In the exercise session we wanted to find a prime ideal in an infinite product of domains, which is not the inverse image of a prime ideal under the projection to one of the factors (in contrast to what happens for finite products). Actually it is a bit more involved than my first try. Here is the example:

We first need the notion of an ultrafilter: Let Λ be a set. Then an *ultrafilter* on Λ is a subset $U \subset \mathbf{P}(\Lambda)$ of the power set of Λ which satisfies the following properties:

(i) $\emptyset \notin U$
(ii) $A \subset B \subset \Lambda$ and $A \in U \Longrightarrow B \in U$
(iii) $A, B \in U \Longrightarrow A \cap B \in U$
(iv) $A \subset \Lambda \Longrightarrow$ either $A \in U$ or $\Lambda \setminus A \in U$

Observe that

$$A \cup B \in U \Longrightarrow A \in U \text{ or } B \in U. \tag{1}$$

(Indeed, if $A \notin U$, then $\Lambda \setminus A \in U$; hence $(A \cup B) \cap (\Lambda \setminus A) \in U$ but it is a subset set of B; hence $B \in U$.) It follows that if U contains a finite set, then it contains a set with one element say $\{\lambda_0\} \in U$. Then

$$U = \{A \subset \Lambda \,|\, \lambda_0 \in A\}. \tag{2}$$

(Indeed, if $\lambda_0 \in A$, then $A \in U$, by (ii); if $\lambda_0 \notin A$, then $A \subset \Lambda \setminus \{\lambda_0\}$ and hence $A \notin U$ by (ii), (i).) Ultrafilters of type (2) are called *principal ultrafilters* and we just saw that an ultrafilter is principal if and only if it contains a finite set. It can be shown using Zorn's Lemma that if Λ is an infinite set, then non-principal ultrafilters on Λ exist.

Claim 1. Let Λ be an infinite set and R_i, $i \in \Lambda$, domains. Set $R = \prod_{i \in \Lambda} R_i$. Let U be a non-principal ultrafilter on Λ and set

$$\mathfrak{p} := \{(a_i)_{i \in \Lambda} \,|\, \exists A \in U \text{ s. t. } a_i = 0 \text{ if and only if } i \in A\} \subset R.$$

Then $\mathfrak{p} \subset R$ is a prime ideal, which is not of the form $\pi_i^{-1}(\mathfrak{p}_i)$, for some $i \in \Lambda$ and $\mathfrak{p}_i \subset R_i$, where $\pi_i : R \to R_i$ is the i-th projection.

Proof. For $\alpha = (a_i) \in R$ define $A(\alpha) := \{i \in \Lambda \,|\, a_i = 0\}$. Then $\alpha \in \mathfrak{p}$ if and only if $A(\alpha) \in U$.

$\mathfrak{p}$ *is an ideal*: Given $\alpha, \beta \in \mathfrak{p}$, $x \in R$, we have $A(\alpha) \cap A(\beta) \subset A(\alpha + x\beta)$. Hence $A(\alpha + x\beta) \in U$, by (ii), (iii), i.e., $\alpha + x\beta \in \mathfrak{p}$.

$\mathfrak{p}$ *is a prime ideal*: Take $\alpha, \beta \in R$ with $\alpha\beta \in \mathfrak{p}$. Then $A(\alpha\beta) \in U$. Since the R_i's are domains we have $A(\alpha\beta) = A(\alpha) \cup A(\beta)$, whence $A(\alpha) \in U$ or $A(\beta) \in U$, by (1). It follows that $\alpha \in \mathfrak{p}$ or $\beta \in \mathfrak{p}$, i.e., $\mathfrak{p}$ is prime.

$\mathfrak{p}$ *is not of the form* $\pi_j^{-1}(\mathfrak{p}_j)$*:* Notice that e.g. the element $(\delta'_{i,j})_{i \in \Lambda} \in \pi_j^{-1}(\mathfrak{p}_j)$, where $\delta'_{i,j} = 1$, if $i \neq j$, and $= 0$, if $i = j$. But this element is not in $\mathfrak{p}$, since U is a non-principal ultrafilter and hence any element $(a_i) \in \mathfrak{p}$ has to have infinitely many $a_i = 0$. $\square$