\mathbb{W}_S\hat{\Omega}^\bullet_{A((t))/\mathbb{Z}}$ can be uniquely written as in [Rül07b, (2.9.1)] and we can define

$$\hat{\mathrm{Res}}^q_{t,n}:W_n\hat{\Omega}^q_{A((t))/\mathbb{Z}}\to W_n\Omega^{q-1}_{A/\mathbb{Z}}$$

as in [Rül07b, (2.9.2)]. (Using A.3, the proof is similar as in [Rül07b, Lem 2.9].)

We define $\mathrm{Res}^q_{t,n}$ as the composition

$$W_n\Omega^q_{A((t))/\mathbb{Z}}\xrightarrow{\text{can.}}W_n\hat{\Omega}^q_{A((t))/\mathbb{Z}}\xrightarrow{\hat{\mathrm{Res}}^q_{t,n}}W_n\Omega^{q-1}_{A/\mathbb{Z}},$$

and if A contains a field and S is a finite truncation set, then we define

$$\mathrm{Res}^q_{t,S}:\mathbb{W}_S\Omega^q_{A((t))}\to\mathbb{W}_S\Omega^{q-1}_A$$

as in [Rül07b, Def 2.11], using that in this case we have $\mathbb{W}_S\Omega^\bullet_A=\mathbb{W}_S\Omega^\bullet_{A/\mathbb{Z}}$ and that the decomposition (A.1.1) also extends to $\mathbb{W}_S\hat{\Omega}$.

For any $\mathbb{Z}_{(p)}$-algebra A the map $\mathrm{Res}^q_{t,n}$ satisfies the properties (i) - (viii) of [Rül07b, Prop 2.12] and [Rül07b, Lem 2.14] holds; if A contains a field the same holds for $\mathrm{Res}^q_{t,S}$, S any finite truncation set. (The case $\mathrm{Res}^q_{t,n}$ is proven as in *loc. cit.*, the case $\mathrm{Res}^q_{t,S}$ follows from this. Note however, that even in the case where A contains a field, the proof of property (iv) and of Lemma 2.14 uses the reduction to a torsion free $\mathbb{Z}_{(p)}$-algebra. Since for $p = 2$ the absolute de Rham-Witt complex is not a dga, we prefer to work with the $\mathbb{W}(\mathbb{Z})$-linear complex.)

Remark A.6. For an $\mathbb{F}_p$-algebra A, the residue $\mathrm{Res}_t : W_n\Omega^q_{A((t))} \to W_n\Omega^{q-1}_A$ was also constructed in [Kat80, §2, Prop 3] using algebraic K-theory and Bloch's approach to the de Rham-Witt complex.

Theorem A.7 ([Rül07a, Thm 2] for $\mathrm{char}(k) \neq 2$)**.** *Let C be a connected regular projective curve over a field k with function field $K = k(C)$. Let S be a finite truncation set. Then*

$$\sum_{P\in C} \mathrm{Res}_P(\omega) = 0, \quad \textit{for all } \omega \in \mathbb{W}_S\Omega^q_K,\ q \geq 1,$$

where the sum is over all closed points in C and $\mathrm{Res}_P : \mathbb{W}_S\Omega^q_K \to \mathbb{W}_S\Omega^{q-1}_k$ is defined as in [Rül07a, Def-Prop 1] *(using $\mathrm{Res}^q_{t,S}$ from A.5 and* Tr *from A.4.)*

Proof. The same proof of [Rül07a, Thm 2], [Rül07b, Thm 2.19] goes through once we made the following remarks: the proof of the well-definedness of Res_P is the same as in [Rül07b, Def-Prop 2.15] since [Rül07b, Lem 1.16] holds in general; [Rül07b, Prop 2.18] holds with the trace from Theorem A.4; at the end of the proof of [Rül07b, Thm 2.19] (on page 140/141) an element is lifted to $\mathbb{W}_S\Omega^1_{A((t))}$, with $A = \mathbb{Z}_{(p)}[z_a, z_b, z_c]$, replace this by the following argument (at least if $p = 2$): first observe that the looked for vanishing can be reduced to the p-typical case by the definition of Res_P; then lift the element $\omega_{2,P}$ to the $W(\mathbb{Z})$-linear complex $W_n\Omega^1_{A((t))/\mathbb{Z}}$ and proceed as in the proof using the Res_t from A.5. □

Proof of Theorem A.2. The proof of [Rül07b, Thm 3.20] (see also [Rül07a]) goes through once we made the following remarks: in [Rül07b, Lem 3.5] replace $\mathbb{W}_m\Omega^r_A$ by $\mathbb{W}_m\Omega^r_{A/\mathbb{Z}}$ (at least if $p = 2$); at the end of the proof of [Rül07b, Thm 3.16] (on page 148) refer to Theorem A.7 instead of [Rül07b, Thm 2.19]; in [Rül07b, Lem 3.15] observe that if $\mathrm{char}(k) = 2$, then $\mathcal{DD}(\alpha) = 0$, since in $K^M_2(k)$ we have $\{a, a\} = \{-1, a\} = 0$, and similar also $\mathcal{F}_r\mathcal{D}\mathcal{V}_r = \mathcal{D}$; this implies that [Rül07b, Prop 3.17] also holds if $\mathrm{char}(k) = 2$; for the rest of the proof of [Rül07b, Thm 3.20] use A.4 for properties of the trace and A.3 instead of [Rül07b, Thm 2.1]. □

References

[Cos08] Viorel Costeanu. On the 2-typical de Rham-Witt complex. *Doc. Math.*, 13:413–452, 2008.

[Hes15] Lars Hesselholt. The big de Rham-Witt complex. *Acta Math.*, 214(1):135–207, 2015.

[HM04] Lars Hesselholt and Ib Madsen. On the De Rham-Witt complex in mixed characteristic. *Ann. Sci. École Norm. Sup. (4)*, 37(1):1–43, 2004.

[Ill79] Luc Illusie. Complexe de de Rham-Witt et cohomologie cristalline. *Ann. Sci. École Norm. Sup. (4)*, 12(4):501–661, 1979.

[Kat80] Kazuya Kato. A generalization of local class field theory by using K-groups. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(3):603–683, 1980.

[Rül07a] Kay Rülling. Erratum to: "The generalized de Rham-Witt complex over a field is a complex of zero-cycles" [J. Algebraic Geom. **16** (2007), no. 1, 109–169; mr2257322]. *J. Algebraic Geom.*, 16(4):793–795, 2007.

[Rül07b] Kay Rülling. The generalized de Rham-Witt complex over a field is a complex of zero-cycles. *Journal of Algebraic Geometry*, 16(1):109–169, 2007.

Freie Universität Berlin, Arnimallee 7, 14195 Berlin, Germany
E-mail address: kay.ruelling@fu-berlin.de